RAD HRVATSKE AKADEMIJE ZNANOSTI I UMJETNOSTI
MATEMATIČKE ZNANOSTI

M. N. Faye, K. N. Adédji and A. Togbé

A generalization of a theorem of Murat Alan

Manuscript accepted for publication

A GENERALIZATION OF A THEOREM OF MURAT ALAN

Mariama Ndao Faye, Kouèssi Norbert Adédji and Alain Togbé

Abstract. Let $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ be the Fibonacci and Lucas sequences respectively. In [5], the author found all Fibonacci and Lucas numbers which are concatenations of two terms of the other sequence. Let $b \geq 2$ be an integer. In this paper, we generalize the results in [5] by considering the following Diophantine equations $F_n = b^d L_m + L_k$ and $L_n = b^d F_m + F_k$ in non-negative integers (n, m, k), where d denotes the number of digits of L_k and F_k in base b, respectively.

1. Introduction

Recall that the generalized Lucas sequence $\{U_n\}_{n\geq 0}$ and its companion sequence $\{V_n\}_{n\geq 0}$ are defined with initial values $U_0 = 0,\ U_1 = 1,\ V_0 = 2,\ V_1 = r$, by

$$U_{n+1} = rU_n + sU_{n-1} \quad \text{and} \quad V_{n+1} = rV_n + sV_{n-1}, \quad \text{for} \quad n \geq 0,$$

where r and s are integers such that $\Delta = r^2 + 4s > 0$. The Binet's formulae are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \tag{1.1}$$

where $\alpha = \dfrac{r + \sqrt{\Delta}}{2}$ and $\beta = \dfrac{r - \sqrt{\Delta}}{2}$. If $r = s = 1$, we get the well-known Fibonacci sequence $\{F_n\}$ and its companion sequence Lucas $\{L_n\}$. Easily, it can be seen by induction that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \quad \text{and} \quad \alpha^{n-1} \leq L_n \leq 2\alpha^n, \tag{1.2}$$

2020 *Mathematics Subject Classification.* 11B36, 11J68, 11J86.

Key words and phrases. Fibonacci numbers, Lucas numbers, b-concatenation, logarithmic height, reduction method.

which holds for all $n \geq 1$ and $n \geq 0$, respectively. There are many papers in the literature which deal with Diophantine equations involving linear recurrent sequences. For more details, see [1], [2], [3], [4], [9], [11], [12], [14]. In 2005, Banks and Luca proved in [6] that if u_n is any binary recurrent sequence of integers then only finitely many terms of the sequence u_n can be written as concatenations of two or more terms of the same sequence u_n under the certain mild hypotheses on u_n. Namely, they found that $13, 21$, and 55 are the only Fibonacci numbers which are non trivial concatenations of two terms of Fibonacci numbers. Later, Alan proved in [5] that $13, 21$, and 34 are the only Fibonacci numbers which are concatenations of two Lucas numbers and $1, 2, 3, 11, 18$, and 521 are the only Lucas numbers which are concatenations of two Fibonacci numbers. In this paper, we give the following concept in view to generalize Alan's result (see [5]).

Definition 1.1. *Let $b \geq 2$ be an integer. Let N be a positive integer and suppose N can be written as*

$$N = a_1 \times b^d + a_2,$$

where a_1 and a_2 are non-negative integers and d is the number of digits of a_2 in base b. Then, we call the number N a b-concatenation of a_1 and a_2.

The goal of this study is to investigate all Fibonacci numbers which are b-concatenations of two Lucas numbers as well as all Lucas numbers which are b-concatenations of two Fibonacci numbers. More precisely, we solve the following two Diophantine equations

$$F_n = b^d L_m + L_k \quad \text{and} \quad L_n = b^d F_m + F_k,$$

in non-negative integers (n, m, k), where d represents the number of digits of L_k and F_k in base b respectively. Therefore, we generalize the results in [5]. The novelty here is that for fixed b, we prove that the considered equations have only finitely many solutions up to the point where all these solutions are found in the range $2 \leq b \leq 10$. Our proofs use a result of Matveev [15] on linear forms in logarithms of algebraic numbers and the reduction method due to Dujella and Pethő [10]. We use a slight modified version of their original result.

2. Preliminary results

In this section, we recall the two key results that we need to prove our main results.

2.1. *Matveev's Theorem.* Let η be an algebraic number of degree t, let $a_0 \neq 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\eta =$

$\eta^{(1)}, \ldots, \eta^{(t)}$ denote its conjugates. The logarithmic height of η is defined by

$$h(\eta) = \frac{1}{t}\left(\log|a_0| + \sum_{j=1}^{t} \log\max\left(1, \left|\eta^{(j)}\right|\right)\right).$$

In the case where p and q are integers such that $q \geq 1$ and $\gcd(p, q) = 1$, then taking $\eta = p/q$ the above definition reduces to $h(\eta) = \log(\max\{|p|, q\})$. We have the following result due to Bugeaud, Mignotte, and Siksek (see [8, Theorem 9.4]) which is an improved version of Matveev's result (see [15]).

Theorem 2.1. *Let $\gamma_1, \ldots, \gamma_s$ be real algebraic numbers and let $b_1, \ldots, b_s$ be nonzero integers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j = \max\{Dh(\gamma_j), |\log\gamma_j|, 0.16\}, \quad \text{for} \quad j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\Lambda := \gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1 \neq 0$, then

$$|\Lambda| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_s). \tag{2.3}$$

2.2. *Dujella-Pethő's Lemma.* Let x be a real number. We denote by $||x|| := \min\{|x - n| : n \in \mathbb{Z}\}$ the distance from x to the nearest integer. Thus, we have the following result that is a slight modified version of the original result due to Dujella and Pethő [10].

Lemma 2.2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of an irrational number τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon = ||\mu q|| - M \cdot ||\tau q||.$$

If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < |m\tau - n + \mu| < AB^{-w},$$

in positive integers m, n and w with

$$m \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. Fibonacci numbers as b-concatenation of two Lucas numbers

Given a real θ, we denote the floor function of θ by $\lfloor\theta\rfloor$ the greatest integer less than or equal to θ. In this subsection, we will prove the following result.

Theorem 3.1. *Let $b \geq 2$ be an integer. Then, the Diophantine equation*

$$F_n = b^d L_m + L_k, \tag{3.4}$$

has only finitely many solutions in non-negative integers (k, b, m, n, d) with $d = \lfloor\log_b L_k\rfloor + 1$. Namely, we have $n < 2.6 \times 10^{29} \cdot \log^4 b$.

3.1. *Proof of Theorem 3.1.* If the Diophantine equation (3.4) holds for $b \geq 2$, we would get from (1.2) that

$$\begin{aligned} d = \lfloor \log_b L_k \rfloor + 1 \leq 1 + \log_b L_k < 1 + \log_b(2\alpha^k) \\ = 1 + k\frac{\log\alpha}{\log b} + \frac{\log 2}{\log b} < k + 2 \end{aligned}$$

and

$$d = \lfloor \log_b L_k \rfloor + 1 > \log_b L_k \geq \log_b(\alpha^{k-1}) = (k-1)\frac{\log\alpha}{\log b}.$$

So, we deduce that

$$(k-1)\frac{\log\alpha}{\log b} < d < k + 2. \tag{3.5}$$

Since

$$L_k = b^{\log_b L_k} < b^d \leq b^{1+\log_b L_k} = b \cdot b^{\log_b L_k} = b \cdot L_k,$$

then we have

$$L_k < b^d \leq b \cdot L_k. \tag{3.6}$$

From the last inequality together with equation (3.4) we can easily see according to (1.2) that

$$\alpha^{n-2} \leq F_n = b^d L_m + L_k \leq b \cdot L_k \cdot L_m + L_k < (b+1) \cdot L_m \cdot L_k$$

and

$$(b+1) \cdot L_m \cdot L_k < (b+1) \cdot 2\alpha^m \cdot 2\alpha^k = 4(b+1)\alpha^{m+k} = \alpha^{\log_\alpha(4(b+1))} \cdot \alpha^{m+k}.$$

Therefore, we obtain

$$\alpha^{n-2} < \alpha^{m+k+\log_\alpha(b+1)+\log_\alpha 4}. \tag{3.7}$$

Also, we get

$$\alpha^{n-1} \geq F_n = b^d L_m + L_k > L_k \cdot L_m + L_k > L_k \cdot L_m \geq \alpha^{m+k-2},$$

which leads to

$$\alpha^{n-1} > \alpha^{m+k-2}. \tag{3.8}$$

We now combine inequalities (3.7) and (3.8) to obtain

$$m + k - 1 < n < m + k + 5 + \frac{\log(b+1)}{\log\alpha}. \tag{3.9}$$

For the rest of the proof we can only consider that $n - k \geq 4$. Let us show it now. Note that $L_k = F_{k+1} + F_{k-1}$, for $k \geq 1$ and $F_{k+3} = 2F_{k+1} + F_{k-1} + F_{k-2}$, for $k \geq 2$. Thus, the Diophantine equation (3.4) becomes

$$F_n = b^d L_m + F_{k+1} + F_{k-1}.$$

Since $L_m \geq 1$ and $b^d > L_k$, we have $F_n = b^d L_m + L_k > b^d + L_k > L_k + L_k = 2L_k$. Thus

$$\begin{aligned} F_n &> 2(F_{k+1} + F_{k-1}) \\ &= (2F_{k+1} + F_{k-1}) + F_{k-1} > 2F_{k+1} + F_{k-1} + F_{k-2} = F_{k+3}. \end{aligned}$$

It follows that $n > k+3$, more precisely

$$n - k \geq 4.$$

We use now Binet's formula for Fibonacci and Lucas sequences in order to rewrite the Diophantine equation (3.4) in the form

$$\frac{\alpha^n - \beta^n}{\sqrt{5}} = (\alpha^m + \beta^m)\, b^d + L_k$$

which leads to

$$\frac{\alpha^n}{\sqrt{5}} - \alpha^m b^d = \frac{\beta^n}{\sqrt{5}} + \beta^m b^d + L_k.$$

Taking absolute values of both sides of the above equality, we get that

$$\left| \frac{\alpha^n}{\sqrt{5}} - \alpha^m b^d \right| < \left| \frac{\beta^n}{\sqrt{5}} \right| + |\beta^m|\, b^d + L_k.$$

Since $\beta = -\alpha^{-1}$, then we have that

$$\left| \frac{\alpha^n}{\sqrt{5}} - \alpha^m b^d \right| < \frac{1}{\alpha^n \sqrt{5}} + \frac{b^d}{\alpha^m} + L_k.$$

Dividing through by $\alpha^n/\sqrt{5}$, we get the inequality

$$\left| 1 - \frac{b^d \sqrt{5}}{\alpha^{n-m}} \right| < \frac{1}{\alpha^{2n}} + \frac{b^d \sqrt{5}}{\alpha^{n+m}} + \frac{\sqrt{5} L_k}{\alpha^n}.$$

Combining now $L_k < b^d \leq b \cdot L_k$ with the inequalities of L_k, given by (1.2), we get the following estimates

$$\begin{aligned} \left| 1 - \frac{b^d \sqrt{5}}{\alpha^{n-m}} \right| &< \frac{1}{\alpha^{2n}} + \frac{b \cdot 2\alpha^k \sqrt{5}}{\alpha^{n+m}} + \frac{2\alpha^k \sqrt{5}}{\alpha^n} \\ &= \frac{1}{\alpha^{2n}} + \frac{2\sqrt{5} b}{\alpha^{n+m-k}} + \frac{2\sqrt{5}}{\alpha^{n-k}} \\ &< \frac{1}{\alpha^{n-k}} + \frac{2\sqrt{5} b}{\alpha^{n-k}} + \frac{2\sqrt{5}}{\alpha^{n-k}} = \frac{1 + 2\sqrt{5} b + 2\sqrt{5}}{\alpha^{n-k}}. \end{aligned}$$

Furthermore, for $b \geq 2$, we have $1 + 2\sqrt{5} b + 2\sqrt{5} < \alpha^5 \cdot b$. Thus, we obtain

$$\left| 1 - \frac{b^d \sqrt{5}}{\alpha^{n-m}} \right| < \frac{b}{\alpha^{n-k-5}}. \tag{3.10}$$

Next, to apply Theorem 2.1, we need to take

$$\Lambda_1 := 1 - b^d \cdot \sqrt{5} \cdot \alpha^{-(n-m)} \tag{3.11}$$

and

$$\begin{array}{lll} \gamma_1 = b, & \gamma_2 = \sqrt{5}, & \gamma_3 = \alpha, \\ b_1 = d, & b_2 = 1, & b_3 = -(n-m). \end{array}$$

Assume that $\Lambda_1 = 0$. We obtain

$$\alpha^{n-m} = \sqrt{5} \cdot b^d. \tag{3.12}$$

Taking the norm in $\mathbb{Q}(\sqrt{5})$ of both sides of (3.12), we get $\pm 1 = 5b^{2d}$, which is impossible. So $\Lambda_1 \neq 0$. We know that $\gamma_1, \gamma_2, \gamma_3$ are elements of $\mathbb{L} = \mathbb{Q}(\sqrt{5})$. Hence $D := [\mathbb{L} : \mathbb{Q}] = 2$. Using the properties of the height function $h(\cdot)$, we get

$$h(\gamma_1) = h(b) = \log b,\ h(\gamma_2) = h(\sqrt{5}) = \frac{1}{2}\log 5,\ h(\gamma_3) = h(\alpha) = \frac{1}{2}\log \alpha.$$

Therefore, in Theorem 2.1 we can take the following values

$$A_1 = 2\log b, \quad A_2 = \log 5 \quad \text{and} \quad A_3 = \log \alpha.$$

Dividing its both sides of $F_n = b^d L_m + L_k$ by L_m, we get

$$b^d < b^d + \frac{L_k}{L_m} = \frac{F_n}{L_m} \leq \alpha^{n-m},$$

and then

$$d < (n-m) \cdot \frac{\log \alpha}{\log b},$$

which also implies that

$$d < n - m, \quad \text{for} \quad b \geq 2. \tag{3.13}$$

We deduce that

$$\max\{|b_1|, |b_1|, |b_3|\} = \max\{d; 1; n-m\} < n - m = B.$$

Taking $s = 3$. and applying Theorem 2.1 to (3.11) lead to

$$\begin{aligned} \log|\Lambda_1| > & -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2) \cdot (1 + \log(n-m)) \times \\ & 2\log b \cdot \log 5 \cdot \log \alpha. \end{aligned}$$

Combining this with (3.10), we get

$$n - k - 5 < 1.6 \times 10^{12} \cdot \log b \cdot (1 + \log(n-m)). \tag{3.14}$$

Then, we rewrite the Diophantine equation (3.4) in the form

$$\frac{\alpha^n - \beta^n}{\sqrt{5}} = b^d L_m + \alpha^k + \beta^k,$$

which also implies that

$$\alpha^n \left(\frac{1}{\sqrt{5}} - \alpha^{k-n} \right) - b^d L_m = \frac{\beta^n}{\sqrt{5}} + \beta^k.$$

Taking the absolute value of both sides of the above equality and using the fact that $\beta = -\alpha^{-1}$, we get

$$\left|\alpha^n\left(\frac{1}{\sqrt{5}} - \alpha^{k-n}\right) - b^d L_m\right| < \left|\frac{\beta^n}{\sqrt{5}}\right| + \left|\beta^k\right| = \frac{1}{\sqrt{5}\alpha^n} + \frac{1}{\alpha^k}.$$

Therefore, we obtain

$$\left|1 - \frac{b^d L_m}{\alpha^n\left(\frac{1}{\sqrt{5}} - \alpha^{k-n}\right)}\right| < \frac{1}{1/\sqrt{5} - \alpha^{k-n}} \cdot \left(\frac{1}{\sqrt{5}\alpha^{2n}} + \frac{1}{\alpha^{n+k}}\right).$$

Moreover, we have

$$\begin{aligned}\left|1 - \frac{b^d L_m}{\alpha^n\left(\frac{1}{\sqrt{5}} - \alpha^{k-n}\right)}\right| &< \frac{1}{1/\sqrt{5} - \alpha^{k-n}} \times \left(\frac{1}{\sqrt{5}\alpha^{2n}} + \frac{1}{\alpha^{n+k}}\right)\\ &= \frac{\sqrt{5}\alpha^{n-k}}{\alpha^{n-k} - \sqrt{5}} \times \left(\frac{1}{\sqrt{5}\alpha^{2n}} + \frac{1}{\alpha^{n+k}}\right)\\ &= \frac{\sqrt{5}}{\alpha^{n-k} - \sqrt{5}} \times \left(\frac{1}{\sqrt{5}\alpha^{n+k}} + \frac{1}{\alpha^{2k}}\right)\\ &< \frac{1.5 \cdot \sqrt{5}}{\alpha^{n-k} - \sqrt{5}} \times \left(\frac{1}{\alpha^{2k}}\right).\end{aligned}$$

In above inequalities, we have used the fact that $n - k \geq 4$, which implies $n > k$. Since $n - k \geq 4$, we have

$$0 < \frac{1.5 \cdot \sqrt{5}}{\alpha^{n-k} - \sqrt{5}} < 1,$$

and therefore

$$\left|1 - \frac{b^d L_m}{\alpha^n\left(\frac{1}{\sqrt{5}} - \alpha^{k-n}\right)}\right| < \frac{1}{\alpha^{2k}}. \tag{3.15}$$

Put

$$\Lambda_2 := 1 - \frac{b^d L_m}{\alpha^n\left(\frac{1}{\sqrt{5}} - \alpha^{k-n}\right)}.$$

We will apply Theorem 2.1 to Λ_2. So, we take the following data

$$\begin{aligned}&\gamma_1 = b, \quad \gamma_2 = \alpha, \quad \gamma_3 = \frac{L_m}{1/\sqrt{5} - \alpha^{k-n}},\\ &b_1 = d, \quad b_2 = -n, \quad b_3 = 1.\end{aligned}$$

Note that γ_1, γ_2 and γ_3 are elements of the real quadratic number field $\mathbb{L} = \mathbb{Q}(\sqrt{5})$. Therefore, we have $D := [\mathbb{L} : \mathbb{Q}] = 2$, the degree of the number field

$\mathbb{L}$. The heights of the algebraic numbers γ_1, γ_2 and γ_3 are defined respectively by $h(\gamma_1) = \log b$, $\quad h(\gamma_2) = \frac{1}{2}\log\alpha$ and

$$\begin{aligned} h(\gamma_3) = h\left(\frac{L_m}{\frac{1}{\sqrt{5}} - \alpha^{k-n}}\right) &\leq h(L_m) + h\left(\frac{1}{\sqrt{5}} - \alpha^{k-n}\right) \\ &\leq \log L_m + h\left(\frac{1}{\sqrt{5}}\right) + h(\alpha^{k-n}) + \log 2 \\ &< m\log\alpha + \frac{1}{2}\log 5 + \frac{n-k}{2}\log\alpha + 2\log 2. \end{aligned}$$

From (3.9), we get $m < (n-k)+1$. It follows that

$$h(\gamma_3) < \frac{3(n-k)+2}{2}\log\alpha + \log(4\sqrt{5}).$$

Therefore, we can take

$$A_1 = 2\log b, \quad A_2 = \log\alpha, \quad A_3 = (3(n-k)+2)\log\alpha + 2\log(4\sqrt{5}).$$

Since $B \geq \max\{|b_i|\} = \max\{d, 1, n\}$ and $d < n - m < n$, then we can take $B = n$. Also, in this case $s = 3$. Thus, combining (3.15) with Theorem 2.1 we see that

$$\begin{aligned} k < 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2) \cdot (1+\log n) \cdot \log b \times \\ (3(n-k)+2)\log\alpha + 2\log(4\sqrt{5}) \end{aligned}$$

which becomes

$$k < 9.7 \times 10^{11} \cdot (1+\log n) \cdot \left[(3(n-k)+2)\log\alpha + 2\log(4\sqrt{5})\right] \cdot \log b. \tag{3.16}$$

Assume that $k \leq m$. Then, $n - m \leq n - k$ and using (3.14) we can write

$$n - k - 5 < 1.6 \times 10^{12} \cdot \log b \cdot (1 + \log(n-k))$$

which implies

$$n - k < 2.8 \times 10^{12} \cdot \log(n-k) \cdot \log b. \tag{3.17}$$

Note also that to obtain inequality (3.17), we have used the fact that

$$1 + \log(n-k) < 1.72\log(n-k)$$

which is valid for $n - k \geq 4$. To get an upper bound of $n - k$ in terms of b, we have to recall the following result [13, Lemma 7].

Lemma 3.2. *If $\ell \geq 1$, $H > \left(4\ell^2\right)^\ell$ and $H > L/(\log L)^\ell$, then*

$$L < 2^\ell H(\log H)^\ell.$$

Thus, we can take $\ell = 1, L = n-k$ and $H = 2.8 \times 10^{12} \cdot \log b$. Therefore, we deduce that

$$\begin{aligned} n-k &< 2 \times 2.8 \times 10^{12} \cdot \log b \times \log(2.8 \times 10^{12} \cdot \log b) \\ &< 5.6 \times 10^{12} \cdot \log b \times (28.7 + \log\log b). \end{aligned}$$

For $b \geq 2$, we can easily see that $28.7 + \log\log b < 41 \log b$. Thus, it follows that

$$n-k < 2.3 \times 10^{14} \cdot \log^2 b \quad \text{and} \quad m-1 < n-k < 2.3 \times 10^{14} \cdot \log^2 b.$$

Since $k \leq m$, from (3.9) we have

$$n < m+k+5+\frac{\log(b+1)}{\log\alpha} \leq 2m+5+\frac{\log(b+1)}{\log\alpha}.$$

Therefore, we obtain

$$n < 4.7 \times 10^{14} \cdot \log^2 b. \tag{3.18}$$

Assume now that $m < k$. Combining the inequalities (3.9) and (3.16), we obtain

$$\begin{aligned} \frac{1}{2}\left[n-5-\frac{\log(b+1)}{\log\alpha}\right] &< k < 9.7 \times 10^{11} \times (1+\log n) \\ &\times \left[(3(n-k)+2)\log\alpha + 2\log(4\sqrt{5})\right] \times \log b. \end{aligned} \tag{3.19}$$

From inequality (3.14), we get

$$\begin{aligned} &(3(n-k)+2)\log\alpha + 2\log(4\sqrt{5}) \\ &= 3(n-k)\log\alpha + 2\log\alpha + 2\log(4\sqrt{5}) \\ &< 1.6 \times 10^{12} \cdot 3\log\alpha \cdot (1+\log n) \cdot \log b + 17\log\alpha + 2\log(4\sqrt{5}) \\ &< 2.4 \times 10^{12} \cdot (1+\log n) \cdot \log b. \end{aligned}$$

Substituting this in (3.19) leads to

$$n < 4.7 \times 10^{24} \cdot (1+\log n)^2 \cdot \log^2 b,$$

which also implies that $n < 9.4 \times 10^{24} \cdot \log^2 n \cdot \log^2 b$ where we have used the fact that $1 + \log n < 2\log n$, for all $n > 1$. To get an upper bound of n in term of b, we need to refer to Lemma 3.2 by putting $\ell = 2$, $L = n$ and $H = 9.4 \times 10^{24} \cdot \log^2 b$. Then, Lemma 3.2 gives

$$n < 2^2 \times 9.4 \times 10^{24} \cdot \log^2 b \times (57.6 + 2\log\log b)^2.$$

Note that for $b \geq 2$, we have $57.6 + 2\log\log b < 83\log b$ and then

$$n < 2.6 \times 10^{29} \cdot \log^4 b. \tag{3.20}$$

Therefore, from (3.18) and (3.20) whether $m \leq k$ or not the bound $n < 2.6 \times 10^{29} \cdot \log^4 b$ is valid in all cases. This completes the proof.

3.2. *Application for* $2 \le b \le 10$. First, in the range $0 \le \max\{m,k\} < 100$, we got only the Fibonacci numbers, which satisfy equation (3.4) and given in the following result. Note also that the upper bounds of the parameters in this range are obtained by referring to the inequalities (3.5) and (3.9).

THEOREM 3.3. *Let* b *be a positive integer such that* $2 \le b \le 10$. *Then, the numbers* 3, 5, 8, 13, 21, 34, 55, 89, 233, 377, 610, *and* 987 *are the only Fibonacci numbers which satisfy the Diophantine equation* (3.4). *More precisely, we have*

$$\begin{array}{rcccl}
3 & = & F_4 & = & 2^1\cdot 1+1,\\
5 & = & F_5 & = & 2^1\cdot 2+1 = 3^1\cdot 1+2 = 4^1\cdot 1+1,\\
8 & = & F_6 & = & 3^1\cdot 2+2 = 5^1\cdot 1+3 = 6^1\cdot 1+2 = 7^1\cdot 1+1,\\
13 & = & F_7 & = & 3^1\cdot 4+1 = 3^2\cdot 1+4 = 4^1\cdot 3+1 = 5^1\cdot 2+3\\
13 & = & F_7 & = & 6^1\cdot 2+1 = 9^1\cdot 1+4 = 10^1\cdot 1+3,\\
21 & = & F_8 & = & 3^2\cdot 2+3 = 5^1\cdot 4+1 = 6^1\cdot 3+3 = 9^1\cdot 2+3,\\
21 & = & F_8 & = & 10^1\cdot 2+1,\\
34 & = & F_9 & = & 3^1\cdot 11+1 = 3^2\cdot 3+7 = 8^1\cdot 4+2 = 9^1\cdot 3+7,\\
34 & = & F_9 & = & 10^1\cdot 3+4,\\
55 & = & F_{10} & = & 3^1\cdot 18+1 = 4^2\cdot 3+7,\\
89 & = & F_{11} & = & 3^1\cdot 29+2 = 8^1\cdot 11+1,\\
233 & = & F_{13} & = & 8^1\cdot 29+1,\\
377 & = & F_{14} & = & 8^1\cdot 47+1,\\
610 & = & F_{15} & = & 8^1\cdot 76+2,\\
987 & = & F_{16} & = & 6^3\cdot 4+123 = 8^1\cdot 123+3.
\end{array}$$

In fact, we will prove that there is no more other solutions of the Diophantine equation (3.4) if $\max\{m,k\} \ge 100$. Thus, let us assume that $\max\{m,k\} \ge 100$. It suffices to prove the following result.

PROPOSITION 3.4. *If the Diophantine equation* (3.4) *holds, then* $m \le 190$. *Moreover, if* $m < k$, *then the set of solution of* (3.4) *is empty.*

PROOF. We assume that $m > 190$. Let

$$\Gamma_1 := d\log b - (n-m)\log\alpha + \log(\sqrt{5}). \tag{3.21}$$

As $184 < m-6 < n-k-5$ and $2 \le b \le 10$, then from (3.11), we obtain

$$|\Lambda_1| := |e^{\Gamma_1} - 1| < \frac{b}{\alpha^{n-k-5}} < \frac{1}{2}.$$

It follows that

$$|\Gamma_1| < \frac{2b}{\alpha^{n-k-5}}.$$

So, from (3.21), we write

$$0 < \left| d\frac{\log b}{\log\alpha} - (n-m) + \frac{\log(\sqrt{5})}{\log\alpha} \right| < \frac{2b/\log\alpha}{\alpha^{n-k-5}}. \tag{3.22}$$

Also, for $2 \le b \le 10$ we have $d < n - m < n < 7.3 \times 10^{30}$. Hence, since the conditions of Lemma 2.2 are satisfied, we may now apply it to inequalities (3.22) with the following data:

$$M := 7.3 \times 10^{30}, \quad A := 2b/\log\alpha, \quad B := \alpha, \quad w := n - k - 5,$$

$$\tau := \frac{\log b}{\log\alpha} \quad \text{and} \quad \mu := \frac{\log(\sqrt{5})}{\log\alpha}.$$

Let q_t be the denominator of the t-th convergent of the continued fraction of τ. So, with the help of Mathematica the results obtained by applying Lemma 2.2 are listed in the following table.

Table 1. The upper bound on $n - k - 5$

b	2	3	4	5	6	7	8	9	10
q_t	q_{71}	q_{63}	q_{70}	q_{66}	q_{62}	q_{70}	q_{62}	q_{60}	q_{65}
$n-k-5<$	160	159	160	161	166	162	163	170	163
$\varepsilon >$	0.42	0.29	0.44	0.48	0.05	0.11	0.34	0.48	0.43

So, we deduce that

$$n - k - 5 < \frac{\log((18/\log\alpha) \cdot q_{60}/0.48))}{\log\alpha} < 170,$$

in all cases. However, this contradicts the fact that $184 < m - 6 < n - k - 5$. Therefore, we conclude that $m \le 190$. For the second part of the proof, we suppose that $m < k$. From (3.9), we obtain

$$n - k < m + 5 + \frac{\log(b+1)}{\log\alpha} < 200.$$

Substituting this upper bound for $n - k$ into the (3.19) and using the fact that $2 \le b \le 10$, we get

$$\frac{1}{2}\left[n - 5 - \frac{\log(b+1)}{\log\alpha}\right] < k < 2.9 \times 10^{14} \times (1 + \log n) \cdot \log b,$$

and then

$$n < 1.4 \times 10^{15} \times (1 + \log n)). \tag{3.23}$$

Therefore, from (3.23), it follows that $n < 5.6 \times 10^{16}$. Next, we put

$$\Gamma_2 := d\log b - n\log\alpha + \log\left(\frac{L_m}{1/\sqrt{5} - \alpha^{k-n}}\right). \tag{3.24}$$

Thus we get

$$|\Lambda_2| := |e^{\Gamma_2} - 1| < \frac{1}{\alpha^{2k}}.$$

Because $m < k$, then $k = \max\{m, k\} \geq 100$. Thus $1/\alpha^{2k} < 1/2$, which also implies

$$|\Gamma_2| < \frac{2}{\alpha^{2k}}. \tag{3.25}$$

By dividing both sides of (3.25) by $\log \alpha$, we get

$$0 < \left| d\frac{\log b}{\log \alpha} - n + \frac{\log\left(\dfrac{L_m}{1/\sqrt{5} - \alpha^{k-n}}\right)}{\log \alpha} \right| < \frac{2/\log \alpha}{\alpha^{2k}}. \tag{3.26}$$

To apply Lemma 2.2, we will use the following parameters:

$$w := 2k, \quad A := 2/\log \alpha, \quad B := \alpha, \quad M := 5.6 \times 10^{16} > n > d,$$

$$\tau := \frac{\log b}{\log \alpha} \quad \text{and} \quad \mu := \frac{\log\left(\dfrac{L_m}{1/\sqrt{5} - \alpha^{k-n}}\right)}{\log \alpha}.$$

Let q_t be the denominator of the t-th convergent of the continued fraction of τ. Now, using Mathematica we see for each $0 \leq m \leq 190$ and $4 \leq n - k < 200$, the following results:

Table 2. The upper bound on $2k$

b	2	3	4	5	6	7	8	9	10
q_t	q_{40}	q_{41}	q_{38}	q_{34}	q_{36}	q_{41}	q_{37}	q_{36}	q_{38}
$2k <$	107	109	107	106	106	111	106	106	114
$\varepsilon >$	10^{-4}	10^{-4}	10^{-5}	10^{-4}	10^{-4}	10^{-5}	0.002	10^{-4}	10^{-6}

Therefore, in all cases we get that $k < 57$, which is a contradiction because of the bound on k. This completes the proof of Proposition 3.4. □

4. Lucas numbers as b-concatenation of two Fibonacci numbers

In this section, we will prove the following result. Here, the method is similar to that developed is Section 3. Therefore, we will avoid some details.

Theorem 4.1. *Let $b \geq 2$ be an integer. Then, the Diophantine equation*

$$L_n = b^d F_m + F_k, \tag{4.27}$$

has only finitely many solutions in non-negative integers (k, b, m, n, d) with $d = \lfloor \log_b L_k \rfloor + 1$. Namely, we have $n < 1.8 \times 10^{30} \cdot \log^4 b$.

4.1. *Proof of Theorem 4.1.* We start by assuming that equation (4.27) holds. First, we determine some relations between the variables n, m, k, and d, where d is the number of digits of F_k in base b, i.e., $d = \lfloor \log_b F_k \rfloor + 1$. Also we assume that $m \neq 0$. The case $m = 0$ will be treated separately in the next subsection. Therefore, using the idea developed at the beginning of subsection 3.1, we easily obtain the following inequalities

$$(k-2)\frac{\log \alpha}{\log b} < d < k, \tag{4.28}$$

$$F_k < b^d \leq b \cdot F_k, \tag{4.29}$$

and

$$m + k - 4 - \frac{\log 2}{\log \alpha} < n \leq m + k - 1 + \frac{\log(b+1)}{\log \alpha}. \tag{4.30}$$

Furthermore, from inequalities (1.2) and (4.27) we get

$$2\alpha^n > L_n = b^d F_m + F_k > F_k \geq \alpha^{k-2}$$

and then

$$n - k > -\left(2 + \frac{\log 2}{\log \alpha}\right) \quad \text{i.e.,} \quad n - k \in \{-3, -2, -1, 0, 1, 2, \cdots\}.$$

Moreover, from (4.27) we get

$$b^d F_m = \begin{cases} F_{k-4} - F_{k-1}, & \text{if} \quad n = k-3, \\ F_{k-3} - F_{k-2}, & \text{if} \quad n = k-2, \end{cases}$$

which leads to a contradiction. Therefore, we will study equation (4.27) in the range

$$n - k \geq -1.$$

To do this, we can focus on the following two cases.
The case $n - k \geq 1$.

With Binet's Formula for Fibonacci and Lucas sequences, we rewrite the Diophantine equation (4.27) into the form

$$\alpha^n + \beta^n = \left(\frac{\alpha^m - \beta^m}{\sqrt{5}}\right) b^d + F_k.$$

So, we get

$$\alpha^n - \frac{\alpha^m}{\sqrt{5}} b^d = \frac{-\beta^n}{\sqrt{5}} b^d - \beta^n + F_k.$$

Hence, we have

$$\left|\alpha^n - \frac{\alpha^m}{\sqrt{5}} b^d\right| < |\beta^n| + \frac{|\beta^m|\, b^d}{\sqrt{5}} + F_k = \frac{1}{\alpha^n} + \frac{b^d}{\sqrt{5}\alpha^m} + F_k. \tag{4.31}$$

Dividing through (4.31) by α^n and using (4.29), we have

$$\left|1 - \frac{b^d}{\sqrt{5}\alpha^{n-m}}\right| < \frac{b+2}{\alpha^{n-k}}. \tag{4.32}$$

Put

$$\Lambda_3 := 1 - \frac{b^d}{\sqrt{5}\alpha^{n-m}}.$$

Applying Theorem 2.1 to Λ_3 by choosing

$$(\gamma_1, b_1) = (b, d), \quad (\gamma_2, b_2) = (\sqrt{5}, -1) \text{ and } (\gamma_3, b_3) = (\alpha, -(n-m)),$$

we obtain

$$\begin{aligned} \log|\Lambda_3| > &-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1+\log 2) \cdot (1+\log(n-m+2)) \times \\ &2\log b \cdot \log 5 \cdot \log\alpha. \end{aligned} \tag{4.33}$$

Combining now (4.32) and (4.33), we get

$$n - k < 3.2 \times 10^{12} \cdot (1 + \log(n-m+2)) \cdot \log b. \tag{4.34}$$

We use (1.1) to rearrange equation (4.27) as

$$\alpha^n + \beta^n = b^d F_m + \frac{\alpha^k - \beta^k}{\sqrt{5}},$$

which also leads to

$$\alpha^n\left(1 - \frac{\alpha^{k-n}}{\sqrt{5}}\right) - b^d F_m = -\beta^n - \frac{\beta^k}{\sqrt{5}}.$$

Taking now absolute value of both sides of the equality above, we get

$$\left|\alpha^n\left(1 - \frac{\alpha^{k-n}}{\sqrt{5}}\right) - b^d F_m\right| < |\beta^n| + \left|\frac{\beta^k}{\sqrt{5}}\right| = \frac{1}{\alpha^n} + \frac{1}{\sqrt{5}\alpha^k}$$

and therefore

$$\left|1 - \frac{b^d F_m}{\alpha^n\left(1 - \frac{\alpha^{k-n}}{\sqrt{5}}\right)}\right| < \frac{1}{\alpha^{2k}}, \tag{4.35}$$

which is valid for $n - k \geq 1$. Put

$$\Lambda_4 := 1 - \frac{b^d F_m}{\alpha^n\left(1 - \frac{\alpha^{k-n}}{\sqrt{5}}\right)}.$$

Next, we can take the data

$$\begin{aligned} &\gamma_1 = b, \quad \gamma_2 = \alpha, \quad \gamma_3 = \frac{F_m}{1 - \frac{\alpha^{k-n}}{\sqrt{5}}}, \\ &b_1 = d, \quad b_2 = -n, \quad b_3 = 1 \end{aligned}$$

and $s = 3$ in view to apply Theorem 2.1 to Λ_4. Using Theorem 2.1 and combining the result obtained with inequality (4.35), we get

$$\begin{aligned}2k\log\alpha &< 1.4\times 30^6\times 3^{4.5}\times 2^2\times(1+\log 2)\times(1+\log(n+2)))\\ &\times 2\log b\times\log\alpha\times[(3(n-k)+6)\log\alpha+3.7].\end{aligned}$$

This implies that

$$k < 9.7\times 10^{11}\cdot(1+\log(n+2))\cdot((3(n-k)+6)\log\alpha+3.7)\cdot\log b. \tag{4.36}$$

• If $k \le m$, then $n-m \le n-k$. Hence, from inequality (4.34) we obtain

$$n-k < 3.2\times 10^{12}\cdot(1+\log(n-k+2))\cdot\log b,$$

which implies

$$n-k+2 < 6.2\times 10^{12}\cdot\log(n-k+2)\cdot\log b, \tag{4.37}$$

where we use the fact that

$$1+\log(n-k+2) < 1.91\log(n-k+2) \quad \text{for} \quad n-k+2\ge 3.$$

Next, to get an upper bound of $n-k$ in terms of b, we have to apply Lemma 3.2 to (4.37) with

$$\ell = 1,\ L = n-k+2 \text{ and } H = 6.2\times 10^{12}\cdot\log b.$$

Thus, we have $n-k < 5.4\times 10^{14}\cdot\log^2 b$. Moreover, from (4.30) we obtain

$$m - \left(4+\frac{\log 2}{\log\alpha}\right) < n-k < 5.4\times 10^{14}\cdot\log^2 b$$

and

$$n \le 2m-1+\frac{\log(b+1)}{\log\alpha} < 1.1\times 10^{15}\cdot\log^2 b. \tag{4.38}$$

• If $m < k$, then from (4.30) and (4.36) we see that

$$\begin{aligned}\frac{1}{2}\left[n-\frac{\log(b+1)}{\log\alpha}+1\right] &< k < 9.7\times 10^{11}\cdot(1+\log(n+2))\\ &\times[(3(n-k)+6)\log\alpha+3.7]\cdot\log b.\end{aligned} \tag{4.39}$$

Now, combining (4.34) with $n-m<n$, we deduce that

$$(3(n-k)+6)\log\alpha+3.7 < 4.7\times 10^{12}\cdot(1+\log(n+2))\cdot\log b.$$

Inserting this into (4.39), we get

$$\begin{aligned}n &< 9.2\times 10^{24}\cdot(1+\log(n+2))^2\cdot\log^2 b\\ &< 5.8\times 10^{25}\cdot\log^2(n+2)\cdot\log^2 b,\end{aligned} \tag{4.40}$$

where we use the fact that $1+\log(n+2) < 2.5\log(n+2)$ which is valid, for all $n\ge 0$. Next we apply Lemma 3.2 to (4.40) with

$$\ell = 2,\quad H = 5.9\times 10^{25}\cdot\log^2 b,\quad L = n+2$$

and we get that

$$n < 1.8 \times 10^{30} \cdot \log^4 b. \tag{4.41}$$

The cases $n - k \in \{-1, 0\}$.

In this case, using (4.30) we easily see that $m \leq 5$. For these cases, we use one more time the identity $L_k = F_{k-1} + F_{k+1}$ to see that the Diophantine equation (4.27) becomes

$$b^d F_m = \begin{cases} F_{k-2}, & \text{if} \quad n = k-1, \\ 2F_{k-1}, & \text{if} \quad n = k. \end{cases}$$

The above equations imply to consider the following Diophantine equations

$$b^d F_m = \lambda F_k + \mu F_{k-1}, \quad \text{with} \quad (\lambda, \mu) \in \{(0,2), (1,-1)\}. \tag{4.42}$$

Inserting Binet's formula of Fibonacci sequence in (4.42) leads to

$$b^d F_m = \lambda \frac{\alpha^k - \beta^k}{\sqrt{5}} + \mu \frac{\alpha^{k-1} - \beta^{k-1}}{\sqrt{5}},$$

which becomes

$$b^d F_m - \alpha^k \left(\frac{\lambda}{\sqrt{5}} + \frac{\mu}{\alpha\sqrt{5}} \right) = -\frac{\lambda\beta^k + \mu\beta^{k-1}}{\sqrt{5}}. \tag{4.43}$$

We take the absolute value of both sides of (4.43) and then dividing the two sides of the inequality obtained by $\alpha^k \left(\frac{\lambda}{\sqrt{5}} + \frac{\mu}{\alpha\sqrt{5}} \right)$, we find

$$\left| b^d \cdot \frac{F_m}{\lambda/\sqrt{5} + \mu/\alpha\sqrt{5}} \cdot \alpha^{-k} - 1 \right| < \frac{5}{\alpha^{k-1}}. \tag{4.44}$$

Let

$$\Lambda_5 := b^d \cdot \frac{F_m}{\lambda/\sqrt{5} + \mu/\alpha\sqrt{5}} \cdot \alpha^{-k} - 1.$$

As in section 3, one can prove that $\Lambda_5 \neq 0$. So, we apply Theorem 2.1 to Λ_5 with $s = 3$,

$$(\gamma_1, b_1) = (b, d), \ (\gamma_2, b_2) = (\alpha, -k), \ (\gamma_3, b_3) = \left(\frac{F_m}{\lambda/\sqrt{5} + \mu/\alpha\sqrt{5}}, 1 \right)$$

and therefore we get

$$k < 4 \times 10^{15} \cdot \log b \quad \text{with} \quad n \in \{k-1, k\}. \tag{4.45}$$

This completes the proof of Theorem 4.1.

4.2. *Application for $2 \le b \le 10$.* Let us start with the case $m = 0$. Then, Diophantine equation (4.27) becomes $L_n = F_k$ which is only possible if $n < k$. Assume now $n > 30$. Using the fact that $L_n F_n = F_{2n}$ we get that $F_{2n} = F_n F_k$ and from (4.30) we have $n < k \le n + 5 < 2n$. It follows from the primitive divisor theorem [7] that the Diophantine equation $F_{2n} = F_n F_k$ has no solution for $2n > 30$. However, in the range $n \le 15$, the solutions of $L_n = F_k$ are easy to find. Assume now that $m \neq 0$. First, we wrote a short computer program to search the parameters d, n, m and k satisfying (4.27) in the range $1 \le \max\{m, k\} \le 200$ and we found only the Lucas numbers given in the following result.

THEOREM 4.2. *Let b be a positive integer such that $2 \le b \le 10$. Then, the numbers 3, 4, 7, 11, 18, 29, 47, 322 and 521 are the only Lucas numbers that satisfy Diophantine equation (4.27). More precisely, we have*

$$\begin{array}{rcccl}
3 & = & L_2 & = & 2^1 \cdot 1 + 1, \\
4 & = & L_3 & = & 3^1 \cdot 1 + 1, \\
7 & = & L_4 & = & 2^1 \cdot 3 + 1 = 2^2 \cdot 1 + 3 = 3^1 \cdot 2 + 1 = 4^1 \cdot 1 + 3, \\
7 & = & L_4 & = & 5^1 \cdot 1 + 2 = 6^1 \cdot 1 + 1, \\
11 & = & L_5 & = & 2^1 \cdot 5 + 1 = 2^2 \cdot 2 + 3 = 3^1 \cdot 3 + 2 = 4^1 \cdot 2 + 3, \\
11 & = & L_5 & = & 5^1 \cdot 2 + 1 = 6^1 \cdot 1 + 5 = 8^1 \cdot 1 + 3 = 9^1 \cdot 1 + 2, \\
11 & = & L_5 & = & 10^1 \cdot 1 + 1, \\
18 & = & L_6 & = & 5^1 \cdot 3 + 3 = 8^1 \cdot 2 + 2 = 10^1 \cdot 1 + 8, \\
29 & = & L_7 & = & 2^3 \cdot 3 + 5 = 2^4 \cdot 1 + 13 = 4^2 \cdot 1 + 13 = 8^1 \cdot 3 + 5, \\
29 & = & L_7 & = & 9^1 \cdot 3 + 2, \\
47 & = & L_8 & = & 9^1 \cdot 5 + 2, \\
322 & = & L_{12} & = & 6^2 \cdot 8 + 34, \\
521 & = & L_{13} & = & 6^3 \cdot 2 + 89 = 10^2 \cdot 5 + 21.
\end{array}$$

For the proof of Theorem 4.2, we assume that $\max\{m, k\} > 200$.

The case $n - k \ge 1$.

Here, it suffices to prove the following result.

PROPOSITION 4.3. *If Diophantine equation (4.27) holds, then $m \le 190$. Moreover, if $m < k$, then the set of solution of (4.27) is empty.*

PROOF. Assume that $m > 190$. Put

$$\Gamma_3 := d \log b - (n - m) \log \alpha + \log(\sqrt{5}). \tag{4.46}$$

Since $184 < m - 4 - \dfrac{\log 2}{\log \alpha} < n - k$ and $2 \le b \le 10$, then from (4.32)

$$|\Lambda_3| := |e^{\Gamma_3} - 1| < \frac{b + 2}{\alpha^{n-k}} < \frac{1}{2},$$

which implies that

$$|\Gamma_3| < \frac{2(b + 2)}{\alpha^{n-k}}.$$

From (4.46), we get

$$0 < \left| d\frac{\log b}{\log \alpha} - (n-m) + \frac{\log(\sqrt{5})}{\log \alpha} \right| < \frac{2(b+2)/\log \alpha}{\alpha^{n-k}}. \tag{4.47}$$

Moreover, for $2 \le b \le 10$, we see that $d < n-m+2 < n+2 < 5.1 \times 10^{31}$. Thus, we can apply Lemma 2.2 to (4.47) with $M := 5.1 \times 10^{31}$, $A := 2(b+2)/\log \alpha$, $B := \alpha$, $w := n-k$,

$$\tau := \frac{\log b}{\log \alpha} \quad \text{and} \quad \mu := \frac{\log(\sqrt{5})}{\log \alpha}.$$

So, with the help of Mathematica we deduce that $n - k < 171$, in all cases. However, this contradicts the fact that $184 < m-4-\dfrac{\log 2}{\log \alpha} < n-k$. Therefore, we conclude that $m \le 190$. Now when $m < k$, from (4.30) we get

$$n - k \le m - 1 + \frac{\log(b+1)}{\log \alpha} \le 194.$$

By substituting this upper bound for $n-k$ into (4.39) and using the fact that $2 \le b \le 10$, we obtain

$$\frac{1}{2}\left[n - \frac{\log(b+1)}{\log \alpha}\right] < k < 2.8 \times 10^{14} \times (1 + \log(n+2)) \cdot \log b,$$

and then $n < 5.2 \times 10^{16}$. Put

$$\Gamma_4 := d \log b - n \log \alpha + \log\left(\frac{F_m}{1 - \alpha^{k-n}/\sqrt{5}}\right). \tag{4.48}$$

So, we have

$$|\Lambda_4| := |e^{\Gamma_4} - 1| < \frac{1}{\alpha^{2k}}.$$

Since $m < k$, then $k = \max\{m, k\} > 200$. Thus $1/\alpha^{2k} < 1/2$, which also implies that

$$|\Gamma_4| < \frac{2}{\alpha^{2k}}. \tag{4.49}$$

Therefore, dividing both sides of (4.49) by $\log \alpha$, we get that

$$0 < \left| d\frac{\log b}{\log \alpha} - n + \frac{\log\left(\dfrac{F_m}{1 - \alpha^{k-n}/\sqrt{5}}\right)}{\log \alpha} \right| < \frac{2/\log \alpha}{\alpha^{2k}}.$$

We apply Lemma 2.2 to the above inequalities and we get in all cases $k < 193$, which is a contradiction.

The case $n - k \in \{-1, 0\}$.

In this case $1 \leq m \leq 5$. Since $\max\{m, k\} > 200$, then we have $k > 200$. Put

$$\Gamma_5 := d \log b - n \log \alpha + \log \left(\frac{F_m}{\lambda/\sqrt{5} + \mu/\alpha\sqrt{5}} \right).$$

Note that for $k > 200$, we get

$$|\Lambda_5| := |e^{\Gamma_5} - 1| < \frac{5}{\alpha^{k-1}} < \frac{1}{2}$$

and so

$$0 < \left| d \frac{\log b}{\log \alpha} - n + \frac{\log \left(\frac{F_m}{\lambda/\sqrt{5} + \mu/\alpha\sqrt{5}} \right)}{\log \alpha} \right| < \frac{10/\log \alpha}{\alpha^{k-1}}. \tag{4.50}$$

Now, we apply Lemma 2.2 to (4.50) and we see in all cases according to the values of b that $k < 177$ which is a contradiction because $k > 200$. This completes the proof of Theorem 4.2. ∎

Acknowledgements.

The authors are grateful to the referee for the useful comments/suggestions that help to improve the quality of the paper. The first author is supported by University Gaston Berger of Senegal, the second author is supported by IMSP, Institut de Mathématiques et de Sciences Physiques de l'Université d'Abomey Calavi. This paper was completed when the third author was visiting Max-Planck-Institut für Mathematik. He thanks the institution for the great working environment, the hospitality and the support.

References

[1] K. N. Adédji, *Balancing numbers which are products of three repdigits in base b*, Bol. Soc. Mat. Mex., 29:45 (2023), https://doi.org/10.1007/s40590-023-00516-0

[2] K. N. Adédji, A. Filipin and A. Togbé, *Fibonacci and Lucas numbers as products of three repdigits in base g*, Rend. Circ. Mat. Palermo, II. Ser (2023), https://doi.org/10.1007/s12215-023-00878-4

[3] K. N. Adédji, F. Luca and A. Togbé, *On the solutions of the Diophantine equation $F_n \pm a(10^m - 1)/9 = k!$*, *J. Number Theory*, **240** (2022), 593-610.

[4] K. N. Adédji, F. Luca and A. Togbé, *On the solutions of the Diophantine equation $P_n \pm a(10^m - 1)/9 = k!$*, European Journal of Mathematics 9, **34** (2023). https://doi.org/10.1007/s40879-023-00626-z

[5] M. Alan, *On Concatenations of Fibonacci and Lucas Numbers*, Bull. Iran. Math. Soc., (2022), https://doi.org/10.1007/s41980-021-00668-7

[6] W. D. Banks, F. Luca, *Concatenations with binary reccurent sequences*, J. Integer Seq. **8 (05)** (2005), 1–3.

[7] Y. Bilu, G. Hanrot, P. M. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers (with Appendix by Mignotte)*, J. Reine Angew. Math. **539** (2001), 75–122.

[8] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential Diophantine equations I. Fibonacci and Lucas powers*, Annals of Mathematics. **163** (2) (2006), 969–1018.
[9] M. Ddamulira, *Tribonacci numbers that are concatenations of two repdigits*, RACSAM 114, **203**, (2020). https://doi.org/10.1007/s13398-020-00933-0
[10] A. Dujella and A. Pethő, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxf. Ser. (2) **49** (1998), 291–306.
[11] F. Erduvan, R. Keskin and Z. Şiar, *Repdigits as product of two Pell or Pell-Lucas numbers*, Acta Math. Univ. Comenian. 88, (2019), 247–256.
[12] F. Erduvan, R. Keskin and Z. Şiar, *Repdigits base b as products of two Pell numbers or Pell-Lucas numbers*, Bol. Soc. Mat. Mex. 27:70 (2021). http://dx.doi.org/10.1007/s40590-021-00377-5
[13] S. Guzmán and F. Luca, *Linear combinations of factorials and s-units in a binary recurrence sequence*, Ann. Math. Qué. **38** (2014), 169–188.
[14] F. Luca, *Fibonacci and Lucas numbers with only one distinct digit*, Portugal. Math. **57**(2) (2000), 243–254.
[15] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in then logarithms of algebraic numbers II*, Izv. Math. **64** (2000), 1217–1269.

Generalizacija teorema Murata Alana

Mariama Ndao Faye, Kouèssi Norbert Adédji i Alain Togbé

Sažetak. Neka su $(F_n)_{n\geq 0}$ i $(L_n)_{n\geq 0}$ Fibonaccijev i Lucasov niz. U [5] autor je pronašao sve Fibonaccijeve i Lucasove brojeve koji su spojevi dva člana drugog niza. Neka je $b \geq 2$ cijeli broj. U ovom radu generaliziramo rezultate iz [5] razmatrajući sljedeće diofantske jednadžbe $F_n = b^d L_m + L_k$ i $L_n = b^d F_m + F_k$ u nenegativnim cijelim brojevima (n, m, k), gdje d označava broj znamenki od L_k i F_k u bazi b.

Mariama Ndao Faye
UFR of Applied Sciences and Technology
University Gaston Berger
Sénégal
E-mail: fayemariamandao@gmail.com

Kouèssi Norbert Adédji
Institut de Mathématiques et de Sciences Physiques
Université d'Abomey-Calavi
Bénin
E-mail: adedjnorb1988@gmail.com

Alain Togbé
Department of Mathematics, Statistics and Computer Science
Purdue University Northwest
2200 169th Street, Hammond, IN 46323 USA
E-mail: atogbe@pnw.edu