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ELLIPTIC CURVES ARISING FROM POLYGONAL NUMBERS

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ABSTRACT. We study a family of Legendre elliptic curves obtained by substituting the n-gonal polynomial $\lambda_n(t) = \frac{n-2}{2}t(t-1) + t$ into the classical Legendre form $y^2 = x(x-1)(x-\lambda)$. Using the geometry of the associated elliptic surface, we show that $E_n(t)$ has generic rank 0 over $\mathbb{Q}(t)$ with torsion $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Despite this, we construct explicit two-parameter subfamilies over $\mathbb{Q}(g, h)$ with positive rank, recovering earlier triangular cases and producing specializations of ranks 2 and 3 for $n = 5$ and $n = 7$. The main result is an explicit 2-Selmer analysis for the special class $n = 2\mathcal{P}$, where $2\mathcal{P} - 1$ and $3\mathcal{P} - 2$ are prime. In this setting we prove the uniform bound $\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E_{2\mathcal{P}}(t_0)) \leq 16$ for all specializations $t_0 \in \mathbb{Q}$, giving a global upper bound on the Mordell–Weil ranks in this subfamily. We complement this with Nagao-type averages and a specialization-level rank heuristic, leading to explicit fibers of positive rank, including an elliptic curve of rank 3 in the family $E_{1908}(t)$.

1. INTRODUCTION

Polygonal numbers have been studied since antiquity, appearing already in Diophantus and later formalised by Fermat. For integers $n > 2$ and $t \geq 1$, the tth n-gonal number is

$$\lambda_n(t) = \frac{(n-2)t(t-1)}{2} + t, \tag{1.1}$$

a quadratic polynomial whose values arrange points in regular polygonal patterns. The classical theorem, anticipated by Fermat and proved by Gauss (for triangular numbers) and by Cauchy (for all n), asserts that every positive integer is a sum of at most n n-gonal numbers. These sequences therefore form one

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of the oldest and structurally most rigid families of quadratic forms in number theory. In recent years, polygonal numbers have begun to appear in the

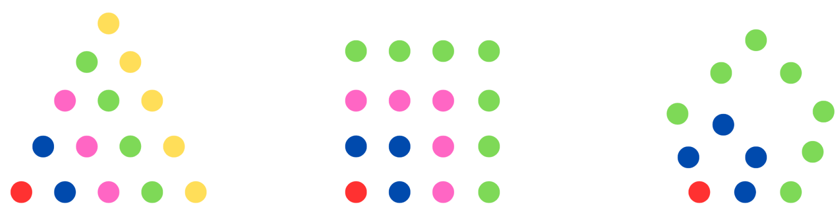

FIGURE 1. Polygonal numbers

study of Diophantine equations through their interaction with elliptic curves. Kim, Park and Pintér (in [7]) used hyperelliptic curves to resolve questions involving polygonal numbers. Juyal, Kumar and Moody (in [6]) analysed the Legendre family parametrised by triangular numbers and showed that it has rank 1 over $\mathbb{Q}(t)$ with explicit higher-rank specialisations. Further contributions include work of Chahal, Griffin and Priddis (see [2]) on when multiples of polygonal numbers are themselves polygonal, and work of Michaud-Rodgers on decagonal numbers through Frey curves in [8]. Together, these studies suggest that figurate parametrisations frequently give rise to rational elliptic surfaces with well controlled singular fibres and amenable descent theory.

The present article considers the general Legendre family

$$E_n(t)\colon y^2 = x(x-1)(x-\lambda_n(t)), \qquad n \geq 3, \tag{1.2}$$

obtained by inserting the n-gonal polynomial directly into the Legendre form. Allowing the index n to vary reveals arithmetic features invisible when n is fixed. The explicit factorisation of $\lambda_n(t)$ into t, $t-1$ and a linear expression in n forces a highly structured set of bad fibres on the associated elliptic surface, and these patterns propagate into the local Kummer conditions governing 2-descent. The controlled geometry of the surface therefore interacts tightly with the combinatorial structure of polygonal numbers, making this family well suited to uniform analysis.

Our first result shows that, despite this structure, the generic behaviour is arithmetically simple. We prove that the Mordell–Weil rank over $\mathbb{Q}(t)$ is always zero and that the torsion subgroup is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Over $\overline{\mathbb{Q}}(t)$ the rank is 1 for $n \neq 4$ and vanishes for $n = 4$. Thus any non-torsion section of a specialised fibre must arise purely from specialisation rather than from a global section. Understanding how and where such specialisations occur forms one of the main themes of the paper.

To investigate this phenomenon, we pursue two directions. The first is an algebraic construction showing that the surface admits explicit two-parameter subfamilies of positive rank. These arise naturally from an elementary manipulation of the equation after imposing that certain coefficients become squares, and they generalise the constructions already known in the triangular case. For instance, for $n = 5$ and $n = 7$ one obtains specialised fibres of ranks 3 and 2 respectively. These examples demonstrate that appreciable rank growth occurs throughout the family despite the generic rank being zero.

The second direction concerns a detailed Selmer-theoretic analysis for a special arithmetic regime. When $n = 2\mathcal{P}$ with $\mathcal{P}$ prime and with $2\mathcal{P} - 1$ and $3\mathcal{P} - 2$ also prime, the pattern of valuations in $\lambda_n(t)$ and $\lambda_n(t) - 1$ becomes particularly rigid. In this setting, the bad fibres are sharply constrained, allowing a complete 2-descent on the specialised curve $E_{2\mathcal{P}}(3)$. A careful examination of the resulting homogeneous spaces, including local solubility conditions at all primes, shows that at most 16 pairs can contribute to the 2-Selmer group. This yields a uniform Selmer bound for the entire family $E_n(t)$ in this regime and proves that

$$\operatorname{rank} E_n(\mathbb{Q}(t)) \leq 16 \qquad \text{whenever } n = 2\mathcal{P} \text{ and } 2\mathcal{P} - 1,\, 3\mathcal{P} - 2 \text{ are prime.}$$

This Selmer computation forms the technical core of the paper and provides a concrete instance in which the structure of $\lambda_n(t)$ allows uniform control over the global Mordell–Weil rank across infinitely many families.

Finally, to complement the descent, we investigate analytic heuristics based on Nagao's averaging of Frobenius traces. By computing these averages along arithmetic progressions in t, one can detect congruence classes that exhibit a persistent positive bias, suggesting the presence of fibres of large analytic rank. For $n = 1908$, this method highlights the progression $t \equiv 8 \pmod 9$, and further refinement identifies a specialisation of rank 3. This example illustrates how the combinatorial rigidity of the polygonal parametrisation aligns naturally with analytic techniques in the search for high-rank fibres.

In summary, polygonal numbers provide a structured and arithmetically natural parametrisation for Legendre-type elliptic surfaces. Their classical rigidity, coupled with the geometric regularity of the associated elliptic surfaces, enables both uniform Selmer bounds and explicit high-rank specialisations within a family of curves whose generic Mordell-Weil rank is identically zero.

2. Basic information and results on Elliptic Curves, Surfaces and Torsion points

An elliptic curve over $\mathbb{Q}(t)$ can be perceived as a one-parameter family of elliptic curves over $\mathbb{Q}$. This family of curves can be also seen as an Elliptic Surface. More generally, for a smooth irreducible curve C over an algebraically closed field k. An elliptic surface $\mathcal{E}$, [5, Chapter V, 6.4] is described using

Kodaira dimensions,

$$\kappa(\mathcal{E}) = \begin{cases} -1, & \text{if } \mathcal{E} \text{ is rational surface,} \\ 0, & \text{if } \mathcal{E} \text{ is a K3 surface,} \\ 1, & \text{for other minimal elliptic surfaces surfaces.} \end{cases}$$

The Kodaira dimension $\kappa(\mathcal{E})$ is an invariant of the surface. An elliptic surface is given by a surjective morphism $\pi : \mathcal{E} \to C$ with a section, such that almost all fibers are non-singular elliptic curves.

DEFINITION 2.1 (Elliptic Surface). An elliptic surface over a smooth projective curve C, over an algebraically closed field k is a pair $(\mathcal{E}, \pi)$ where $\mathcal{E}$ is a smooth, irreducible, projective surface over k, and $\pi : \mathcal{E} \to C$ is an elliptic fibration having singular fibers and a non-zero section.

For example, in a projective model one may write $\pi([X, Y, Z], t) = t$, though in general the fibration need not be a split product. In this case, it is clear that, the fibers for almost every point $t \in C$, the fiber (see [13, pp 200–201]) is given by

$$\mathcal{E}_t = \pi^{-1}(t) = \{P \in \mathcal{E} : \pi(P) = t\}.$$

For an elliptic curve E over the function field $k(C)$ of C, the existence and uniqueness of the elliptic surface $f : \mathcal{E} \to C$ with generic fiber E are given by the work of Kodaira and Néron.

THEOREM 2.2 ([13, Chapter IV, Theorem 8.2]). *Let E be an elliptic curve over $\overline{\mathbb{Q}}(t)$. Let $\mathcal{S} \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ be the set of points of bad reduction of E. Let $G(F_v)$ denote the group generated by the simple components of the fiber F_v at $v \in \mathcal{S}$. Then there is an injective homomorphism*

$$\phi : E(\overline{\mathbb{Q}}(t))_{\mathrm{tors}} \longrightarrow \prod_{v \in \mathcal{S}} G(F_v). \tag{2.3}$$

If F_v is of multiplicative type I_n in Kodaira notation, the corresponding group is $\mathbb{Z}/n\mathbb{Z}$, if it is of additive type I_{2n}^*, the group is $(\mathbb{Z}/2\mathbb{Z})^2$.

These surfaces are classified into Kodaira types [13, Chapter IV, Theroem 6.1], which determine several arithmetic properties, including contributions to the Mordell–Weil group. The following theorem provides us with a relationship between the curve and the associated surface using the singular fibers over points of bad reduction.

THEOREM 2.3 (Shioda-Tate formula [12, Corollary 5.3]). *Let (S, f) be an elliptic surface over C. For each point v of C having a singular fiber, let m_v denote the number of irreducible components of the singular fiber above v. Let E denote the generic fiber of S. The rank $\rho(S)$ of the Néron-Severi group of S is given by the equality*

$$\rho(S) = \operatorname{rank} E(k(C)) + 2 + \sum_v (m_v - 1), \tag{2.4}$$

where the summation ranges over the points v of C under singular fibers.

The above results help us in computing the torsion group and get an estimate for some families of elliptic curves but do not solve the problem of computing the exact rank of the family $E/\mathbb{Q}(t)$. In attempts to evaluate $E/\mathbb{Q}(t)$ we find Silverman's specialization theorem very useful where he states that the specialization map $\sigma : E/\mathbb{Q}(t) \mapsto E/\mathbb{Q}$ is injective in all but finitely many places.

THEOREM 2.4 ([3, Theorem 1.1]). *Let E be a non-constant elliptic curve over $\mathbb{Q}(t)$ given by the equation*

$$E : y^2 = (x - e_1)(x - e_2)(x - e_3) \qquad e_i \in \mathbb{Z}[t] \tag{2.5}$$

If t_0 satisfies the following condition:

For every non-constant square-free divisor h in $\mathbb{Z}[t]$ of

$$\begin{aligned} &(e_1 - e_2)(e_1 - e_3) \quad \textit{or,} \\ &(e_2 - e_1)(e_2 - e_3) \quad \textit{or,} \\ &(e_3 - e_1)(e_3 - e_2) \end{aligned}$$

the rational number $h(t_0)$ is not a square in $\mathbb{Q}$. Then the specialization map is injective at t_0.

We will use the specialization theorem (Theorem 2.4) to choose parameters t_0 for which the specialization map is injective. This allows us to transfer information about $E(\mathbb{Q}(t))$ to the specialized curves $E_{t_0}(\mathbb{Q})$, which will be essential in our rank computations in later sections.

3. MORDELL-WEIL STRUCTURE OF $E_n(t)$

THEOREM 3.1. *Let $\lambda_n(t)$ be the n-gonal polynomial defined in* (1.1). *The elliptic curve*

$$E_n(t) : y^2 = x(x-1)\big(x - \lambda_n(t)\big)$$

has Mordell–Weil rank 0 over the function field $\mathbb{Q}(t)$. Over $\overline{\mathbb{Q}}(t)$ the rank is 1 for $n \neq 4$ and 0 for $n = 4$. Moreover, the torsion subgroup of $E_n(t)$ over $\mathbb{Q}(t)$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

We now determine the generic Mordell–Weil rank and torsion subgroup of the family $E_n(t)$, thereby proving Theorem 3.1. For each fixed integer $n \geq 3$, the equation

$$E_n(t) : y^2 = x(x-1)(x - \lambda_n(t))$$

defines a one-parameter elliptic curve over $\mathbb{Q}(t)$, and hence a rational elliptic surface over $\mathbb{P}^1_t$. Our strategy is to analyse the singular fibres of this surface and apply the Shioda–Tate formula to compute the generic Mordell–Weil rank.

In this section, two separate cases are considered, one where $n \neq 4$ and the other for $n = 4$. To apply the Shioda-Tate formula, we first compute the singular fibers of the elliptic surface associated to $E_n(t)$. For this, we rewrite the equation in short Weierstrass form. For $n \neq 4$, through some standard change of variables we get a Short Weierstrass form.

(3.6)

$E_n(t) : y^2 = x^3 + A_\lambda(t)x + B_\lambda(t),$

where,

$$A_n(t) = -\frac{1}{12}(n^2-4n+4)t^4 + \frac{1}{6}(n^2-6n+8)t^3 \\ - \frac{1}{12}(n^2-10n+20)t^2 - \frac{1}{6}(n-4)t - \frac{1}{3},$$

and,

$$B_n(t) = -\frac{1}{108}(n^3-6n^2+12n-8)t^6 + \frac{1}{36}(n^3-8n^2+20n-16)t^5 \\ - \frac{1}{36}(n^3-11n^2+36n-36)t^4 + \frac{1}{108}(n^3-18n^2+84n-112)t^3 \\ + \frac{1}{36}(n^2-6n+12)t^2 - \frac{1}{18}(n-4)t - \frac{2}{27}.$$

The curve has discriminant $\Delta_n(t) = (nt-n-2t+4)^2(nt-2t+2)^2(t-1)^2t^2$. Therefore, the singular fibers occur at $t = 0, 1, -\frac{2}{n-2}, \frac{n-4}{n-2}$ and $t = \infty$. The number of irreducible components m_v of the singular fiber at v in the associated elliptic surface $\mathcal{E}$ are provided in the following table.

TABLE 1. Kodaira types and m_v of the fibers at v

v	$\mathrm{Ord}_{t=v}A_n(t)$	$\mathrm{Ord}_{t=v}B_n(t)$	$\mathrm{Ord}_{t=v}\Delta_n(t)$	Kodaira Type	$m_v - 1$
0	0	0	2	I_2	1
1	0	0	2	I_2	1
$-\frac{2}{n-2}$	0	0	2	I_2	1
$\frac{n-4}{n-2}$	0	0	2	I_2	1
∞	0	0	4	I_4	3

Then, the following lemma describes the rank of $E_n(t)$.

LEMMA 1. *The rank of* $E_n(t)$ *in* (3.6) *over* $\overline{\mathbb{Q}}(t)$ *is* 1.

PROOF. Observe that $\deg(a_i(t)) \leq i$, where $a_i(t)$ represents the i-th coefficient of the Weierstrass equation in (3.6). By Shioda's result in [12, Equation 10.14], the associated elliptic surface is rational. Therefore, the rank of the Néron-Severi group of the associated elliptic surface $\mathcal{E}$ is $\rho(\mathcal{E}) = 10$.

Then, referring to the Table 1 and using the Shioda-Tate formula in Theorem 2.3, the rank of $E_n(t)$ over $\overline{\mathbb{Q}}(t)$ can be computed.

$$10 = \mathrm{rank}(E_n(\overline{\mathbb{Q}}(t))) + 2 + 1 + 1 + 1 + 1 + 3,$$

corresponding to the four I_2 fibers and one I_4 fiber identified in Table 1. Therefore, the rank of $E_n(t)$ over $\overline{\mathbb{Q}}(t))$ is 1. $\square$

Shioda, in [12, Lemma 10.9], showed that the generators of the Mordell-Weil group over $\overline{\mathbb{Q}}(t)$ are of the form $(a_2t^2 + a_1t + a_0, b_3t^3 + b_2t^2 + b_1t + b_0)$ for $a_i, b_i \in \overline{\mathbb{Q}}$. In this case, the generator of $E_n(t)$ over $\overline{\mathbb{Q}}(t)$ can be chosen as,

$$P = \left(\frac{1}{6}(2-n)t^2 - \frac{1}{3}(n-1)t - \frac{1}{3}, \frac{t^2}{2\sqrt{2}}(n-2)\sqrt{-(n-2)} + \frac{t}{\sqrt{2}}\sqrt{-(n-2)}\right).$$

Clearly, $\mathrm{rank}(E_n(\mathbb{Q}(t))) \leq 1$. The following Lemma evaluates the rank over $\mathbb{Q}(t)$.

Lemma 2. *The rank of $E_n(t)$ over $\mathbb{Q}(t)$ is* 0.

Proof. The generator of $E_n(\overline{\mathbb{Q}}(t))$ found above involves $\sqrt{-(n-2)}$, and hence is not defined over $\mathbb{Q}(t)$ unless $n-2$ is a negative rational square. Thus no non-torsion point descends to $\mathbb{Q}(t)$. On the contrary, if the rank were 1 over $\mathbb{Q}(t)$, then we could construct a non-trivial character σ of the canonical Galois representation ρ defined by

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(\overline{\mathbb{Q}}(t) \otimes \mathbb{Z}),$$

and

$$\sigma(\sqrt{-1}) = -\sqrt{-1}, \qquad \sigma(n-2) = n-2,$$

which acts trivially on $(E_n(\mathbb{Q}(t)))$. This is a contradiction to the fact that $(E_n(\mathbb{Q}(t))) \otimes \mathbb{Z} = (E_n(\overline{\mathbb{Q}}(t))) \otimes \mathbb{Z}$. So, the rank of $(E_n(t)$ over $\mathbb{Q}(t))$ is 0. $\square$

The following lemma explicitly describes the torsion subgroup of $E_n(t)$.

Lemma 3. *The torsion subgroup of the curve $E_n(t)$ in* (3.6) *is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. Since the curve $E_n(t)$ is in Legendre form, there are exactly three points of order two, namely, $(0,0), (1,0)$, and $(\lambda_n(t), 0)$. By [14, Chapter VIII, 7.5], the torsion subgroup of the Mordell-Weil group $E_n(t)$ over $\mathbb{Q}(t)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ for $N = 2$ or 4, as by Theorem 2.2, it is embedded in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$. If the torsion subgroup of $E_n(\mathbb{Q}(t))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ then there must exist points of order 4. If $P = (x, y) \in E_n(t)(\mathbb{Q}(t))$ is a point of order 4, then $2P$ is of order two. But through some simple calculations using SAGE [10], it is easy to see that $2P$ cannot be any of the points $(0,0), (1,0)$ and $(\lambda_n(t), 0)$ as there are no $x, y \in \mathbb{Q}(t)$ satisfying these conditions. Hence, the torsion subgroup is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. $\square$

The case for $n = 4$, must be treated separately because the substitution used to obtain the Weierstrass form for $n \neq 4$ becomes singular. For $n = 4$, the curve takes the Weierstrass form

$$E_4(t) : y^2 = x^3 + A_4(t)x + B_4(t), \tag{3.7}$$

where,

$$A_4(t) = -\frac{1}{3}t^4 + \frac{1}{3}t^2 - \frac{1}{3},$$

and

$$B_4(t) = -\frac{2}{27}t^6 + \frac{1}{9}t^4 + \frac{1}{9}t^2 - \frac{2}{27}.$$

The discriminant $\Delta_4(t)$ is given by

$$\Delta_4(t) = 16(t+1)^2(t-1)^2t^2.$$

Clearly, the singular fibers of $E_4(t)$ are at $v = 0, -1, 1$ and ∞. Again, Table 2 lists the irreducible components.

TABLE 2. Kodaira types and m_v of the fibers at v

v	$\mathrm{Ord}_{t=v}A_4(t)$	$\mathrm{Ord}_{t=v}B_4(t)$	$\mathrm{Ord}_{t=v}\Delta_4(t)$	Kodaira Type	$m_v - 1$
0	0	0	4	I_4	3
1	0	0	2	I_2	1
-1	0	0	2	I_2	1
∞	0	0	4	I_4	3

Using Theorem 2.3, the rank over $\overline{\mathbb{Q}}(t)$ can be obtained.

$$10 = \mathrm{rank}(E_4(t)) + 2 + 3 + 1 + 1 + 3$$

The above equation implies that $\mathrm{rank}(E_4(\overline{\mathbb{Q}}(t))) = 0$. Now, by a similar argument to that in Lemma 2, the rank over $\mathbb{Q}(t)$ becomes 0. Again, for the torsion subgroup, through an argument similar to the case $n \neq 4$, it is clear that $E_{\lambda_4}(t)_{\mathrm{tors}}$ over $\mathbb{Q}(t)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Since, the choice of n was arbitrary in $\mathbb{N} \setminus \{4\}$ in Lemma 1, the rank of $E_n(t)$ over $\overline{\mathbb{Q}}(t)$ is 0 for $n = 4$ and 1 for $n \neq 4$. Further, $\mathrm{rank}(E_n(t)/\mathbb{Q}(t))$ is 0.

Although the generic rank is 0, there are certain subfamilies over $\mathbb{Q}(s,t)$ having rank at least one. The following section provides an explicit construction of a subfamily with rank at least one.

The arguments above establish the fibre configuration and yield the Mordell–Weil group as claimed. Hence, Theorem 3.1 follows.

4. Subfamilies and Rank Growth in the Family $E_n(t)$

In Section 3, we proved that the generic Mordell-Weil rank of the family $E_n(t)$ in Equation (1.2) is equal to 0 for all $n \geq 3$. Consequently, any non-torsion point on a specialized curve $E_{n,t_0}(\mathbb{Q})$ arises entirely from specialization rather than from a global section of the elliptic fibration. This makes the behavior of

$$t_0 \longmapsto \operatorname{rank} E_{n,t_0}(\mathbb{Q})$$

both clean and arithmetically meaningful.

In this section, we develop two complementary approaches for finding positive-rank specializations. An algebraic method producing two-parameter subfamilies of rank at least 1, and an analytic method based on Selmer bounds, Nagao averages, and specialization-level heuristics, culminating in an explicit specialization of rank 3 in the family $E_{1908}(t)$. Despite generic rank 0, these results show that the family contains explicit fibers of unusually high rank.

4.1. *Algebraic Construction of Positive-Rank Subfamilies.* Substituting the value of $\lambda_n(t)$ from Equation (1.1) into Equation (1.2) yields

$$y^2 = -(x^2 - x)\frac{n-2}{2}t^2 + (x^2 - x)\left(\frac{n-2}{2} - 1\right)t + (x^2 - x)x. \tag{4.8}$$

To obtain a rational point on the surface, we force the coefficient of t^2 to be a square in $\mathbb{Q}(x)$. Setting

$$x = \frac{2(n-2)}{h^2 + 2(n-2)}$$

gives

$$m^2 = \left(\frac{(n-2)h}{h^2 + 2(n-2)}\right)^2,$$

so that (4.8) becomes

$$(y - tm)(y + tm) = -m^2\frac{n-4}{n-2}t - \frac{2m^2}{n-2}x.$$

Introducing $p = y - tm$ and writing $g = p/m$, we obtain explicit rational functions

$$t = -\frac{g^2h^2 + 2g^2(n-2) + 4}{(h^2 + 2(n-2))\left(2g + \frac{n-4}{n-2}\right)}, \tag{4.9}$$

$$y = \frac{(n-2)h}{h^2 + 2(n-2)}\left(g - \frac{g^2h^2 + 2g^2(n-2) + 4}{(h^2 + 2(n-2))\left(2g + \frac{n-4}{n-2}\right)}\right). \tag{4.10}$$

Thus $(x(g,h), y(g,h), t(g,h))$ yields a rational point on $E_n(t(g,h))$ for all g, h, showing that the two-parameter family

$$E_n(t(g,h))/\mathbb{Q}(g,h)$$

has rank at least 1.

For $n = 3$, the construction reproduces the positive-rank family of Juyal-Kumar-Moody [6].

EXAMPLE 4.1 (For, $n = 5$). For this case, substituting $n = 5$ in the Equations (4.9) and (4.10), yields,

$$t = -\frac{3(g^2h^2 + 6g^2 + 4)}{(h^2 + 6)(6g + 1)},$$

and

$$y = 3h\frac{3g^2h^2 + gh^2 + 18g^2 + 6g - 12}{(h^2 + 6)^2(6g + 1)}.$$

Specializing at $(g, h) = (1, 1)$ and using SAGE [10] we get an infinite-order point $(\frac{6}{7}, -\frac{48}{343})$ of the curve of rank 3 in equation (4.11) given by

$$y^2 = x^3 - \frac{4843}{2401}x^2 + \frac{2442}{2401}x. \tag{4.11}$$

EXAMPLE 4.2 (For, $n = 7$). Again, we substitute $n = 7$ in Equations (4.9) and (4.10) to get

$$t = -\frac{5(g^2h^2 + 10g^2 + 4)}{(h^2 + 10)^2(10g + 3)}$$

and,

$$y = 5h\frac{(5g^2h^2 + 3gh^2 + 50g^2 + 30g - 20)}{(h^2 + 10)^2(10g + 3)}.$$

Now specializing at $(g, h) = (1, 1)$, and using SAGEMath [10] we get an infinite ordered point $\left(\frac{10}{11}, \frac{340}{1573}\right)$ on the curve of rank 2 over $\mathbb{Q}$ given by the equation

$$y^2 = x^3 - \frac{50599}{20449}x^2 + \frac{30150}{20449}x.$$

REMARK 4.3. In example 4.2, if g and h are further transformed using the transformations

$$g \mapsto u^3 + 1, \qquad h \mapsto u^2 + u + 1,$$

then, specializing the curve at $u = \frac{2}{5}$ yields the elliptic curve

$$y^2 = x^3 - \frac{4642523329}{2463831769}x^2 + \frac{2178691560}{2463831769}x$$

over the rational field $\mathbb{Q}$. Performing a `simon_two_descent` in SAGE-Math [10] gives the output $\left(2, 8, \left[\frac{6250}{7091}, \frac{6097250}{351975976}\right]\right)$. Thus, it is easy to see that the 2-Selmer rank is bounded above by 8.

These algebraic constructions already demonstrate substantial rank growth within the family.

4.2. *Selmer Groups of Specialized Fibers.* In the special case where $n = 2\mathcal{P}$ with $2\mathcal{P}-1$ and $3\mathcal{P}-2$ both prime, the local conditions on the 2-coverings admit a uniform analysis carried out in Section 5. In this setting, one obtains a global bound on the 2-Selmer ranks of all specializations.

Since the generic rank is zero, the appearance of non-torsion points on $E_{n,t_0}(\mathbb{Q})$ is governed entirely by the local Kummer conditions on the 2-coverings. For each odd prime ℓ, the structure of

$$\mathrm{Sel}_2(E_{n,t_0}/\mathbb{Q})$$

depends only on valuations of

$$\lambda_n(t_0), \qquad \lambda_n(t_0)-1, \qquad t_0(t_0-1),$$

and the corresponding reduction types. As proved in Section **??**, this analysis yields the uniform bound

$$\dim_{\mathbb{F}_2} \mathrm{Sel}_2(E_{n,t_0}/\mathbb{Q}) \leq 16, \tag{4.12}$$

valid for all $t_0 \in \mathbb{Q}$ when n satisfies the above conditions. While this upper bound is unlikely to be sharp, it is extremely useful in practice. Within a given congruence class for t_0, this upper bound rules out large portions of the search space and allows the analytic heuristics of the next subsections to be applied much more efficiently.

We emphasize that the special case $n = 1908$ considered later does not fall under this restricted Selmer analysis. In that case, the bound (4.12) is not invoked. Instead, the specialization-level heuristics of Sections 4.3 and 4.4 drive the search for high-rank fibers.

4.3. *Nagao Averages Along Arithmetic Progressions.* To locate unusually large rank fibers, we apply Nagao's heuristic, relating averaged Frobenius traces to analytic rank. For a fixed n,

$$a_p(E_{n,t}) = p+1-\#E_{n,t}(\mathbb{F}_p) = -\sum_{x\in\mathbb{F}_p}\left(\frac{x(x-1)(x-\lambda_n(t))}{p}\right).$$

Fix integers $m \geq 2$ and $0 \leq a < m$, and restrict the parameter to $t = mu+a$. For each prime p, define the local average

$$A_p(n;m,a) = \frac{1}{\#\{t\in\mathbb{F}_p : t \equiv a (\mathrm{mod}\ m)\}} \sum_{\substack{t\in\mathbb{F}_p \\ t\equiv a(\mathrm{mod}\ m)}} a_p(E_{n,t}),$$

and the finite Nagao average

$$N_X(n;m,a) = \frac{1}{X}\sum_{\substack{p\leq X \\ p \text{ prime}}} -A_p(n;m,a)\log p. \tag{4.13}$$

Large values of (4.13) predict large rank specializations within $t \equiv a (\mathrm{mod}\ m)$.

A scan with cutoff $X = 50$ identified $n_0 = 1908$ with noticeably large global Nagao value $N_{50}(1908) \approx 0.72$. Refining over congruence classes $2 \le m \le 12$, the progression $t \equiv 8 \pmod 9$ produced the much larger value $N_{50}(1908; 9, 8) \approx 2.41$, indicating that the subfamily $E_{1908,\,9u+8}$ merited detailed study.

4.4. *A Specialization-Level Rank Heuristic.* To extract individual specializations from $t = 9u + 8$, we use the smoothed trace average

$$H_X(t_0) = \frac{1}{\log X} \sum_{\substack{p \le X \\ p \text{ prime}}} \frac{-a_p(E_{n_0,t_0}) \log p}{p}. \tag{4.14}$$

This expression behaves like a smoothed Nagao sum, with large values correlating strongly with large analytic rank.

Evaluating (4.14) for $t_0 = 9u + 8$ with $|u| \le 50$ and $50 \le X \le 200$ identified

$$t_0 \in \{386,\ -361,\ -388,\ -154,\ 431,\ 116\}$$

as the top candidates. For each such t_0, we formed

$$E_{n_0,t_0} : \quad y^2 = x^3 - (1 + \lambda_{n_0}(t_0))\, x^2 + \lambda_{n_0}(t_0)\, x, \qquad \lambda_{n_0}(t_0) = 953t_0^2 - 952t_0,$$

and computed the minimal model and rank using Magma [1].

4.5. *A Rank 3 Specialization.* Among the candidates above, $t_0 = -154$ yields

$$\lambda_{1908}(-154) = 22747956,$$

so

$$E_{1908,-154} : \quad y^2 = x(x-1)\big(x - 22747956\big).$$

After minimization, the global minimal model is

$$E^{\min}_{1908,-154} : \qquad y^2 = x^3 - x^2 - 172489826476660\, x - 871953475146450797408.$$

A simple check using `Rank(E)` in Magma [1] gives

$$\operatorname{rank} E_{1908,-154}(\mathbb{Q}) = 3.$$

Since the generic rank is zero, all three independent points arise purely from specialization.

This example illustrates how the combined algebraic and analytic framework developed in this section detects deep rank growth within a generically rank-zero family.

5. A particular case of $E_n(t)/\mathbb{Q}(t)$

Theorem 5.1. *Let $n = 2\mathcal{P}$, where $\mathcal{P}$ is prime and $2\mathcal{P} - 1$ and $3\mathcal{P} - 2$ are also prime. Then the elliptic curve $E_n(t)$ has 2-Selmer rank at most 16. Consequently,*

$$\operatorname{rank} E_n(\mathbb{Q}(t)) \leq 16.$$

Let $\mathcal{P}$ be a prime such that both $p = 2\mathcal{P} - 1$ and $q = 3\mathcal{P} - 2$ are prime. Thus $\mathcal{P}, p, q$ form a Cunningham chain of the second kind. Set $n = 2\mathcal{P}$ and let S denote the set of bad places of the curve $E_n(t)$. For the specialization $t = 3$, Theorem 2.4 shows that the specialization map

$$\sigma_3 \colon E_n(\mathbb{Q}(t)) \longrightarrow E_{n,3}(\mathbb{Q})$$

is injective. Since the torsion subgroup does not change under this specialization, [4] implies that σ_3 is in fact an isomorphism.

The commutative diagram below shows that the image of $E_n(\mathbb{Q}(t))/2E_n(\mathbb{Q}(t))$ injects into $E_{n,3}(\mathbb{Q})/2E_{n,3}(\mathbb{Q})$ under specialization. Consequently, any upper bound for the 2-Selmer rank of $E_{n,3}(\mathbb{Q})$ also bounds the Mordell–Weil rank of $E_n(\mathbb{Q}(t))$.

$$\begin{tikzcd}
E_n(\mathbb{Q}(t))/2E_n(\mathbb{Q}(t)) \arrow[r] \arrow[d, "\sigma_3"'] & S^2(E_n(\mathbb{Q}(t))) \arrow[r] & Ш(E_n(\mathbb{Q}(t)))[2] \\
E_{n,3}(\mathbb{Q})/2E_{n,3}(\mathbb{Q}) \arrow[r] & S^2(E_{n,3}(\mathbb{Q})). &
\end{tikzcd}$$

Therefore, it is sufficient to bound the 2-Selmer rank of the specialized curve.

To bound the 2-Selmer rank of the specialized curve, firstly, define $\mathbb{Q}(S, 2)$ as,

$$\mathbb{Q}(S, 2) := \{b \in \mathbb{Q}^*/(\mathbb{Q}^*)^2 : \operatorname{ord}_\ell(b) \equiv 0 (\operatorname{mod} 2) \text{for all finite primes } \ell \neq 2, 3, p, q\}.$$

or,

$$\mathbb{Q}(S, 2) = \{\pm 1, \pm 2, \pm 3, \pm p, \pm q, \pm 6, \pm 2p, \pm 2q, \pm 3p, \pm 3q, \pm pq, \pm 6p, \pm 6q, \pm 2pq, \pm 3pq, \pm 6pq\}.$$

The descent map

$$\phi : E(\mathbb{Q})/2E(\mathbb{Q}) \longrightarrow \mathbb{Q}(S, 2) \times \mathbb{Q}(S, 2),$$

is given by,

$$(x, y) \mapsto \begin{cases} (x, x-1), & \text{if } x \neq 0, 1, \\ (3p, -1), & \text{if } x = 0, \\ (1, -2q) & \text{if } x = 1, \\ (1, 1) & \text{if } x = \infty, \end{cases}$$

and sends the torsion points to

$$\{(3p, -1), (1, -2q), (3p, 2q), (1, 1)\}. \tag{5.15}$$

By [14, Chapter X, Proposition 1.4], a pair $(b_1, b_2) \in \mathbb{Q}(S,2) \times \mathbb{Q}(S,2)$ lies in the image of ϕ precisely when there exist $(z_1, z_2, z_3) \in \mathbb{Q}^3$ satisfying

$$b_1 z_1^2 - b_2 z_2^2 = 1, \tag{5.16}$$

$$b_1 z_1^2 - b_1 b_2 z_3^2 = 3p, \tag{5.17}$$

$$b_2 z_2^2 - b_1 b_2 z_3^2 = 2q. \tag{5.18}$$

Thus any solution of (5.16)–(5.18) corresponds to an infinite-order point on $E_{n,3}(\mathbb{Q})$.

5.1. *Local Solutions.* We know that any ℓ-adic number a can be written as $a = \ell^n \cdot u$ where $n \in \mathbb{Z}$ and $u \in \mathbb{Z}_\ell^*$. So, we consider here that $v_\ell(z_i) = k_i$, and thus we write $z_i = \ell^{k_i} u_i$.

Lemma 4. *If the system of equations* (5.16) *and* (5.17) *has a solution in* $\mathbb{Q}_\ell \times \mathbb{Q}_\ell \times \mathbb{Q}_\ell$ *with* $v_\ell(z_i) < 0$ *for any one* $i = \{1, 2\}$ *then*

$$v_\ell(z_1) = v_\ell(z_2) = v_\ell(z_3) = -k < 0,$$

for some integer k.

Proof. Let us consider that for ℓ prime $z_1, z_2, z_3 \in \mathbb{Q}_\ell$ Let $v_\ell(z_i) = k_i$ for all $i \in \{1, 2, 3\}$. If $v_\ell(z_1) = k_1 < 0$ then, from Equation (5.16) we have

$$b_1 u_1^2 - b_2 u_2^2 \ell^{2(k_2 - k_1)} = \ell^{-2k_1}. \tag{5.19}$$

Now, if $k_2 > k_1$ then from Equation (5.19) we get $\ell | b_1^2$, but this is a contradiction as $b_1 \in \mathbb{Q}(S, 2)$ is square-free. Again, if $0 > k_1 > k_2$ then, from Equation (5.16) we get

$$b_1 u_1^2 \ell^{2(k_1 - k_2)} - b_2 u_2^2 = \ell^{-2k_2}, \tag{5.20}$$

implying $\ell^2 | b_2$ which is again a contradiction.

Thus, we get that if $k_1 < 0$ then, $v_\ell(z_1) = v_\ell(z_2) = -k < 0$ for some integer k. Similarly, if $v_ell(z_2) = k_2 < 0$ then we get $v_\ell(z_1) = v_\ell(z_2) = -k < 0$.

Now, if $k_1 < 0$ and $k_1 < k_3$ then, from Equation (5.17) we get,

$$b_1 u_1^2 - b_1 b_2 u_3^2 \ell^{2(k_3 - k_1)} = 3p\ell^{-2k_1}. \tag{5.21}$$

Equation (5.21) implies $\ell^2 | b_1$ which is again a contradiction. Further, if $k_3 < k_1 < 0$ then from Equation (5.17) we get,

$$b_1 u_1^2 \ell^{2(k_1 - k_3)} - b_1 b_2 u_3^2 = 3p\ell^{-2k_3}. \tag{5.22}$$

Therefore, from Equation (5.22) we get $\ell^2 | b_1 b_2$. Without loss of generality we look at $\ell = 2, 3, p$ or q. If $\ell = p$ then from Equation (5.22) we get that $p^3 | b_1 b_2$ which is a contradiction. If $\ell = 3$ then again from Equation (5.22) we get that $3^3 | b_1 b_2$ which is a contradiction.

Now, if $\ell = q$ then from Equation (5.18) we get,

$$b_2 u_2^2 \ell^{2(k_2 - k_3)} - b_1 b_2 u_3^2 = 2q\ell^{-2k_3}. \tag{5.23}$$

If $k_3 < k_2$ them from Equation (5.23) we conclude $q^3 || b_1 b_2$ which is again a contradiction. Now, in the case $k_2 \leq k_3 < 0$ we see that $k_1 = k_2$, and $k_1 < k_3$ cannot happen. Furthermore, if $\ell = 2$ similarly $K_3 < k_2$ and $k_2 < k_3 < 0$ cannot happen.

Thus, we conclude that if $v_\ell(z_1) = k_1 < 0$ or $v_\ell(z_2) = k_2 < 0$ then $v_\ell(z_1) = v_\ell(z_2) = v_\ell(z_3) = -k < 0$ for some integer k. □

Assume (b_1, b_2) admits a local solution at ℓ; we derive contradictions.

Lemma 5. *Let $(b_1, b_2) \in \mathbb{Q}(S,2) \times \mathbb{Q}(S,2)$. Then the following have no solutions in $\mathbb{Q}_\ell$.*

(a) *If $b_1 < 0$, for $\ell = \infty$.*
(b) *If $2 \mid b_1$, for $\ell = 2$.*
(c) *If $q \mid b_1$, for $\ell = q$.*
(d) *If $3 \mid b_2$, for $\ell = 3$.*
(e) *If $p \mid b_2$, for $\ell = p$.*

Proof. Let $b_1, b_2 \in \mathbb{Q}(S,2)$ be points for which there are $z_1, z_2, z_3 \in \mathbb{Q}$ satisfying Equations (5.16) and (5.17).

(a) Let $b_1 < 0$ such that the homogeneous space corresponding to (b_1, b_2) have solutions. If $b_2 > 0$ then, from Equation (5.16) we get $1 < 0$ which is absurd. Again, if $b_2 < 0$ then from Equation (5.17) we get that $3p < 0$, which is a contradiction.

Therefore, there are no solutions to the homogeneous space if $b_1 < 0$.

(b) Let $(b_1, b_2) \in \mathbb{Q}(S,2) \times \mathbb{Q}(S,2)$ is a solution of the homogeneous space such that $2|b_1$. Let $b_1 = 2\tilde{b}_1$, where $\tilde{b}_1$ is odd, as $b_1 \in \mathbb{Q}(S,2)$ is square-free.

If $2 \nmid b_2$ then from Equation (5.17) we get,

$$2\tilde{b}_1 z_1^2 - 2\tilde{b}_1 b_2 z_3^2 = 3p. \tag{5.24}$$

If $v_2(z_1) \geq 0$ and $v_2(z_3) \geq 0$ then, $2|3p$ but p is prime and thus we have a contradiction. Again, if $v_2(z_1) = k_1 \geq 0$ and $v_2(z_3) = k_3 < 0$ then, from Equation (5.17) we get,

$$\tilde{b}_1 u_1^2 2^{2(k_1-k_3)} - \tilde{b}_1 b_2 u_3^2 = 3p2^{2k_3-1}. \tag{5.25}$$

Therefore, from Equation (5.25) we get that, $2|\tilde{b_1} b_2 u_3^2$ which is a contradiction.

Now, if $v_2(z_1) < 0$ then from Lemma 4 we have that $v_2(z_1) = v_2(z_2) = -k < 0$ for some positive integer k. So, from Equation (5.16) we get that

$$2\tilde{b}_1 u_1^2 - b_2 u_2^2 = 2^{2k}$$

which in turn gives us $2|b_2 u_2^2$ which is again a contradiction.

Again, let $2|b_2$. If $v_2(z_1), v_2(z_2) \geq 0$ then from Equation (5.16) we get $2|1$ which is absurd. Further, let us consider, $b_2 = 2\tilde{b}_2$, where $\tilde{b}_2$ is odd, as $b_2 \in \mathbb{Q}(S, 2)$. If $v_1(z_1) < 0$ then by Lemma 4 we know that $v_2(z_1) = v_2(z_2) = v_2(z_3) = -k < 0$ for some positive integer k. Thus, from Equation (5.17) we get

$$\tilde{b}_1 u_1^2 - 2\tilde{b}_1\tilde{b}_2 u_3^2 = 2^{2k-1}3p. \tag{5.26}$$

Thus, from Equation (5.26) we get $2|\tilde{b_1} u_1^2$ which is again a contradiction. Therefore, if $2|b_1$ there are no solutions of the homogeneous space for $\ell = 2$.

(c) Similar to part (b).

(d) Let $(b_1, b_2) \in \mathbb{Q}(S, 2) \times \mathbb{Q}(S, 2)$ be a solution of the homogeneous spaces such that $3|b_2$. Then we can assume $b_2 = 3\tilde{b}_2$ where $\tilde{b}_2$ is odd, since b_2 is square-free.

Let, $3 \nmid b_1$. If $v_3(z_2), v_3(z_3) \geq 0$, then from Equation (5.18) we get that $3|2q$, which is a contradiction. Again, if $v_3(z_2) \geq 0$ and $v_3(z_3) \leq 0$ then from Equation (5.18) we get,

$$\tilde{b}_2 u_2^2 3^{2(k_2-k_3)} - b_1\tilde{b}_2 u_3^2 = 2q3^{-2k_3}. \tag{5.27}$$

From Equation (5.27) we can conclude that $3|b_1\tilde{b}_2 u_3^2$ which is a contradiction. Further, if $v_3(z_2) < 0$ then from Lemma 4 we have $v_3(z_1) = v_2(z_2) = v_3(z_3) = -k$ for some positive integer k. Then from Equation (5.16), we have

$$b_1 u_1^2 - 3\tilde{b}_2 u_2^2 = 3^{2k}. \tag{5.28}$$

Thus, from Equation (5.28) we get $3|b_1 u_1^2$ which is again a contradiction.

Again, let $3|b_2$ and $3|b_1$, then if $v_3(z_2), v_3(z_1) \geq 0$. So, from Equation (5.16) we get $3|1$ which is absurd. From Lemma 4 we know that if $v_3(z_1)$ or $v_3(z_2) < 0$ then $v_3(z_1) = v_3(z_2) = v_3(z_3) = -k < 0$ for some positive integer k. So, from Equation (5.18) we have the following equation.

$$3\tilde{b}_2 u_2^2 - 3\tilde{b}_1\tilde{b}_2 u_3^2 = 2q3^{2k-1}. \tag{5.29}$$

Therefore, from Equation (5.29) we get $3|\tilde{b}_2 u_2^2$.

Hence, we conclude that if $3|b_2$ then there are no solutions to the homogeneous spaces for $\ell = 3$.

(e) Similar to part (d).

$\square$

Since each b_i has 32 possible values in $\mathbb{Q}(S,2)$, there are initially 1024 pairs (b_1, b_2). Lemma 5 rules out all pairs that admit no ℓ-adic solutions, reducing the viable set to 32 candidates.

Now, we remove the images of the torsion subgroup which leaves us with only 28 pairs. In the next section we study the Equations (5.16) and (5.17) over $\mathbb{Q}$ and further reduce the number of possibilities.

5.2. *Diophantine Analysis.* It is easy to observe that few pairs are interdependent and thus, if a solution exists for one, then there is a solution exists for the other points.

The following diagrams lists the inter-dependencies between the pairs.

$$\begin{tikzcd}
(3,\pm 1) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & (p,\mp 1) \arrow[d, leftrightarrow] \\
(p,\pm 2q) \arrow[r, leftrightarrow] & (3,\mp 2q)
\end{tikzcd}
\qquad
\begin{tikzcd}
(3p,\pm q) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & (1,\pm 2) \arrow[d, leftrightarrow] \\
(3p,\mp 2) \arrow[r, leftrightarrow] & (1,\mp q)
\end{tikzcd}$$

$$\begin{tikzcd}
(3p,\pm q) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & (1,\pm 2) \arrow[d, leftrightarrow] \\
(3p,\mp 2) \arrow[r, leftrightarrow] & (1,\mp q)
\end{tikzcd}
\qquad
\begin{tikzcd}
(3p,\pm 1) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & (1,\pm 21) \arrow[d, leftrightarrow] \\
(3p,\mp 2q) \arrow[r, leftrightarrow] & (1,\mp 1)
\end{tikzcd}$$

One of the above relations are solidified in Lemma 6, and the rest follows in a similar manner.

LEMMA 6. *Let there be a solution $z_1, z_2, z_3 \in \mathbb{Q}$ for Equations* (5.16) *and* (5.17) *for $(b_1, b_2) \in \mathbb{Q}(S,2) \times \mathbb{Q}(S,2)$. Then,*

(a) *Let $(b_1, b_2) = (3, \pm 1)$, then if there are z_1, z_2, z_3 in $\mathbb{Q}$ satisfying Equations* (5.16) *and* (5.17) *then there are $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \in \mathbb{Q}$ satisfying the equations for the case $(b_1, b_2) = (p, \mp 1)$.*

(b) *Let $(b_1, b_2) = (3, \pm 2q)$, then if z_1, z_2, z_3 in $\mathbb{Q}$ satisfies Equations* (5.16) *and* (5.17) *then there exist $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \in \mathbb{Q}$ satisfying the equations for $(b_1, b_2) = (p, \pm 1)$.*

(c) *Let $(b_1, b_2) = (p, \pm 2q)$, then if z_1, z_2, z_3 in $\mathbb{Q}$ satisfies Equations* (5.16) *and* (5.17) *then there exist $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \in \mathbb{Q}$ satisfying the equations for $(b_1, b_2) = (3, \pm 1)$.*

PROOF. Let there exist solution $(z_1, z_2, z_3) \in \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}^*$ for $b_1 = 3$ and $b_2 = 1$ in (5.16) and (5.17). Then observe that,

$$\tilde{z}_1 = \frac{1}{z_1}, \quad \tilde{z}_2 = \frac{z_3}{z_1}, \quad \tilde{z}_3 = \frac{z_2}{z_3}$$

forms a solution for $b_1 = p$ and $b_2 = -1$. The proof for the case $(b_1, b_2) = (3, -1)$ is similar to the above.

Again, if there is a solution $(z_1, z_2, z_3) \in \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}^*$ for $b_1 = p$ and $b_2 = 2q$ in (5.16) and (5.17). Then

$$\tilde{z}_1 = \frac{z_2}{z_3} \quad \tilde{z}_2 = \frac{z_1}{z_3} \quad \tilde{z}_3 = \frac{1}{z_3},$$

gives us solutions for $(b_1, b_2) = (3, 1)$. Again for the negative case, the proof is similar.

Further, if $(z_1, z_2, z_3) \in \mathbb{Q}^* \times \mathbb{Q}^* \times \mathbb{Q}^*$ for $b_1 = 3$ and $b_2 = 2q$ is a solution then,

$$\tilde{z}_1 = \frac{z_2}{z_3} \quad \tilde{z}_2 = \frac{z_1}{z_3} \quad \tilde{z}_3 = \frac{1}{z_3},$$

forms a solution for $(b_1, b_2) = (p, 1)$. The negative case is again similar. ∎

As mentioned in 5.15, the torsion points are $(3p, -1), (1, -2q), (3p, 2q)$ and $(1, 1)$. The number of significant points is reduced to 28.

Lemma 7. *If $(b_1, b_2) = (1, -1)$ then there are no solutions to the Equations* (5.16) *and* (5.17) *in* $\mathbb{Q}$.

Proof. Consider $(z_1, z_2, z_3) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ such that the Equations

$$z_1^2 + z_2^2 = 1, \tag{5.30}$$

$$z_1^2 + z_3^2 = 3p, \tag{5.31}$$

are satisfied. Without loss of generality consider $z_1 = \frac{x}{z}$ and $z_3 = \frac{y}{z}$. Then Equation (5.31) transforms into

$$x^2 + y^2 = 3pz^2,$$

where x, y and z are all in $\mathbb{Z}$. From here it is clear that Equation (5.31) has no rational solutions. ∎

Further analysis into this direction eliminates two more points.

Lemma 8. *If $(b_1, b_2) = (p, -1)$ then there are no solutions to Equations* (5.16) *and* (5.17) *in* $\mathbb{Q}$.

Proof. Similar to the Lemma 6, consider $z_1, z_2, z_3 \in \mathbb{Q}$ satisfying the Equations

$$pz_1^2 + z_2^2 = 1 \tag{5.32}$$

$$z_1^2 + z_3^2 = 3. \tag{5.33}$$

Assuming, $z_1 = \frac{x}{z}$ and $z_3 = \frac{y}{z}$, Equation (5.33) becomes

$$x^2 + y^2 = 3z^2.$$

By a similar argument as in Lemma 6, it is clear that Equation (5.33) has no rational solutions. ∎

Another approach evaluate solutions is to check for the general point on a conic sections.

Lemma 9. *If* $(b_1, b_2) = (1, 2)$ *then there are no solutions to Equations* (5.16) *and* (5.17) *in* $\mathbb{Q}$.

Proof. Let there be $z_1, z_2, z_3 \in \mathbb{Q}$ satisfying Equations (5.16) and (5.18) when $b_1 = 1$ and $b_2 = 2$. Then the system of Equations transformed into the following,

$$z_1^2 - 2z_2^2 = 1, \tag{5.34}$$

$$z_2^2 - z_3^2 = q. \tag{5.35}$$

Note that, $(1, 0)$ forms a trivial solution for Equation (5.34). Therefore, a line passing through $(0, 1)$ with slope m will have the equation

$$y = m(x - 1).$$

Substituting the value of y in the place of z_2 in Equation (5.34) implies that $x = 1$ or, $x = \frac{2m^2+1}{2m^2-1}$. Thus, $y = \frac{2m}{2m^2-1}$, when substituted in place of z_2 in Equation (5.35) yields $z_3^2 < 0$, which is a contradiction.

Therefore, there are no rational solutions of the Equations (5.16) and (5.17) if $b_1 = 1$ and $b_2 = 2$. □

Combining Lemmas 5 and 6 with Hensel's local–global principle shows that at most 32 pairs $(b_1, b_2) \in \mathbb{Q}(S, 2) \times \mathbb{Q}(S, 2)$ can satisfy (5.16)–(5.18). Lemmas 7, 8 and 9 eliminate three of these pairs, and each such pair rules out three additional dependent pairs. After removing the images of torsion, only 16 viable pairs remain.

Therefore the 2-Selmer rank of $E_{2\mathcal{P}}(3)$ is at most 16, and hence the Mordell–Weil rank of $E_{2\mathcal{P}}(\mathbb{Q}(t))$ is also at most 16. Since the specialization map at $t = 3$ is an isomorphism ([4]), this upper bound holds uniformly for the family $E_{2\mathcal{P}}(t)$, completing the proof of Theorem 5.1.

6. Concluding remarks

In this work we studied the Legendre family

$$E_n(t) :\; y^2 = x(x-1)\big(x - \lambda_n(t)\big), \qquad \lambda_n(t) = \tfrac{n-2}{2} t(t-1) + t,$$

arising from the n-gonal polynomial. Using the geometry of the associated elliptic surface and the Shioda-Tate formula, we proved that for every $n \geq 3$ the generic Mordell-Weil rank over $\mathbb{Q}(t)$ is 0, so that all non-torsion points on specialized fibres arise purely from specialization.

Although the global rank is zero, we constructed explicit two-parameter subfamilies $E_n(t(g, h))$ of rank at least 1. Specializations of these subfamilies recover known examples (e.g. for $n = 3$) and produce new ones with ranks 2 and 3. This shows that the polygonal parametrization accommodates substantive rank growth inside a generically trivial family.

We also carried out a detailed 2-descent for the special subfamily $n = 2\mathcal{P}$, where $2\mathcal{P} - 1$ and $3\mathcal{P} - 2$ are prime. A combination of local solubility analysis and Diophantine arguments bounds the 2-Selmer rank of $E_{2\mathcal{P}}(3)$ by 16, and by specialization this bound holds uniformly for $E_{2\mathcal{P}}(t)$. This provides a rare instance of a parametrized Legendre family with a transparent, uniform Selmer bound.

Finally, we used Nagao averages and a specialization-level heuristic to search for large-rank fibres. For $n = 1908$ the progression $t \equiv 8 \pmod 9$ showed a strong analytic signal and yielded an explicit specialization of Mordell–Weil rank 3. This illustrates how Selmer-theoretic and analytic tools complement each other in detecting rank growth within parametrized Legendre families.

Several directions suggest themselves: extending the Selmer analysis to broader classes of n, comparing rank distributions in fixed progressions with random-matrix predictions, and studying analogous families attached to other figurate sequences. We hope the framework developed here will be useful in such investigations.

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