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The non-abelian group of order 26 acting on Steiner 2–designs $S(2, 6, 91)$

(Running head: $Frob_{26}$ acting on 2–$(91, 6, 1)$ designs)

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Abstract

There are only four known Steiner 2–designs $S(2, 6, 91)$, the Mills design, the McCalla design and two designs found by C. J. Colbourn and M. J. Colbourn. All these designs admit a cyclic automorphism of order 91. In 1991, Z. Janko and V. D. Tonchev showed that any point-transitive Steiner 2–design $S(2, 6, 91)$ with an automorphism group of order larger than 91 is one of the four known designs. It is an open question whether there exists a Steiner 2–design $S(2, 6, 91)$ with full automorphism group of order smaller than 91. In this paper we show that any Steiner 2–design $S(2, 6, 91)$ having a non-abelian automorphism group of order 26 (i.e. the Frobenius group Frob_{26}) is isomorphic to one of the known designs, the McCalla design having the full automorphism group isomorphic to $C_{91} : C_{12}$ or the Colbourn and Colbourn design having the full automorphism group isomorphic to $C_{91} : C_4$.

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1 Introduction

There are only four known Steiner 2–designs $S(2, 6, 91)$, i.e. designs with parameters 2–$(91, 6, 1)$. The designs have been found by Mills, C.J. Colbourn and M.J. Colbourn in [3], [4] and [16]. Each design is cyclic, i.e. having a cyclic automorphism group acting transitively on points. Two of the designs are the Mills design and the so called McCalla design (named after Gordon McCalla, but constructed by C.J. Colbourn and M.J. Colbourn in [4]), which are the only block-transitive 2–$(91, 6, 1)$ designs, and also point-imprimitive. Their full automorphism groups were computed by S.D. Stoichev and V.D. Tonchev in

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[19]. The Mills and the McCalla design have the full automorphism group $C_7 : (C_{13} : C_3)$ and $C_7 : (C_{13} : C_{12})$ respectively, so the full automorphism group of the McCalla design is reaching the bound $91 \cdot 12$ obtained in [19]. Their base blocks for the action of the automorphism group C_{91} generated by the permutation $(0, 1, \ldots, 90)$, are as follows:

$\mathcal{D}_1$) The Mills design - base blocks: $[7, 8, 10, 19, 52, 71]$, $[7, 11, 47, 61, 76, 81]$, $[7, 13, 20, 45, 80, 88]$.

$\mathcal{D}_2$) The McCalla design - base blocks: $[1, 13, 26, 31, 64, 73]$, $[7, 13, 17, 39, 84, 87]$, $[13, 52, 54, 81, 88, 89]$.

The other two designs have the full automorphism group $C_7 : (C_{13} : C_4)$ and C_{91}, and their base blocks for the action of the automorphism group C_{91} generated by the permutation $(0, 1, \ldots, 90)$, are as follows:

$\mathcal{D}_3$) Base blocks: $[1, 13, 26, 31, 64, 73]$, $[7, 13, 17, 39, 84, 87]$, $[11, 13, 52, 67, 68, 75]$.

$\mathcal{D}_4$) Base blocks: $[7, 8, 10, 19, 52, 71]$, $[7, 11, 28, 33, 48, 62]$, $[7, 13, 20, 45, 80, 88]$.

Camina and Di Martino in [1] proved that any automorphism group of a point-transitive 2–$(91, 6, 1)$ design is the natural split extension of a cyclic group of order 91 by a cyclic group of order d, where d divides 12.

Hence, any point-transitive 2–$(91, 6, 1)$ design is cyclic, and all additional automorphisms of that design are multipliers forming a cyclic group of order at most 12. Further, in 1991, Z. Janko and V.D. Tonchev showed that any point-transitive 2–$(91, 6, 1)$ design with an automorphism group of order larger than 91 is one of the four known designs (see [12]).

It is an open question whether there exists a 2–$(91, 6, 1)$ design with full automorphism group of order smaller than 91. Since the non-abelian group Frob_{26} is the automorphism group of the two known 2–$(91, 6, 1)$ designs (the McCalla design $\mathcal{D}_2$ and the Colbourn and Colbourn design $\mathcal{D}_3$), in this paper we observe the action of that group on a 2–$(91, 6, 1)$ design.

The paper is organized as follows. After a brief overview of the basic concepts related to designs and their automorphisms, given in the next section, in Section 3 we outline the construction of designs via orbit matrices. Besides that, we give an additional constraint on the columns of point orbit matrices of quasi-symmetric designs and a cyclic automorphism group that correspond to orbits of odd length. In Sections 4 and 5 we examine the action of the automorphism groups C_{13} and C_2 on a 2–$(91, 6, 1)$ design. The classification of 2–$(91, 6, 1)$ designs having the non-abelian group of order 26 as an automorphism group is given in Section 6. It is shown that there exist exactly two pairwise nonisomorphic Steiner 2–designs $S(2, 6, 91)$ having a non-abelian automorphism group of order 26, which are isomorphic to the known designs (the McCalla design $\mathcal{D}_2$ and the Colbourn and Colbourn design $\mathcal{D}_3$).

Computation in this paper consisted of programs written for GAP [10].

2 Preliminaries

A t–(v, k, λ) *design* is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, where $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$, with $|\mathcal{P}| = v$ points with the property that each element of $\mathcal{B}$ (called a block) is incident with

exactly k points and every t distinct points are incident with exactly λ blocks. If a design is simple (has no repeated blocks), then a block can be identified with a subset of the point set $\mathcal{P}$. A *Steiner system* $S(t,k,v)$, or a *Steiner t–design* $S(t,k,v)$, is a t–$(v,k,1)$ design with $t \geq 2$.
In a 2–(v,k,λ) design (called a block design) every point is incident with exactly $r = \frac{\lambda(v-1)}{k-1}$ blocks, and r is called the replication number of the design. The number of blocks is $b = \frac{vr}{k}$. If a 2–(v,k,λ) design is *symmetric* (i.e. $b = v$), then $r = k$, and any two distinct blocks B and B' are incident with exactly λ common points, i.e. $|B \cap B'| = \lambda$. More generally, a t–design is called *quasi-symmetric* with intersection numbers x and y, for nonnegative integers $x < y$, if any two blocks intersect in either x or y points. So, any symmetric design is quasi-symmetric design with $x = \lambda$ and y is arbitrary. Any Steiner 2-design ($\lambda = 1$) is a quasi-symmetric design with intersection numbers $x = 0$ and $y = 1$. For more information on quasi-symmetric designs we refer the reader to [17] and [18].

The point-by-block *incidence matrix* of a t–(v,k,λ) design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a $(v \times b)$ matrix whose rows are indexed by points and columns by blocks, with the entry in row P and column B being 1 if $(P,B) \in \mathcal{I}$, and 0 otherwise.

An isomorphism from one design to an other is a bijective mapping of points to points and blocks to blocks which preserves incidence. An isomorphism from a design $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted by $\mathrm{Aut}(\mathcal{D})$. The concept of G–isomorphism was introduced by V.Ćepulić in [2], and it is very useful in the isomorph rejection during the construction of designs by using orbit matrices. An isomophism α from a design $\mathcal{D}_1 = (\mathcal{P}, \mathcal{B}, \mathcal{I}_1)$ onto a design $\mathcal{D}_2 = (\mathcal{P}, \mathcal{B}, \mathcal{I}_2)$ is called a *G-isomorphism* from $\mathcal{D}_1$ onto $\mathcal{D}_2$, for $G \leq \mathrm{Aut}(\mathcal{D}_1) \cap \mathrm{Aut}(\mathcal{D}_2)$, if there is an automorphism $\tau : G \to G$ such that $\alpha(P)\tau(g) = \alpha(Q) \Leftrightarrow Pg = Q$, for each $P, Q \in \mathcal{P}$ and each $g \in G$, where Pg denotes the action of g on the point P. In other words, a G–isomorphism is an isomorphism from one onto antoher design which preserves G-orbits on the set of points and also on the set of blocks. If $\mathcal{I}_1 = \mathcal{I}_2$, α is called a G–automorphism of $\mathcal{D}_1$. In [6], the authors proved that a permutation $\alpha \in S = S(\mathcal{P}) \times S(\mathcal{B})$ is a G–isomorphism from $\mathcal{D}_1$ onto $\mathcal{D}_2$ if and only if α is in the normalizer $N_S(G)$. For more details on G–isomorphisms we refer to [2] and [6].

3 Orbit matrices and indexing

The method of tactical decomposition (and orbit matrices) is a well known method which has been used for constructions of 2–designs with presumed automorphism groups for the last 40 years (see [5, 6, 11, 12, 13]). Actually, orbit matrices were first used by Dembowski (see [8]). Below, we give the notations and properties for point orbit matrices that we used in the construction of 2–$(91,6,1)$ designs.

Let $\mathcal{D}$ be a 2–(v,k,λ) design with $G \leq \mathrm{Aut}(\mathcal{D})$. We denote G-orbits of points and blocks by $\mathcal{P}_1, \ldots, \mathcal{P}_m$ and $\mathcal{B}_1, \ldots, \mathcal{B}_n$ respectively, and put $|\mathcal{P}_i| = \omega_i$, $|\mathcal{B}_j| = \Omega_j$, $1 \leq i \leq m$, $1 \leq j \leq n$. It holds that $\sum_{i=1}^{m} \omega_i = v$, and also $\sum_{j=1}^{n} \Omega_j = b$. This action of G divides the incidence matrix M of $\mathcal{D}$ into $m \cdot n$ submatrices M_{ij}, for $1 \leq i \leq m,\ 1 \leq j \leq n$.

Further, by t_{ij} we denote the number of blocks in $\mathcal{B}_j$ incident with the representative of the point orbit $\mathcal{P}_i$, i.e. $t_{ij} = |\{B \in \mathcal{B}_j \mid (P,B) \in \mathcal{I},\ P \in \mathcal{P}_i\}|$. Analogously, b_{ij}

is the number of points in $\mathcal{P}_i$ incident with the representative of the block orbit $\mathcal{B}_j$, i.e. $b_{ij} = |P \in \mathcal{P}_i \mid (P, B) \in \mathcal{I},\ B \in \mathcal{B}_j|$. The numbers t_{ij} and b_{ij} do not depend on the choice of the representatives of point and block orbits, respectively. By determing the cardinal number of the set $\{(P, B) \mid (P, B) \in \mathcal{I},\ P \in \mathcal{P}_i,\ B \in \mathcal{B}_j\}$ in two different ways, it follows that $\omega_i t_{ij} = \Omega_j b_{ij}$.
The following conditions hold for t_{ij} (see [6, 11]):

$$0 \leq t_{ij} \leq \Omega_j, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n, \tag{1}$$

$$\sum_{j=1}^{n} t_{ij} = r, \quad 1 \leq i \leq m, \tag{2}$$

$$\sum_{i=1}^{m} \frac{\omega_i}{\Omega_j} t_{ij} = k, \quad 1 \leq j \leq n, \tag{3}$$

$$\sum_{j=1}^{n} \frac{\omega_s}{\Omega_j} t_{sj} t_{s'j} = \lambda \omega_s + \delta_{ss'} \cdot (r - \lambda), \quad 1 \leq s, s' \leq m. \tag{4}$$

Definition 3.1. A $(m \times n)$ matrix $T = [t_{ij}]$ with entries satisfying conditions (1) – (4) is called a *point orbit matrix* of a 2–(v, k, λ) design with orbit lengths distributions $(\omega_1, \ldots, \omega_m)$ and $(\Omega_1, \ldots, \Omega_n)$.

Definition 3.2. A $(l \times n)$ matrix $[t_{ij}]$, for $l < m$, with entries satisfying conditions (1), (2), (4), and the condition

$$\sum_{i=1}^{l} \frac{\omega_i}{\Omega_j} t_{ij} \leq k, \quad 1 \leq j \leq n,$$

is called a *partial point orbit matrix* of a 2–(v, k, λ) design with orbit lengths distributions $(\omega_1, \ldots, \omega_m)$ and $(\Omega_1, \ldots, \Omega_n)$.

Remark 3.3. By replacing t_{ij} with $\frac{\Omega_j}{\omega_i} b_{ij}$ in the equations (2)–(4) we get the $(m \times n)$ matrix $B = [b_{ij}]$ which is called a *block orbit matrix* of a 2–(v, k, λ) design satisfying the given equations, where $0 \leq b_{ij} \leq \omega_i,\ 1 \leq i \leq m,\ 1 \leq j \leq n$. So, the entry b_{ij} corresponds to the sum of a column in M_{ij}, and the entry t_{ij} corresponds to the sum of a row in M_{ij}. Obviously, if an automorphism group acts semi-standardly on $\mathcal{D}$ i.e. with the same orbit lengths distribution on the set of points and on the set of blocks, then $\mathcal{D}$ is a symmetric design and $T = B$.

The construction of designs based on orbit matrices consists of two basic steps (see [11]). The first step is the construction of orbit matrices for the presumed automorphism group G of a design and orbit lengths distributions for an action of G on the set of points and blocks. The second step is the construction of incidence matrices of designs corresponding to the orbit matrices obtained. This step is often called the *indexing* of orbit matrices. So, during the indexing of orbit matrices, one have to determine which blocks are incident with the representative of a point orbit for the presumed action of G on a design. That suggests the notion of an index set. The set of indices of blocks in the

G-orbit $\mathcal{B}_j$ indicating which blocks of $\mathcal{B}_j$ are incident with the representative of the point G-orbit $\mathcal{P}_i$ is called the *index set* for an entry t_{ij} in a point orbit matrix, $i \in \{1, \ldots m\}$ and $j \in \{1, \ldots, n\}$.

For the indexing, and also for the construction of orbit matrices we have used reverse lexicographical order, as described in [2].

In many cases, the indexing of orbit matrices is very hard to implement (due to many possiblities for index sets that should be determined for each t_{ij}, where $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Hence, in some cases, it is convenient to add an additional step in the construction of designs, the so called refinement of orbit matrices for a solvable group $G \leq Aut(\mathcal{D})$ onto orbit matrices for an action of a normal subgroup $H \trianglelefteq G$, since each G-orbit of points or blocks of $\mathcal{D}$ decomposes to one or more H-orbits of the same size, and the group G/H acts transitively on that set of H-orbits. In such a way, each orbit matrix for the group G decomposes to orbit matrices for the group H (for more information see [6, 7]).

An orbit matrix does not have to produce a design, while, on the other hand, many designs can be constructed from a single orbit matrix. Moreover, different orbit matrices may produce isomorphic designs. Thus, constructed designs need to be checked additionally for isomorphisms. However, the isomorph rejection, i.e. the elimination of (partial) orbit matrices which will produce isomorphic designs is very useful technique to reduce the number of constructed isomorphic designs. In this way, the computational time and memory required to execute the construction is reduced and contributes significantly to its implementation. So, to reduce the number of isomorphic designs during the construction, elements of the normalizer $N_{S(\mathcal{P})\times S(\mathcal{B})}(G)$, for a design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ and a group $G \leq \mathrm{Aut}(\mathcal{D})$ can be used. In particular, during the construction of (partial) orbit matrices, for the elimination of isomorphic ones, all permutations of their rows and columns, i.e. the elements from $S_m \times S_n$ which satisfy the conditions from Proposition 3.4 can be used (see [6] and [7]).

Proposition 3.4. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a 2-(v, k, λ) design, $G \leq Aut(\mathcal{D})$, and let the $(m \times n)$ matrix T be a point orbit matrix of the design $\mathcal{D}$ with respect to the group G. Then, let $g = (\alpha, \beta) \in S = S_m \times S_n$ with the following properties:*

1. *if $\alpha(s) = t$, then the stabilizer G_{P_s} is conjugate to G_{P_t}, where $P_s, P_t \in \mathcal{P}$, $\mathcal{P}_s = P_sG$ and $\mathcal{P}_t = P_tG$,*

2. *if $\beta(i) = j$, then G_{x_i} is conjugate to G_{x_j}, where $x_i, x_j \in \mathcal{B}$, $\mathcal{B}_i = x_iG$, $\mathcal{B}_j = x_jG$.*

Then there exists a permutation $g^* \in C_{S(\mathcal{P})\times S(\mathcal{B})}(G)$, such that

$$\alpha(s) = t \quad \textit{if and only if} \quad g^*(\mathcal{P}_s) = \mathcal{P}_t, \quad \textit{and}$$

$$\beta(i) = j \quad \textit{if and only if} \quad g^*(\mathcal{B}_i) = \mathcal{B}_j.$$

Definition 3.5. Two orbit matrices T and T' are *isomorphic* if there is a permutation $g = (\alpha, \beta) \in S_m \times S_n$ (called an isomorphism) from T onto $T' = Tg$ that satisfies the conditions from Proposition 3.4. If $T = Tg$ then the permutation g is called an *automorphism* of the orbit matrix T.

All automorphisms of the orbit matrix T form the full automorphism group of T denoted by $\mathrm{Aut}(T)$. During the construction of orbit matrices, for the isomorph rejection one can use the isomorphisms on the set of (partial) orbit matrices. But, during the indexing only the automorphisms of orbit matrices and also, more generally, the G-isomorphisms of designs (i.e. elements from the normalizer $N_{S(\mathcal{P})\times S(\mathcal{B})}(G)$) can be used (see [6]).

3.1 Orbit matrices of quasi-symmetric designs

The method of constructing designs by using orbit matrices has been applied also for quasi-symmetric designs (see [9, 14]). V. Krčadinac and R. Vlahović Kruc in [14] used an additional property that can be applied on the columns of block orbit matrices for quasi-symmetric designs. In the following proposition, we give a similar property of point orbit matrices of quasi-symmetric designs.

Proposition 3.6. *Let $\mathcal{D}$ be a quasi-symmetric 2–(v,k,λ) design with intersection numbers x and y, $0\le x<y$, and $G\le Aut(\mathcal{D})$ acting with orbit lengths distributions $(\omega_1,\ldots,\omega_m)$ and $(\Omega_1,\ldots,\Omega_n)$ on the set of points and blocks, respectively. Then the point orbit matrix $T=[t_{ij}]$ has an additional property:*

$$\frac{1}{\Omega_j}\sum_{i=1}^{m}\omega_i t_{ij}t_{ij'}=\begin{cases}\xi x+(\Omega_{j'}-\xi)y, & \text{for } j\neq j',\ 0\le\xi\le\Omega_{j'}\\ k+\xi x+(\Omega_{j'}-1-\xi)y, & \text{for } j=j',\ 0\le\xi<\Omega_{j'}\end{cases}\tag{5}$$

Proof. For a fixed block B in a block orbit $\mathcal{B}_j$ let us determine the number of points that are incident with B and B', for all blocks B' in a block orbit $\mathcal{B}_{j'}$, i.e., count the number of elements in the set $\{(P,B')\in\mathcal{P}\times\mathcal{B}_{j'}\mid(P,B)\in\mathcal{I},(P,B')\in\mathcal{I}\}$. Hence, it holds that $\sum_{i=1}^{m}b_{ij}t_{ij'}=\sum_{B'\in\mathcal{B}_{j'}}|\langle B\rangle\cap\langle B'\rangle|=\xi x+(\Omega_{j'}-\xi)y+(k-y)\delta_{jj'}$, where $\langle B\rangle$ represents the set of points incident with the block B, and nonnegative integer $\xi\ge 0$ such that $\xi\le\Omega_{j'}$ for $j\neq j'$, and $\xi\le\Omega_{j'}-1$ for $j=j'$. Since $\omega_i t_{ij}=\Omega_j b_{ij}$, the property (5) holds. $\square$

Ding et al. in [9] made the classification of quasi-symmetric 2–$(28,12,11)$ designs with intersection numbers $x=4$, $y=6$, having an automorphism of order 7 without fixed points or blocks. In that paper, the authors considered all possible values of the sums $\sum_{i=1}^{m}b_{ij}^2$ for each column $j\in\{1,\ldots,n\}$ in the constructed block orbit matrices $B=[b_{ij}]$, in order to eliminate some of them. Observing such an action of an automorphism of order 7 on an orbit of blocks, the authors have set fewer options for the parameter ξ in (5) for $j=j'$, so they got less number of candidates for columns in block orbit matrices. In the sequel, we make a generalization of their approach for any quasi-symmetric design and its cyclic automorphism group G having an orbit of blocks of odd length.

Let M be an incidence matrix of a quasi-symmetric 2–(v,k,λ) design $\mathcal{D}$ having intesection numbers x and y with a presumed cyclic automorphism group G acting on $\mathcal{D}$. Let $\mathcal{P}_1,\ldots\mathcal{P}_m$ be point orbits and $\mathcal{B}_1,\ldots,\mathcal{B}_n$ block orbits of $\mathcal{D}$ with respect to the action of G. Then, the corresponding $(m\times n)$ point orbit matrix $T=[t_{ij}]$ refines to an incidence matrix M of $\mathcal{D}$ that is divided into $m\cdot n$ circulant submatrices M_{ij}, for $1\le i\le m, 1\le j\le n$, as follows:

$$T = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ t_{m1} & \cdots & t_{mn} \end{bmatrix} \Rightarrow M = \begin{bmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{m1} & \cdots & M_{mn} \end{bmatrix}.$$

Let M_j be a $(v \times \Omega_j)$ submatrix of M corresponding to the block orbit $\mathcal{B}_j$, for $j \in \{1, \ldots, n\}$. Then, it is easy to see that $M_j^T M_j = \sum_{i=1}^m M_{ij}^T M_{ij}$, and the elements of $M_j^T M_j$ are the intersection numbers of blocks from the orbit $\mathcal{B}_j$. The matrix $M_j^T M_j$ is both symmetric and circulant, and its elements are from the set $\{k, x, y\}$ with k on the diagonal, since the intersection of a block with itself is equal to k. Label the entries of the first row in $M_j^T M_j$ by $m_1, \ldots, m_{\Omega_j}$. If Ω_j is odd, for some $j \in \{1, \ldots, m\}$, then

$$m_2 = m_{\Omega_j},\ m_3 = m_{\Omega_j - 1},\ \ldots\ ,\ m_{\frac{\Omega_j+1}{2}} = m_{\frac{\Omega_j+3}{2}},$$

since $M_j^T M_j$ is a symmetric and circulant matrix. So, there are $\frac{\Omega_j - 1}{2}$ pairs of equal elements (x or y) in each row of $M_j^T M_j$. It is obvious that $m_1 = k$. Therefore, the first row sum in $M_j^T M_j$ is

$$\sum_{i=1}^{\Omega_j} m_i = k + 2u \cdot x + (\Omega_j - 1 - 2u)y = k + (\Omega_j - 1)y + 2u(x - y), \tag{6}$$

for $u = 0, \ldots, \frac{\Omega_j - 1}{2}$.

On the other hand, the sum of the entries in the first row of $M_j^T M_j$ represents the sum of block intersections in the block orbit $\mathcal{B}_j$, i.e. the sum $\sum_{B' \in \mathcal{B}_j} | \langle B \rangle \cap \langle B' \rangle |$ for a block $B \in \mathcal{B}_j$. Hence, from (6) and the proof of Proposition 3.6, the additional restriction on the columns of a point orbit matrix $T = [t_{ij}]$ follows. This restriction is given in the following proposition.

Proposition 3.7. *Let $\mathcal{D}$ be a quasi-symmetric 2-(v, k, λ) design with the intersection numbers x and y, $0 \le x < y$, and a cyclic automorphism group $G \le Aut(\mathcal{D})$ acting on $\mathcal{D}$ with orbit lengths distributions $(\omega_1, \ldots, \omega_m)$ and $(\Omega_1, \ldots, \Omega_n)$. A point orbit matrix $T = [t_{ij}]$ has the following property (QS-property).*

1. *If $j \neq j'$, then*

$$\frac{1}{\Omega_j} \sum_{i=1}^m \omega_i t_{ij} t_{ij'} = u \cdot x + (\Omega_{j'} - u)y, \quad 0 \le u \le \Omega_{j'}. \tag{7}$$

2. *If $j = j'$ then*

$$\frac{1}{\Omega_j} \sum_{i=1}^m \omega_i t_{ij}^2 = k + 2u \cdot x + (\Omega_j - 1 - 2u)y, \quad 0 \le u \le \frac{\Omega_j - 1}{2}, \text{ for } \Omega_j \text{ odd}, \tag{8}$$

$$\frac{1}{\Omega_j} \sum_{i=1}^m \omega_i t_{ij}^2 = k + u \cdot x + (\Omega_j - 1 - u)y, \quad 0 \le u < \Omega_j, \text{ for } \Omega_j \text{ even}. \tag{9}$$

Remark 3.8. The number of potential values for the parameter u in (8) is twice as small as the number of possible values for the parameter ξ in (5) for $j = j'$ and an $\Omega_j > 1$ odd, where $j \in \{1, 2, \ldots, n\}$. In the case when Ω_j is even, then the entries of the first row in $Mj^T M_j$ are $m_1 = k$, $m_2 = m_{\Omega_j}, m_3 = m_{\Omega_j - 1}, \ldots, m_{\frac{\Omega_j}{2}} = m_{\frac{\Omega_j}{2}+2}$ and $m_{\frac{\Omega_j}{2}+1}$. So, in this case the first row sum in $Mj^T M_j$ doest not give any restriction on the parameter $0 \leq \xi < \Omega_j$ for $j = j'$ in Proposition 3.6.

Remark 3.9. Note that the QS-property from Proposition 3.7 can be modified for a block orbit matrix $B = [b_{ij}]$ by using the equality $\omega_i t_{ij} = \Omega_j b_{ij}$.

Remark 3.10. We checked whether the orbit matrices obtained in Section 6.1 satisfy the conditions outlined in Propositions 3.6 and 3.7. Since all the constructed orbit matrices satisfy these conditions, that did not lead to a reduction of the number of the orbit matrices.

4 The action of C_{13} on a 2–$(91, 6, 1)$ design

To determine the action of the non-abelian group Frob_{26} on a 2–$(91, 6, 1)$ design, first we need to examine the action of its normal subgroup C_{13} on a 2–$(91, 6, 1)$ design. To ascertain the number of points fixed by an automorphism of a prime order we use the following theorem, given in [15].

Theorem 4.1. *Let α be an automorphism of prime order p of a 2–$(v, k, 1)$ design. If α has f_T fixed points, then*

$$f_T \leq \begin{cases} r + k - p - 1, & \text{if } p \leq k - 1 \\ r - \frac{p-1}{k-1}, & \text{if } p \geq k. \end{cases}$$

Proposition 4.2. *An automorphism of order 13 acts fixed point and fixed block free on a 2–$(91, 6, 1)$ design.*

Proof. If a 2–$(91, 6, 1)$ design admits an automorphism ρ of order 13, then the number of points fixed by ρ is $f_T \in \{0, 13\}$, since ρ acts on the set of points (and blocks) in orbits of length 1 and 13, and by Theorem 4.1 it holds that $f_T \leq 15$. The replication number is $r = 18$, hence each ρ–fixed point is incident with 5 or 18 blocks fixed by ρ. Let there exists a ρ–fixed point incident with 18 fixed blocks. Since each ρ–fixed block is incident with $k = 6$ fixed points and $f_T \leq 13$, there will always be a pair of ρ–fixed blocks that intersect in more than one fixed point, which is a contradiction. So, for $f_T = 13$, each ρ–fixed point should be incident with exactly 5 ρ–fixed blocks. If we consider the fixed part of a design and count the incident pairs of points and blocks fixed by ρ, i.e. $|\{(P, B) \in \mathcal{P} \times \mathcal{B} \mid (P, B) \in \mathcal{I},$ where P and B are fixed by $\rho\}|$, in two ways, then we obtain $f_B \cdot 6 = 13 \cdot 5$. That implies that $f_B \notin \mathbb{N}_0$, which is a contradiction. Hence, $f_T = f_B = 0$. $\square$

5 The action of C_2 on a 2–$(91,6,1)$ design

Proposition 5.1. *If an involutory automorphism acts on a* 2–$(91,6,1)$ *design, then the number of fixed points is* $f_T \in \{1,7,13,15,17,19,21\}$.

Proof. If σ is an involutory automorphism of a 2–$(91,6,1)$ design, then it follows from Theorem 4.1 that $f_T \leq 21$. Obviously, $f_T \equiv 1 (\mathrm{mod}\ 2)$, since σ acts on the set of points (and blocks) in orbits of length 1 or 2, and $v = 91$. Hence, $f_T \in \{1,3,5,7,9,11,13,15,17,19,21\}$. The decomposition of a point orbit matrix A of a design for an action of the automorphism σ, in general, is given in (10), where the first column corresponds to the point and the first row corresponds to the block orbit lengths distribution.

$$\begin{array}{c|ccc:ccc} A & 1 & \cdots & 1 & 2 & \cdots & 2 \\ \hline 1 & & & & & & \\ \vdots & & A_{11} & & & A_{12} & \\ 1 & & & & & & \\ \hdashline 2 & & & & & & \\ \vdots & & A_{21} & & & A_{22} & \\ 2 & & & & & & \end{array} \tag{10}$$

Each block fixed by an involution is incident with an even number of non–fixed points. Since $k = 6$, a fixed block is also incident with an even number of fixed points. For the number of σ–fixed blocks it holds that $f_B = n'_0 + n'_2 + n'_4 + n'_6$, where n'_i is the number of fixed blocks incident with i fixed points, for $i \in \{0,2,4,6\}$.

Now, we consider all possible point and block orbit lengths distributions, which are determined by the number of points and blocks fixed by σ.

1) Let $\underline{f_T = 1}$. Then the point orbit matrix A has 46 rows. In this case, a fixed block is incident with 6 non–fixed points ($f_B = n'_0$), so the sum of all elements in the submatrix A_{21} is equal to $3 \cdot f_B$. Since two non-fixed points from the same orbit are incident with exactly one common fixed block, each row in A_{21} is a permutation of $(1,0,0,\ldots,0)$. Thus, the sum of the elements in A_{21} is $45 = 3 \cdot f_B$, which gives us $\underline{f_B = 15}$.

2) In the case of $f_T = 3$, the point orbit matrix A has $3 + 44 = 47$ rows. A fixed block is incident with 0 or 2 fixed points, so $f_B = n'_0 + n'_2$. Further, $n'_2 = 3$, since each pair of fixed points must be incident with one common fixed block. Each of the n'_2 fixed blocks is incident with 4 non-fixed points, and each of the n'_0 fixed blocks is incident with 6 non-fixed points, i.e. the sum of all elements in A_{21} is $2n'_2 + 3n'_0 = 6 + 3n'_0$. Analogously as in the first case, each row in the submatrix A_{21} is a permutation of $(1,0,0,\ldots,0)$, hence the sum of elements in A_{21} is $44 \cdot 1 = 6 + 3n'_0$, which is not possible.

3) If $f_T = 5$, then the point orbit matrix A has $5 + 43 = 48$ rows. A fixed block is incident with 0, 2 or 4 fixed points, so $f_B = n'_0 + n'_2 + n'_4$. Each pair of fixed points is incident with one common fixed block (a block incident with 2 or 4 fixed points), so the number of pairs of fixed points is $\binom{5}{2} = n'_4 \cdot \binom{4}{2} + n'_2 \cdot 1$. It is easy to check that the solutions are $n'_4 = 1, n'_2 = 4$ and $n'_4 = 0, n'_2 = 10$. However, neither of the possibilities meets the condition $3 \cdot n'_0 + 2 \cdot n'_2 + 1 \cdot n'_4 = 43$ (i.e. the sum of elements in A_{21}).

4) Let $\underline{f_T = 7}$. Then the matrix A has $7 + 42 = 49$ rows. A fixed block is incident with 0,

2, 4 or 6 fixed points, so $f_B = n'_0 + n'_2 + n'_4 + n'_6$. For the number of pairs of fixed points the following holds:

$$21 = \binom{7}{2} = n'_6 \cdot \binom{6}{2} + n'_4 \cdot \binom{4}{2} + n'_2 \Rightarrow n'_6 \leq 1. \tag{11}$$

The sum of elements in A_{21} is equal to

$$3 \cdot n'_0 + 2 \cdot n'_2 + 1 \cdot n'_4 = 42. \tag{12}$$

4.1) If $n'_6 = 1$, then $n'_4 = 0$, since $f_T = 7$ and $\lambda = 1$. So, from (11) and (12) it follows that $n'_2 = 6$ and $n'_0 = 10$, hence $\underline{f_B = 17}$.

4.2) If $n'_6 = 0$, then it is easy to see that $n'_4 \leq 2$, since $\lambda = 1$. If $n'_4 = 2 \overset{(11)}{\Longrightarrow} n'_2 = 9$ (also if $n'_4 = 1 \overset{(11)}{\Longrightarrow} n'_2 = 15$), but from (12) it follows that $n'_0 \notin \mathbb{N}$. Hence, $n'_4 = 0$, so $n'_2 = 21$ and $\underline{f_B = 21}$.

5) Let $f_T = 9$, then the number of σ-fixed blocks is $f_B = n'_0 + n'_2 + n'_4 + n'_6$, and the matrix A has $9 + 41 = 50$ rows. Here, the following conditions on the sum of the elements in A_{21} and on the number of pairs of fixed points hold:

$$3 \cdot n'_0 + 2 \cdot n'_2 + 1 \cdot n'_4 = 41 \;\Rightarrow\; n'_2 \leq 20, \tag{13}$$

$$36 = \binom{9}{2} = n'_6 \cdot \binom{6}{2} + n'_4 \cdot \binom{4}{2} + n'_2. \tag{14}$$

Since $\lambda = 1$, it follows that $n'_6 \leq 1$. If $n'_6 = 1$, then $n'_4 = 1$ (so $n'_2 = 15$), which follows from (13), (14) and $\lambda = 1$. However, from (13) it follows that $n'_0 \notin \mathbb{N}$. If $n'_6 = 0$, then $n'_4 \leq 3$, i.e. there are at most 3 fixed blocks incident with 4 fixed points, since any pair of fixed points are incident with a common fixed block. However, it is easy to check that for each $n'_4 \leq 3$, the equations (13) and (14) lead to a conclusion that $n'_0 \notin \mathbb{N}_0$.

6) If $f_T = 11$, then $f_B = n'_0 + n'_2 + n'_4 + n'_6$, and the matrix A has $11 + 40 = 51$ rows. The following conditions on the sum of the elements in A_{21} and on the number of pairs of fixed points hold:

$$3 \cdot n'_0 + 2 \cdot n'_2 + 1 \cdot n'_4 = 40 \;\Rightarrow\; n'_2 \leq 20, \tag{15}$$

$$55 = \binom{11}{2} = n'_6 \cdot \binom{6}{2} + n'_4 \cdot \binom{4}{2} + n'_2. \tag{16}$$

It is easy to check that $n'_6 \leq 2$, since $\lambda = 1$. If $n'_6 = 2$, then $n'_4 = 0$ and $n'_2 = 25$, but (15) leads to a contradiction, since $n'_2 \leq 20$. In the case of $n'_6 = 1$, then $n'_4 \leq 2$, but the equations (15) and (16) also lead to a contradiction. If $n'_6 = 0$, then from (15) and (16) follows that $n'_4 \geq 6$. Since $\lambda = 1$, it can be checked easily that in this case there are at most 6 fixed blocks having exactly 4 fixed blocks. So, $n'_4 = 6$, which implies that $n'_2 = 19$. Now the equation (15) gives that $n'_0 \notin \mathbb{N}_0$.

7) If $\underline{f_T = 13}$, then $f_B = n'_0 + n'_2 + n'_4 + n'_6$, and the matrix A has $13 + 39 = 52$ rows.

The following conditions on the sum of the elements in A_{21} and on the number of pairs of fixed points hold:

$$3 \cdot n'_0 + 2 \cdot n'_2 + 1 \cdot n'_4 = 39 \;\Rightarrow\; n'_2 \leq 19, \tag{17}$$

$$78 = \binom{13}{2} = n'_6 \cdot \binom{6}{2} + n'_4 \cdot \binom{4}{2} + n'_2. \tag{18}$$

There are at most 2 fixed blocks incident with 6 fixed points, i.e. $n'_6 \leq 2$. If $n'_6 = 2$, then it is easy to check that $n'_4 \leq 1$, but then the equations (17) and (18) are not satisfied. If $n'_6 = 1$, then the equation (18) and $n'_2 \leq 19$ imply that $n'_4 \geq 8$. In this case, the maximal number of fixed blocks having exactly 4 fixed points is 6, which is a contradiction. If $n'_6 = 0$, then from (17) and (18) follows that $n'_4 = 12$, $n'_2 = 6$ and $n'_0 = 5$. Hence, there are 23 fixed blocks, i.e. $\underline{f_B = 23}$.

8) The results for $f_T \in \{15, 17, 19, 21\}$ are presented in Table 1. $\square$

$f_T = 15$	
Conditions	Solutions
The sum in A_{21}:	1) If $n_6' = 3$, then $n_4' = 8, n_2' = 12, n_0' = 2 \Rightarrow \boxed{f_B = 25}$
$3n_0' + 2n_2' + n_4' = 38$	2) If $n_6' = 2$, then $n_4' = 11, n_2' = 9, n_0' = 3 \Rightarrow \boxed{f_B = 25}$
#Pairs of fixed points:	3) If $n_6' = 1$, then $n_4' = 14, n_2' = 6, n_0' = 4 \Rightarrow \boxed{f_B = 25}$
$n_2' + 6n_4' + 15n_6' = 105$	4) If $n_6' = 0$, then $n_4' = 17, n_2' = 3, n_0' = 5 \Rightarrow \boxed{f_B = 25}$
Upper-bound for n_6': $n_6' \leq 3$	
$f_T = 17$	
Conditions	Solutions
The sum in A_{21}:	1) If $n_6' = 3$, then $n_4' = 14, n_2' = 7, n_0' = 3 \Rightarrow \boxed{f_B = 27}$
$3n_0' + 2n_2' + n_4' = 37$	2) If $n_6' = 2$, then $n_4' = 17, n_2' = 4, n_0' = 4 \Rightarrow \boxed{f_B = 27}$
#Pairs of fixed points:	3) If $n_6' = 1$, then $n_4' = 20, n_2' = 1, n_0' = 5 \Rightarrow \boxed{f_B = 27}$
$n_2' + 6n_4' + 15n_6' = 136$	4) If $n_6' = 0$, then there are no solutions.
Upper-bound for n_6': $n_6' \leq 3$	
$f_T = 19$	
Conditions	Solutions
The sum in A_{21}:	1) If $n_6' = 4$, then $n_4' = 18, n_2' = 3, n_0' = 4 \Rightarrow \boxed{f_B = 29}$
$3n_0' + 2n_2' + n_4' = 36$	2) If $n_6' = 3$, then $n_4' = 21, n_2' = 0, n_0' = 5 \Rightarrow \boxed{f_B = 29}$
#Pairs of fixed points:	3) If $n_6' = 2$, then there are no solutions.
$n_2' + 6n_4' + 15n_6' = 171$	4) If $n_6' = 1$, then there are no solutions.
Upper-bound for n_6': $n_6' \leq 4$	5) If $n_6' = 0$, then there are no solutions.
$f_T = 21$	
Conditions	Solutions
The sum in A_{21}:	1) If $n_6' = 7$, then $n_4' = 17, n_2' = 3, n_0' = 4 \Rightarrow \boxed{f_B = 31}$
$3n_0' + 2n_2' + n_4' = 35$	2) If $n_6' = 6$, then $n_4' = 20, n_2' = 0, n_0' = 5 \Rightarrow \boxed{f_B = 31}$
#Pairs of fixed points:	3) If $n_6' = 5$, then there are no solutions.
$n_2' + 6n_4' + 15n_6' = 210$	4) If $n_6' = 4$, then there are no solutions.
Upper-bound for n_6': $n_6' \leq 7$	5) If $n_6' = 3$, then there are no solutions.
	5) If $n_6' = 2$, then there are no solutions.
	5) If $n_6' = 1$, then there are no solutions.
	5) If $n_6' = 0$, then $n_4' = 35, n_2' = 0, n_0' = 0 \Rightarrow \boxed{f_B = 35}$

Table 1: The numbers of blocks fixed by an involution, for $f_T \in \{15, 17, 19, 21\}$.

The construction of 2–$(91, 6, 1)$ designs assuming only an action of an involutory automorphism or an automorphism of order 13, would be a very difficult task if approached using the method of tactical decomposition and orbit matrices, due to a very large number of possibilities for rows in orbit matrices to be determined, and then also for their indexing. However, for a larger group, such as Frob_{26}, the classification of 2–$(91, 6, 1)$ designs can be carried out, as it is shown in the next section.

6 The group Frob_{26} acting on a 2–$(91, 6, 1)$ design

In this section we study the action of the non-abelian group $Frob_{26} \cong C_{13} : C_2$ on a 2–$(91, 6, 1)$ design $\mathcal{D}$. The group $Frob_{26}$ acts as an automorphism group of two known 2–$(91, 6, 1)$ designs, the McCalla design $\mathcal{D}_2$ and the Colbourn and Colbourn design $\mathcal{D}_3$.

Let the group $G \cong \mathrm{Frob}_{26}$ be presented as follows

$$G = \left\langle \rho, \sigma \mid \rho^{13} = \sigma^2 = 1,\ \sigma\rho\sigma = \rho^{-1} \right\rangle .$$

Let us denote the points of $\mathcal{D}$ as $\mathcal{P} = \{P_j \mid P = 1, 2, \dots, 7,\ j = 0, 1, \dots, 12\}$ and the blocks of $\mathcal{D}$ as $\mathcal{B} = \{B_j \mid B = 1, 2, \dots, 21,\ j = 0, 1, \dots, 12\}$. By Proposition 4.2, we may assume that the automorphism ρ of order 13 acts on the set of points and blocks of $\mathcal{D}$ as $(P_0, P_1, \dots, P_{12})$, $P = 1, 2, \dots, 7$, and $(B_0, B_1, \dots, B_{12})$, $B = 1, 2, \dots, 21$, respectively.

The involutory automorphism σ acts on a ρ–orbit of points or blocks of $\mathcal{D}$ in one of the following ways.

1. Let σ be the stabilizer of the representative P_0 of the ρ–orbit of points $\{P_0, P_1, \dots, P_{12}\}$, for $P \in \{1, 2, \dots, 7\}$, and also of the representative B_0 of the ρ–orbit of blocks $\{B_0, B_1, \dots, B_{12}\}$, for $B \in \{1, 2, \dots, 21\}$. Then, these ρ–orbits are also G–orbits and the permutation σ acts on the indices of a point and block ρ–orbit in one of the following ways:

$$\sigma = \begin{cases} (0)(1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7) \\ (0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)(12). \end{cases} \tag{19}$$

Let us take a look on the action of $\sigma = (0)(1, 12)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)$ on the ρ–orbit of points $\{P_0, P_1, \dots, P_{12}\}$ and blocks $\{B_0, B_1, \dots, B_{12}\}$. In that case, such an action of σ on incident pairs is given as follows

$$(B_i, P_j) \mapsto (B_{(12 \cdot i) \bmod 13}, P_{(12 \cdot j) \bmod 13}),\ \text{for all } i, j \in \{0, 1, \dots, 12\}. \tag{20}$$

Hence, the corresponding part of the incidence matrix is given as follows:

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
B_0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_6	x_5	x_4	x_3	x_2	x_1
B_1	x_1	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_6	x_5	x_4	x_3	x_2
B_2	x_2	x_1	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_6	x_5	x_4	x_3
B_3	x_3	x_2	x_1	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_6	x_5	x_4
B_4	x_4	x_3	x_2	x_1	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_6	x_5
B_5	x_5	x_4	x_3	x_2	x_1	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_6
B_6	x_6	x_5	x_4	x_3	x_2	x_1	x_0	x_1	x_2	x_3	x_4	x_5	x_6
B_7	x_6	x_6	x_5	x_4	x_3	x_2	x_1	x_0	x_1	x_2	x_3	x_4	x_5
B_8	x_5	x_6	x_6	x_5	x_4	x_3	x_2	x_1	x_0	x_1	x_2	x_3	x_4
B_9	x_4	x_5	x_6	x_6	x_5	x_4	x_3	x_2	x_1	x_0	x_1	x_2	x_3
B_{10}	x_3	x_4	x_5	x_6	x_6	x_5	x_4	x_3	x_2	x_1	x_0	x_1	x_2
B_{11}	x_2	x_3	x_4	x_5	x_6	x_6	x_5	x_4	x_3	x_2	x_1	x_0	x_1
B_{12}	x_1	x_2	x_3	x_4	x_5	x_6	x_6	x_5	x_4	x_3	x_2	x_1	x_0

where the elements $x_i \in \{0, 1\}, \forall i \in \{0, 1, \dots, 6\}$, represent incidences between blocks and points.

For an entry t_{ij} in a point orbit matrix T, corresponding to a G-orbit of points and blocks of length 13, it holds that $t_{ij} \in \{0, 1, \ldots, 4\}$, since $\lambda = 1$. Besides that, if σ is the stabilizer of the representative a ρ–orbit of points and a ρ-orbit of blocks fixing exactly one point and block, as given in (19), it is easy to see that

$$t_{ij} \in \{0, 1, 2\}. \tag{21}$$

Namely, to index a point orbit matrix having an entry $t_{ij} = 4$ (i.e. to determine the corresponding index set), it can be done in exactly 15 ways, as follows (only first rows are listed since the other 12 rows are their cyclic permutations):

$$\begin{array}{l}
[0,1,1,0,0,0,0,0,0,0,0,1,1], [0,1,0,1,0,0,0,0,0,0,1,0,1], [0,1,0,0,1,0,0,0,0,1,0,0,1],\\
[0,1,0,0,0,1,0,0,1,0,0,0,1], [0,1,0,0,0,0,1,1,0,0,0,0,1], [0,0,1,1,0,0,0,0,0,0,1,1,0],\\
[0,0,1,0,1,0,0,0,0,1,0,1,0], [0,0,1,0,0,1,0,0,1,0,0,1,0], [0,0,1,0,0,0,1,1,0,0,0,1,0],\\
[0,0,0,1,1,0,0,0,0,1,1,0,0], [0,0,0,1,0,1,0,0,1,0,1,0,0], [0,0,0,1,0,0,1,1,0,0,1,0,0],\\
[0,0,0,0,1,1,0,0,1,1,0,0,0], [0,0,0,0,1,0,1,1,0,1,0,0,0], [0,0,0,0,0,1,1,1,1,0,0,0,0]
\end{array}$$

But, none of the 15 possibilities above satisfy the intersection condition $\lambda = 1$, since for each possibility there is a pair of points from the G–orbit that is incident with more than 1 common block. Analogously, if an entry in a point orbit matrix is $t_{ij} = 3$, it cannot be indexed because none of the following 6 candidates have adequate intersections of rows obtained by cyclic permutations:

$$\begin{array}{l}
[1,1,0,0,0,0,0,0,0,0,0,0,1],\ [1,0,1,0,0,0,0,0,0,0,0,1,0],\ [1,0,0,1,0,0,0,0,0,0,1,0,0],\\
[1,0,0,0,1,0,0,0,0,1,0,0,0],\ [1,0,0,0,0,1,0,0,1,0,0,0,0],\ [1,0,0,0,0,0,1,1,0,0,0,0,0].
\end{array}$$

If σ is the stabilizer of the representative of a ρ–orbit of points fixing exactly one point, and the stabilizer of the representative of a ρ–orbit of blocks fixing all blocks (or vice versa), then the corresponding entry in a point orbit matrix for the action of G on a 2–$(91, 6, 1)$ design must be equal to 0.

2. Let us observe the case when σ does not stabilize any point form some ρ–orbit of points, i.e. σ maps a ρ–orbit $\{P_0, P_1, \ldots, P_{12}\}$ onto a ρ–orbit $\{P'_0, P'_1, \ldots, P'_{12}\}$, for $P \neq P'$.

   2.1) If σ is the stabilizer of the representative of a ρ–orbit of blocks $\{B_0, B_1, \ldots, B_{12}\}$, acting on the set of indices as $\sigma = (0)(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$, then σ acts on incident pairs as follows

$$(B_i, P_j) \mapsto (B_{(12 \cdot i) \bmod 13}, P'_{(12 \cdot j) \bmod 13}), \text{ for all } i, j \in \{0, 1, \ldots, 12\}.$$

   Hence, the representative B_0 of the G–orbit of blocks $\{B_0, B_1, \ldots, B_{12}\}$ is as follows (the remaining blocks are obtained by cyclic permutations), where $x_i \in \{0, 1\}, \forall i \in \{0, 1, \ldots, 12\}$:

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P'_0	P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8	P'_9	P'_{10}	P'_{11}	P'_{12}
B_0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_0	x_{12}	x_{11}	x_{10}	x_9	x_8	x_7	x_6	x_5	x_4	x_3	x_2	x_1

   For an entry b_{ij} in a block orbit matrix B that corresponds to the G–orbit of blocks of length 13, and to the G–orbit of points of length 26, it can be shown that $b_{ij} \in \{0, 2\}$, since $\lambda = 1$ and the refinement of b_{ij} for the action of ρ requires its decomposition on $[1, 1]$ or $[0, 0]$. Namely, the action $i \mapsto 12i \bmod 13$ on the indices can be written as $i \mapsto -i \bmod 13$, and since $j - i \equiv -((-j) - (-i)) \bmod 13$, there exist two columns in the incidence matrix of the design, one corresponding to $\{P_0, P_1, \ldots, P_{12}\}$ and the other corresponding to $\{P'_0, P'_1, \ldots, P'_{12}\}$, having two ones in the positions corresponding to two rows, i.e. the corresponding blocks intersect in more than one point. This is a contradiction with $\lambda = 1$. Hence, $b_{ij} \neq 4$, i.e. it does not decompose as $[2, 2]$. That also implies that b_{ij} can not be decomposed as $[3, 3]$ or $[4, 4]$. Hence, $b_{ij} \in \{0, 2\}$.

2.2) If σ maps a ρ–orbit $\{B_0, B_1, \ldots, B_{12}\}$ onto a ρ–orbit of blocks $(B'_0, B'_1, \ldots, B'_{12})$, for $B \neq B'$, then for all $i, j \in \{0, 1, \ldots, 12\}$ σ acts as follows

$$(B_i, P_j) \mapsto (B'_{(12 \cdot i) \bmod 13}, P'_{(12 \cdot j) \bmod 13}),$$
$$(B_i, P'_j) \mapsto (B'_{(12 \cdot i) \bmod 13}, P_{(12 \cdot j) \bmod 13}).$$

Hence, the representatives B_0 and B'_0 of the G-orbits of blocks $\{B_0, B_1, \ldots, B_{12}\}$ and $\{B'_0, B'_1, \ldots, B'_{12}\}$, respecitvely, for $x_i, y_i \in \{0, 1\}, \forall i \in \{0, 1, \ldots, 12\}$, are as follows:

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P'_0	P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8	P'_9	P'_{10}	P'_{11}	P'_{12}
B_0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	y_{12}
B'_0	y_0	y_{12}	y_{11}	y_{10}	y_9	y_8	y_7	y_6	y_5	y_4	y_3	y_2	y_1	x_0	x_{12}	x_{11}	x_{10}	x_9	x_8	x_7	x_6	x_5	x_4	x_3	x_2	x_1

3. If σ is the stabilizer of the representative of a ρ–orbit of points $\{P_0, \ldots, P_{12}\}$, but does not stabilize a block from a ρ–orbit of blocks, i.e. maps a ρ–orbit $\{B_0, B_1, \ldots, B_{12}\}$ onto a ρ-orbit of blocks $\{B'_0, B'_1, \ldots, B'_{12}\}$, for $B \neq B'$, then the action of σ is given by
$$(B_i, P_j) \mapsto (B'_{(12 \cdot i) \bmod 13}, P_{(12 \cdot j) \bmod 13}), \text{ for all } i, j \in \{0, 1, \ldots, 12\}.$$

   Hence, the representatives B_0 and B'_0 of the G–orbit of blocks $\{B_0, B_1, \ldots, B_{12}\}$ and $\{B'_0, B'_1, \ldots, B'_{12}\}$, respectively, for $x_i \in \{0, 1\}, \forall i \in \{0, 1, \ldots, 12\}$, are as follows:

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
B_0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
B'_0	x_0	x_{12}	x_{11}	x_{10}	x_9	x_8	x_7	x_6	x_5	x_4	x_3	x_2	x_1

   Here, an entry b_{ij} in a block orbit matrix B that corresponds to the G–orbit of blocks $\{B_0, B_1, \ldots, B_{12}, B'_0, B'_1, \ldots, B'_{12}\}$ of length 26 and to the G–orbit of points $(P_0, P_1, \ldots, P_{12})$ of length 13, is equal to

$$b_{ij} \in \{0, 1\}. \tag{22}$$

   The argumentation is analogous to the case 2.1).

As stated above, point and block orbit lengths for an action of G on $\mathcal{D}$ are 13 and 26. However, there are two types of (point or block) orbits of size 13, since the involutory automorphism σ could act in two different ways, fixing only one or all elements in a ρ–orbit, as given in (19). If the permutation representation of σ in an G–orbit of length 13 is $(0)(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)$, i.e. fixes only one element, then its length will be denoted by 13. Otherwise, if σ fixes all elements in the G-orbit, then its length will be denoted by $\underline{13}$.

Thus, based on the study of possible actions of the automorphism group G on $\mathcal{D}$ given above, here we give an overview of the entries which may appear in a point orbit matrix T. The first row and the first column represent a block and a point G–orbit lengths distribution, respectively:

$$T = \begin{array}{c|c:c:c} & \underline{13}\cdots\underline{13} & 13\cdots 13 & 26\cdots 26 \\ \hline \underline{13} & & & \\ \vdots & 0,1,\ldots 4 & 0 & 0 \\ \underline{13} & & & \\ \hdashline 13 & & & \\ \vdots & 0 & 0,1,2 & 0,2 \\ 13 & & & \\ \hdashline 26 & & & \\ \vdots & 0 & 0,1 & 0,1,\ldots,6 \\ 26 & & & \end{array} \tag{23}$$

By Propositions 4.2 and 5.1, one can conclude that there are 12 possible orbit lengths distributions for an action of $G \cong \mathrm{Frob}_{26}$ on a 2–$(91,6,1)$ design $\mathcal{D}$. These distributions are denoted by F1, F2, …, F12 and shown in Table 2. In the third column we marked if a distribution is good, which means that it could produce at least one point orbit matrix. The distributions F1, F3, F6, …, F12 will not produce orbit matrices, since for a G–orbit of blocks of length $\underline{13}$ there is at most one G–orbit of points of length $\underline{13}$, so the corresponding column sum in block orbit matrices will be less than 6.

σ-fixed points/blocks	G-orbit lengths distributions	Good
$f_T = 1$	F1: $(\omega_1,\dots,\omega_4) = (13,26,26,26)$, $(\Omega_1,\dots,\Omega_{11}) = (\underline{13},13,13,\overbrace{26,\dots,26}^{\times 9})$	No
$f_B = 15$	F2: $(\omega_1,\dots,\omega_4) = (13,26,26,26)$, $(\Omega_1,\dots,\Omega_{18}) = (\overbrace{13,\dots,13}^{\times 15},26,26,26)$	Yes
$f_T = 7$	F3: $(\omega_1,\dots,\omega_7) = (\overbrace{13,\dots,13}^{\times 7})$, $(\Omega_1,\dots,\Omega_{13}) = (\underline{13},13,13,13,13,\overbrace{26,\dots,26}^{\times 8})$	No
$f_B = 17$	F4: $(\omega_1,\dots,\omega_7) = (\overbrace{13,\dots,13}^{\times 7})$, $(\Omega_1,\dots,\Omega_{19}) = (\overbrace{13,\dots,13}^{\times 17},26,26)$	Yes
$f_T = 7$	F5: $(\omega_1,\dots,\omega_7) = (\overbrace{13,\dots,13}^{\times 7})$, $(\Omega_1,\dots,\Omega_{21}) = (\overbrace{13,\dots,13}^{\times 21})$	Yes
$f_B = 21$	F6: $(\omega_1,\dots,\omega_7) = (\overbrace{13,\dots,13}^{\times 7})$, $(\Omega_1,\dots,\Omega_{15}) = (\underline{13},\overbrace{13,\dots,13}^{\times 8},\overbrace{26,\dots,26}^{\times 6})$	No
$f_T = 13$ $f_B = 23$	F7: $(\omega_1,\dots,\omega_4) = (\underline{13},26,26,26)$, $(\Omega_1,\dots,\Omega_{16}) = (\underline{13},\overbrace{13,\dots,13}^{\times 10},\overbrace{26,\dots,26}^{\times 5})$	No
$f_T = 15$ $f_B = 25$	F8: $(\omega_1,\dots,\omega_5) = (\underline{13},13,13,26,26)$, $(\Omega_1,\dots,\Omega_{17}) = (\underline{13},\overbrace{13,\dots,13}^{\times 12},\overbrace{26,\dots,26}^{\times 4})$	No
$f_T = 17$	F9: $(\omega_1,\dots,\omega_6) = (\underline{13},\overbrace{13,\dots,13}^{\times 4},26)$, $(\Omega_1,\dots,\Omega_{12}) = (\underline{13},\underline{13},13,\overbrace{26,\dots,26}^{\times 9})$	No
$f_B = 27$	F10: $(\omega_1,\dots,\omega_6) = (\underline{13},\overbrace{13,\dots,13}^{\times 4},26)$, $(\Omega_1,\dots,\Omega_{18}) = (\underline{13},\overbrace{13,\dots,13}^{\times 14},26,26,26)$	No
$f_T = 19$	F11: $(\omega_1,\dots,\omega_7) = (\underline{13},\overbrace{13,\dots,13}^{\times 6})$, $(\Omega_1,\dots,\Omega_{13}) = (\underline{13},\underline{13},13,13,13,\overbrace{26,\dots,26}^{\times 8})$	No
$f_B = 29$	F12: $(\omega_1,\dots,\omega_7) = (\underline{13},\overbrace{13,\dots,13}^{\times 6})$, $(\Omega_1,\dots,\Omega_{19}) = (\underline{13},\overbrace{13,\dots,13}^{\times 16},26,26)$	No
$f_T = 21$ $f_B = 31$ or 35	A distribution does not exist.	

Table 2: Orbit lengths distributions for the action of $G \cong \mathrm{Frob}_{26}$ on a 2–$(91,6,1)$ design.

6.1 Construction of orbit matrices and designs

In the previous section, all point and block orbit lengths distributions for the action of $G \cong \mathrm{Frob}_{26}$ on a 2–$(91,6,1)$ design are determined. Now, for each good distribution (i.e. the distributions F2, F4 and F5) we construct up to isomorphism all point orbit matrices. Then, from each point orbit matrix obtained, we'll construct the corresponding designs.

F2) For G–orbit lengths distribution $(\omega_1,\dots,\omega_4) = (13,26,26,26)$, $(\Omega_1,\dots,\Omega_{18}) = (13,\dots,13,26,26,26)$, all possible candidates (up to permutations of columns that correspond to the block orbits of the same length) for the rows of a point orbit matrix T that correspond to a point orbit of length 13 and 26 are shown in Table 3. They satisfy the conditions (1), (2), (4) for $s = s'$, and (23).

13	26
1) [2,2,2,2,2,2,1,1,1,1,1,1,0,0,0,0,0,0]	1) [1,1,1,1,1,1,1,1,1,1,1,1,1,0,0,4,1,0]
2) [2,2,2,2,2,2,1,1,1,1,0,0,0,0,0,2,0,0]	2) [1,1,1,1,1,1,1,1,1,1,0,0,0,0,4,2,1]
3) [2,2,2,2,2,2,1,1,0,0,0,0,0,0,0,2,2,0]	
4) [2,2,2,2,2,2,0,0,0,0,0,0,0,0,0,2,2,2]	

Table 3: Candidates for the rows of a point orbit matrix for the distribution F2.

An entry in the first row of T should not be equal to 1, since $k = 6$ and each entry in the submatrix of T denoted below by (*) must be equal to 0 or 1 regarding (23). So, the first three candidates given in the first column of Table 3 can be eliminated. Hence, the first row in T is as given below.

$$T = \begin{array}{c|cccccc:ccccccccc:ccc}
 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 26 & 26 & 26 \\
\hline
13 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\
\hdashline
26 & & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\
26 & & & & (*) & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \\
26 & & & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & &
\end{array}$$

By summing up the elements of columns in (*) the value $6 \cdot 2 = 12$ is obtained, since $k = 6$. On the other hand, summing up the elements of all possible rows in (*) we get that the total sum of 12 can be obtained only for the first candidate in the second column (corresponding to point orbits of length 26) in Table 3. Therefore, one can notice that there is only one point orbit matrix for the distribution F2, up to isomorphisms, i.e. permutations of rows and columns corresponding to G–orbits of the same length (see Proposition 3.4), as presented in (24).

$$T = \begin{array}{c|cccccccccccccccccc}
 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 26 & 26 & 26 \\
\hline
13 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\
26 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 1 & 0 \\
26 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 4 \\
26 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 4 & 1
\end{array} \qquad (24)$$

Let us consider the refinements of T for the action of the automorphism group $\langle \rho \rangle \trianglelefteq G$ of order 13. Up to permutations of the columns, there are exactly 4 candidates for rows in point orbit matrices for $\langle \rho \rangle$ satisfying the conditions (1), (2) and (4) for $s = s'$, as listed here:

[4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0], [3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0], [3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0] and [2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0].

Let us take a look on the first three rows of the matrix T in (24). The rows could be refined for the action of the automorphism ρ, as shown here:

$$\begin{array}{c|ccccccccccccccc:cc:cc:cc}
 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 \\
\hline
13 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hdashline
13 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 & 0 & 1 & 0 & 0 & 0 \\
13 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 4 & 0 & 1 & 0 & 0 \\
\hdashline
13 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 4 & 0 \\
13 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 4
\end{array} \qquad (25)$$

The second and the forth row in (25) do not satisfy the condition (4) for the action of the automorphism ρ on a 2–$(91,6,1)$ design. More generally, there will always be a pair of rows in a refined matrix for the action of ρ, obtained from T, which does not satisfy the equation (4), for $s \neq s'$. Therefore, we can conclude that there are no 2–$(91,6,1)$ designs admitting the action of the group $G \cong \mathrm{Frob}_{26}$ with the orbit lengths distribution F2.

F4) Now, let us take a look at the orbit lengths distribution $(\omega_1, \ldots, \omega_7) = (13, \ldots, 13)$, $(\Omega_1, \ldots, \Omega_{19}) = (13, \ldots, 13, 26, 26)$ for the action of the automorphism group G on the set of points and blocks of a 2–$(91,6,1)$ design, respectively. The candidates (up to permutations of columns corresponding to the block orbits of the same length) for rows in a point orbit matrix $T = [t_{ij}]$ are given in Table 4, and they satisfy the conditions (1), (2), (4) for $s = s'$ and (23).

13
1) [2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0]
2) [2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 2, 0]
3) [2, 2, 2, 2, 2, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2]

Table 4: Candidates for the rows of a point orbit matrix for the distribution F4.

It is easy to see that only the first and the third candidate in Table 4 are compatible, i.e. they satisfy the condition (3) and (4) for $s \neq s'$.
The next step in the construction of designs is to obtain all pairwise nonisomorphic point orbit matrices for the action of G on $\mathcal{D}$ with the distribution F4. During the construction, we were eliminating partial point orbit matrices having two rows with two 1s in the same columns, since in that case the action of the involutory automorphism σ in the orbits of length 13 given in (20) implies there are two blocks from G–orbits of length 13 that intersect in at least two points. For isomorph rejection we were using the permutations of rows and columns of (partial) point orbit matrices that satisfy the conditions from Proposition 3.4. The construction gave us exactly 6 pairwise nonisomorphic point orbit matrices that satisfy the conditions $(1)-(4)$, that are given in Table 5.

$$T_1 = \begin{bmatrix} 2&2&2&2&2&2&1&1&1&1&1&1&0&0&0&0&0&0&0 \\ 2&2&0&0&0&0&2&2&1&0&0&0&2&2&1&0&0&2&2 \\ 2&2&0&0&0&0&0&0&0&2&2&1&0&0&1&2&2&2&2 \\ 0&0&2&2&0&0&2&0&0&1&0&2&2&0&1&2&0&2&2 \\ 0&0&2&2&0&0&0&1&2&0&2&0&0&2&1&0&2&2&2 \\ 0&0&0&0&2&2&1&0&2&2&0&0&2&0&1&0&2&2&2 \\ 0&0&0&0&2&2&0&2&0&0&1&2&0&2&1&2&0&2&2 \end{bmatrix} \quad T_2 = \begin{bmatrix} 2&2&2&2&2&2&1&1&1&1&1&1&0&0&0&0&0&0&0 \\ 2&2&0&0&0&0&2&2&1&0&0&0&2&2&1&0&0&2&2 \\ 2&2&0&0&0&0&0&0&0&2&2&1&0&0&1&2&2&2&2 \\ 0&0&2&2&0&0&2&0&0&1&0&2&2&0&1&2&0&2&2 \\ 0&0&2&0&2&0&1&0&2&2&0&0&0&2&1&0&2&2&2 \\ 0&0&0&2&0&2&0&2&0&0&1&2&0&2&1&0&2&2&2 \\ 0&0&0&0&2&2&0&1&2&0&2&0&2&0&1&2&0&2&2 \end{bmatrix} \quad T_3 = \begin{bmatrix} 2&2&2&2&2&2&1&1&1&1&1&1&0&0&0&0&0&0&0 \\ 2&2&0&0&0&0&2&2&1&0&0&0&2&2&1&0&0&2&2 \\ 2&2&0&0&0&0&0&0&0&2&2&1&0&0&1&2&2&2&2 \\ 0&0&2&2&0&0&2&0&0&1&0&2&2&0&1&2&0&2&2 \\ 0&0&2&0&2&0&1&0&2&2&0&0&0&2&1&0&2&2&2 \\ 0&0&0&2&0&2&0&1&2&0&2&0&2&0&1&0&2&2&2 \\ 0&0&0&0&2&2&0&2&0&0&1&2&0&2&1&2&0&2&2 \end{bmatrix}$$

$$T_4 = \begin{bmatrix} 2&2&2&2&2&2&1&1&1&1&1&1&0&0&0&0&0&0&0 \\ 2&2&0&0&0&0&2&2&1&0&0&0&2&2&1&0&0&2&2 \\ 2&2&0&0&0&0&0&0&0&2&2&1&0&0&1&2&2&2&2 \\ 0&0&2&2&0&0&2&0&0&1&0&2&2&0&1&2&0&2&2 \\ 0&0&2&0&2&0&0&2&0&0&1&2&0&2&1&0&2&2&2 \\ 0&0&0&2&0&2&0&1&2&0&2&0&2&0&1&0&2&2&2 \\ 0&0&0&0&2&2&1&0&2&2&0&0&0&2&1&2&0&2&2 \end{bmatrix} \quad T_5 = \begin{bmatrix} 2&2&2&2&2&2&1&1&1&1&1&1&0&0&0&0&0&0&0 \\ 2&2&0&0&0&0&2&2&1&0&0&0&2&2&1&0&0&2&2 \\ 2&0&2&0&0&0&2&0&0&2&1&0&0&0&1&2&2&2&2 \\ 0&2&0&2&0&0&0&0&0&1&2&2&2&0&1&2&0&2&2 \\ 0&0&2&0&2&0&0&1&2&0&0&2&2&0&1&0&2&2&2 \\ 0&0&0&2&0&2&1&0&2&0&2&0&0&2&1&0&2&2&2 \\ 0&0&0&0&2&2&0&2&0&2&0&1&0&2&1&2&0&2&2 \end{bmatrix} \quad T_6 = \begin{bmatrix} 2&2&2&2&2&2&1&1&1&1&1&1&0&0&0&0&0&0&0 \\ 2&2&0&0&0&0&2&2&1&0&0&0&2&2&1&0&0&2&2 \\ 2&0&2&0&0&0&1&0&2&2&0&0&0&0&1&2&2&2&2 \\ 0&2&0&2&0&0&0&0&0&2&2&1&2&0&1&2&0&2&2 \\ 0&0&2&0&2&0&0&2&0&0&1&2&2&0&1&0&2&2&2 \\ 0&0&0&2&0&2&2&0&0&1&0&2&0&2&1&0&2&2&2 \\ 0&0&0&0&2&2&0&1&2&0&2&0&0&2&1&2&0&2&2 \end{bmatrix}$$

Table 5: All pairwise nonisomorphic point orbit matrices for the action of G on a 2–$(91,6,1)$ design with the orbit lengths distribution F4.

The next step is to construct designs from the obtained orbit matrices $T_1,\dots,T_6$, so called indexing of orbit matrices. Cleary, if $t_{ij}=1$ or $t_{ij}=0$, for $i\in\{1,\dots,7\}$ and $j\in\{1,\dots,17\}$, then t_{ij} are indexed in a unique way, due to (20). However, there are exactly 6 possibilities for indexing when $t_{ij}=2$, for $i\in\{1,\dots,7\}$, $j\in\{1,\dots,17\}$, (see (20)), which are listed here (only the first rows since the others can be obtained by cyclic permutations):

$$\begin{gathered} [0,1,0,0,0,0,0,0,0,0,0,0,1],[0,0,1,0,0,0,0,0,0,0,0,1,0],[0,0,0,1,0,0,0,0,0,0,1,0,0], \\ [0,0,0,0,1,0,0,0,0,1,0,0,0],[0,0,0,0,0,1,0,0,1,0,0,0,0],[0,0,0,0,0,0,1,1,0,0,0,0,0]. \end{gathered} \tag{26}$$

There are six entries equal to 2 in the first 17 columns of each row in $T_1,\dots,T_6$, and each of them must be indexed in a different way, since $\lambda=1$. Further, there are 13 possibilities to index an entry $t_{ij}=2$ for $i\in\{1,\dots,7\}$ and $j\in\{18,19\}$, i.e. an element in matrices $T_1,\dots T_6$ that corresponds to a G–orbit of points of length 13 and to a G–orbit of blocks of length 26, with respect to the action described through the case 3 in Section 6. Namely, a G–orbit of blocks of length 26 decomposes into two ρ–orbits of length 13, for the action of the automorphism of order 13, so $t_{ij}=2$ refines to $[1,1]$ for the action of ρ, which can further be indexed in $13\cdot 1$ ways. For the isomorph rejection during the indexing, we have used the elements of the normalizer of the group G in the group $S(\mathcal{P})\times S(B)$, denoted in [6] by α_l, for $2\le l\le 12$. So, the entry $t_{11}=2$ is indexed in this unique way: $[0,1,0,0,0,0,0,0,0,0,0,0,1]$.
The indexing of the point orbit matrices $T_1,\dots,T_6$ was carried out quickly. The result of the indexing of the orbit matrices $T_1,\dots,T_6$, conducted using the computer program that we developed for that purpose, is that none of the matrices produce a 2–$(91,6,1)$ design.

F5) If a 2–$(91,6,1)$ design $\mathcal{D}$ has an automorphism group $G\cong \mathrm{Frob}_{26}$ acting on its points and blocks with the orbit lengths distribution $(\omega_1,\dots,\omega_7)=(13,\dots,13)$, $(\Omega_1,\dots,\Omega_{21})=(13,\dots,13)$, respectively, then there is only one candidate (up to permutations of its columns) for a row of a point orbit matrix $T=[t_{ij}]$:

$$[2,2,2,2,2,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0], \tag{27}$$

by the conditions (1), (23) and the equations $\sum_{j=1}^{21} t_{ij} = 18$, $\sum_{j=1}^{21} t_{ij}^2 = 13+17 = 30$.

As in the case of the distribution F4, we can eliminate point orbit matrices with two rows having two 1s on the same position. Besides that, the QS-property from Proposition 3.7 can be applied in the construction of (partial) point orbit matrices. So, for $j = j'$, the following equations hold $\sum_{i=1}^{7} t_{ij}^2 = 6+0+(13-1-2u)\cdot 1 = 18-2u$, for $u = 0, \ldots, 6$, and $j = 1, \ldots, 21$. Hence,

$$\sum_{i=1}^{7} t_{ij} t_{ij'} \in \begin{cases} \{0, 1, \ldots, 13\}, & \text{for } j \neq j' \\ \{6, 8, 10, 12, 14, 16, 18\}, & \text{for } j = j'. \end{cases} \tag{28}$$

The candidates for columns of a point orbit matrix (up to the permuting the rows) are:

$$\begin{matrix} 2 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

and it is easy to check that the columns satisfy the property (28). Therefore in this case the QS-property from Proposition 3.7 was not included into our computations.

There are $\binom{21}{6} \cdot \binom{15}{6} = 271591320$ candidates for each row of a point orbit matrix. Hence, without the elimination of pairwise isomorphic (partial) point orbit matrices, it wouldn't be possible to carry on with the construction. For the isomorph rejection during the construction of (partial) point orbit matrices we used all permutations of their rows and columns (i.e. the permutations $(\alpha, \beta) \in S_7 \times S_{21}$), since all of them satisfy the conditions from Proposition 3.4.
By using the above mentioned eliminations, we constructed 3865485 pairwise nonisomorphic point orbit matrices that satisfy the conditions (1)$-$(4). The number of pairwise nonisomorphic partial point orbit matrices per row grows rapidly. The first row produces only 1 partial orbit matrix, then the second row gives 9, the third 827, the forth 101704, the fifth row gives around 6.5 million and the sixth row even more of pairwise nonisomorphic partial point orbit matrices.

The next step in the construction of designs is indexing, i.e. the construction of designs from the obtained point orbit matrices. If an entry of a point orbit matrix is $t_{ij} = 0$ or $t_{ij} = 1$, clearly it should be indexed in a unique way, due to (20), but there are 6 possibilities for indexing an entry $t_{ij} = 2$, as listed in (26).
There are six entries equal to 2 in each row of point orbit matrices, and each of them must be indexed in a different way, since $\lambda = 1$. Likewise, if there are two entries equal to 2 in a column of a point orbit matrix, then it is easy to see that they must be indexed differently as well. Therefore, there are 6! candidates for each row of a point orbit matrix, except for the first row for which there $1\cdot 5! = 120$ possibilites, since the entry $t_{11} = 2$ can be indexed in the unique way: $[0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]$, due to the isomorph rejection with the elements of the normalizer of G in $S(\mathcal{P}) \times S(\mathcal{B})$. The indexing of the point orbit matrices constructed was carried out very quickly.

The sole point orbit matrix that yielded the desired designs is as follows:

$$OM = \begin{bmatrix} 2&2&2&2&2&2&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0 \\ 2&1&1&0&0&0&2&2&1&0&0&0&2&2&2&1&1&1&0&0&0 \\ 1&2&0&1&0&0&0&0&0&2&2&1&2&1&0&2&1&0&2&1&0 \\ 1&0&0&0&2&1&2&0&0&1&0&2&0&0&1&1&2&2&2&0&1 \\ 0&1&0&0&1&2&0&1&2&0&2&0&0&1&2&0&2&0&1&2&1 \\ 0&0&2&1&1&0&0&2&0&0&1&2&1&0&1&2&0&1&0&2&2 \\ 0&0&1&2&0&1&1&0&2&2&0&0&1&2&0&0&0&2&1&1&2 \end{bmatrix}.$$

In the end, four 2–$(91, 6, 1)$ designs are constructed from the point orbit matrix OM, but only two of them are pairwise nonisomorphic. By using GAP, we established that the designs are isomorphic to the McCalla design $\mathcal{D}_2$ and the Colbourn and Colbourn design $\mathcal{D}_3$.

Finally, we summarise the above results in the following theorem.

Theorem 6.1. *Let $\mathcal{D}$ be a Steiner 2–design $S(2, 6, 91)$ having a non-abelian automorphism group G of order 26. Then, a subgroup of G of order 13 acts without fixed points and fixed blocks, an involution fixes 7 points and 21 blocks, and $\mathcal{D}$ is isomorphic either to the McCalla design $\mathcal{D}_2$ or to the Colbourn and Colbourn design $\mathcal{D}_3$.*

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Steinerovi 2–dizajni $S(2,6,91)$ s nekomutativnom grupom automorfizama reda 26

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Sažetak

Do sada su poznata samo četiri Steinerova 2–dizajna s parametrima $S(2,6,91)$. To su Millsov dizajn, McCalla dizajn i dva dizajna koja su konstruirali C. J. Colbourn i M. J. Colbourn. Svaki od njih imaju cikličku grupu automorfizama reda 91. Z. Janko i V. D. Tonchev dokazali su da je svaki Steinerov 2–dizajn $S(2,6,91)$ s grupom automorfizama reda većeg od 91, koja djeluje tranzitivno na skupu točaka, izomorfan s jednim od četiri poznata dizajna. Još uvijek je neodgovoreno pitanje postoji li Steinerov 2–dizajn $S(2,6,91)$ s punom grupom automorfizama reda manjeg od 91. U ovom smo radu pokazali da je svaki Steinerov 2–design $S(2,6,91)$ s nekomutativnom grupom automorfizama reda 26 (odnosno Frobeniusovom grupom Frob_{26}) izomorfan s jednim od dva poznata dizajna, s McCalla dizajnom koji ima punu grupu automorfizama izomorfnu s $C_{91} : C_{12}$ ili s Colbourn i Colbourn dizajnom koji ima punu grupu automorfizama izomorfnu s $C_{91} : C_4$.

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