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Elliptic curves associated to a spacelike curve in the Lorentz plane

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ELLIPTIC CURVES ASSOCIATED TO A SPACELIKE CURVE IN THE LORENTZ PLANE

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Abstract. We associate an elliptic curve to each point of a natural parametrization of a spacelike naturally parametrized curve C in the Lorentz plane. The main tool of our study is the curvature of C and the equilateral hyperbola appears as a remarkable example since its elliptic curve is a CM one. Other two elliptic curves are obtained with this approach, the second one corresponding to the adjoint differential equation of the first, which in turn, is associated to a Lorentzian version of the logarithmic spiral.

1. Introduction

The natural appearance of elliptic curves in various mainstream domains of Mathematics is excellently illustrated in the paper [7]. Due to this fact, it is desirable to have many methods to obtain or to derive elliptic curves from other (geometrical) objects.

We propose here such a way to associate an elliptic curve $\Gamma(t)$ to any point of a given spacelike curve C in the Lorentz plane $\mathbb{L}^2$, a space which naturally appears in the general theory of four-dimensional Minkowski space M^4 when only one space dimension is important. A main hypothesis is that C is naturally parameterized by t while the main tool in this approach is an ordinary differential equation (of third order) satisfied by C through its curvature k_L of constant sign. In fact, we concentrate our study in finding remarkable examples of pairs $(C, \Gamma(t))$ and, if possible, of universal (i.e. not depending on t) elliptic curves Γ. So, we obtain three such elliptic curves, the first two corresponding to the unit equilateral hyperbola respectively the

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Lorentzian counterpart of the logarithmic spiral. The last elliptic curve is derived by using the adjoint differential operator of the second one.

We point out that some difficult computations are performed using WolframAlpha, especially to prove that the general elliptic curve $\Gamma(t)$ can not be expressed in a normal form $y^2 = x(x-1)(x-\lambda(t))$ with $\lambda(t) \notin \{0,1\}$. Also, we remark that the first elliptic curve, namely that corresponding to the unit equilateral hyperbola, is a CM one; we recall that CM means *complex multiplication.*

2. Elliptic curves associated to points of a given spacelike plane curve

The setting of this paper is provided by the Lorentz (or Minkowski) plane $\mathbb{L}^2 := (\mathbb{R}^2, \langle \cdot, \cdot \rangle_L)$:

$$\begin{cases} \langle u, v\rangle_L = -u^1v^1 + u^2v^2, \quad u = (u^1, u^2) \in \mathbb{R}^2, \quad v = (v^1, v^2) \in \mathbb{R}^2, \\ 0 \leq \|u\|_L^2 = |\langle u, u\rangle_L|. \end{cases} \tag{2.1}$$

Fix an open interval $I \subseteq \mathbb{R}$ and consider $C \subset \mathbb{R}^2$ a smooth spacelike parametrized curve of equation:

$$\begin{cases} C : r(t) = (x(t), y(t)) = x(t)\bar{i} + y(t)\bar{j}, \quad \bar{i} = (1,0), \bar{j} = (0,1), \\ \langle r'(t), r'(t)\rangle_L > 0, \quad t \in I. \end{cases} \tag{2.2}$$

The appropriate algebraic structure of the Lorentz plane is the two-dimensional paracomplex algebra $(\mathbb{R}^2, j)$, $j^2 = 1$, [3]. So, the Frenet apparatus (T, N, k_L) of the curve C is provided by:

$$\begin{cases} T(t) = \frac{r'(t)}{\|r'(t)\|_L}, N(t) = j \cdot T(t) = \frac{1}{\|r'(t)\|_L}(y'(t), x'(t)), \\ \langle T(t), T(t)\rangle_L = 1 = -\langle N(t), N(t)\rangle_L \\ k_L(t) = \frac{1}{\|r'(t)\|_L}\langle T'(t), N(t)\rangle_L = \frac{1}{\|r'(t)\|_L^3}\langle r''(t), jr'(t)\rangle_L \\ k_L(t) = \frac{1}{\|r'(t)\|_L^3}[x'(t)y''(t) - y'(t)x''(t)]. \end{cases} \tag{2.3}$$

Hence T is an unit spacelike vector field along C while N is an unit timelike vector field along C. The curvature function can be expressed using a 2×2 determinant:

$$k_L(t) = \frac{1}{\|r'(t)\|_L^3} \det \begin{pmatrix} x'(t) & y'(t) \\ x''(t) & y''(t) \end{pmatrix} \tag{2.4}$$

and the difference to the Euclidean curvature k_E consists in the ratio in front of this determinant; in the Euclidean case is the Euclidean norm $\|r'(t)\|_E^{-3} > 0$. The Lorentz rotated curve $jC : r_j(t) := j \cdot r(t) = (y(t), x(t))$ is a timelike curve with $< r_j'(t), r_j'(t) >_L = - < r'(t), r'(t) >_L$ and the same curvature k_L. Note that the timelike curves represent physically the motion of particles with nonzero mass.

Our main assumptions from now are:
H1) k_L has a constant sign; from $k_L \neq 0$ it results that C is a not a line,

H2) r is a natural parametrization of C, i.e. $\|r'(t)\|_L = +1$, and then a similar result to the fundamental theorem of Euclidean plane curves gives the expression of the derivative:

$$\begin{cases} r'(t) = (\sinh(-K_L(t)), \cosh(-K_L(t))) = (-\sinh(K_L(t)), \cosh(K_L(t))), \\ K_L(t) := \int_{t_0}^{t} k_L(s)ds, \quad fixed\ t_0 \in I, \quad k_L(t) = k_E(t) \cdot \|r'(t)\|_E^3. \end{cases} \tag{2.5}$$

The last relation (2.5) says that the curvature k_L and k_E have the same signature. The starting point of this study is (see also [4]):

Proposition 2.1 *Under the hypothesis H1-H2 the components functions x, y of r satisfy the third-order linear ordinary differential equation*:

$$\mathcal{E}^3 : U''' - \frac{k_L'}{k_L}U'' - k_L^2 U' = 0. \tag{2.6}$$

Proof The two derivatives in (2.5) give:

$$\begin{cases} r''(t) = (-k_L(t)\cosh(K_L(t)), k_L(t)\sinh(K_L(t))), \\ r'''(t) = \left(-k_L'\cosh(K_L(t)) - k_L^2\sinh(K_L(t)), k_L'\sinh(K_L(t)) + k_L^2\cosh(K_L(t))\right), \end{cases} \tag{2.7}$$

and these relations yields the conclusion. Note that the curve r'' is timelike and:

$$\langle r'''(t), r'''(t)\rangle_L = k_L^4(t) - (k_L')^2(t) > -(k_L')^2(t). \tag{2.8}$$

□

The idea of our work, one inspired by (2.6), is to associate to a given point $r(t) \in C$ the cubic curve:

$$\Gamma(t) : y^2 = x^3 - \frac{k_L'(t)}{k_L(t)}x^2 - k_L^2(t)x = x\left(x^2 - \frac{k_L'(t)}{k_L(t)}x - k_L^2(t)\right). \tag{2.9}$$

The Euclidean version of this project is considered in [5]. The method to associate an object to every point of a given manifold M is the starting point of the theory of *bundles over* M. Recall, that a general cubic curve $\Gamma : y^2 = x^3 + ax^2 + bx + c$ is an *elliptic curve* if the polynomial function of the right-hand-side is nonsingular i.e. it has *distinct* roots. For our $\Gamma(t)$ since $k_L^2 > 0$ the root $x_1 = 0$ can not be a multiple root. Hence is necessary to discuss the sign of the discriminant:

$$\tilde{\Delta}(t) = \left(\frac{k_L'(t)}{k_L(t)}\right)^2 + 4k_L^2(t) \tag{2.10}$$

and again since $k_L^2(t) > 0$ for all t we have a bundle of groups over our curve C.

This discriminant $\tilde{\Delta}$ can be expressed in terms of K_L as:

$$\tilde{\Delta} = \left(\frac{K_L''}{K_L'}\right)^2 + 4(K_L')^2. \tag{2.11}$$

Recall also that for a function $F \in C^3(I, \mathbb{R})$ its *Schwarzian derivative* is:

$$S_F := \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 \tag{2.12}$$

and hence, the discriminant is:

$$\tilde{\Delta} = \frac{2}{3}\left(\frac{K_L'''}{K_L'} - S_{K_L}\right) + 4(K_L')^2. \tag{2.13}$$

We finish this section with the remark that if t_0 is a vertex of C i.e. $k_L'(t_0) = 0$ then the elliptic curve is:

$$\Gamma(t_0) : y^2 = x(x - k_L(t_0))(x + k_L(t_0)). \tag{2.14}$$

3. The geometry of the elliptic curve $\Gamma(t)$

We start this section through an example; we point out that other very interesting examples are considered in [1] and [2] while the case of a shortening flow is studied in [9]:

Example 3.1 The corresponding of the circles $\mathcal{C}(O, R > 0)$ of Euclidean plane geometry is provided by the equilateral hyperbola (all the points are vertices):

$$\begin{cases} H_e(R) : x^2 - y^2 = R^2, \quad r_e^{\pm}(t) = R\left(\pm\cosh\frac{t}{R}, \sinh\frac{t}{R}\right), \quad t \in \mathbb{R} \\ k_L^{\pm} = constant = \mp\frac{1}{R}, \quad \tilde{\Delta} = \frac{4}{R^2} > 0. \end{cases} \tag{3.1}$$

We note that $H_e(R)$ is called *pseudo-circle* in [8, p. 110] and is denoted $H^1(-R)$. Recall that the infinitesimal generator of the Lorentz rotations in $\mathbb{R}^2_1$ is the linear vector field:

$$\xi_L(u) := u^2\frac{\partial}{\partial u^1} + u^1\frac{\partial}{\partial u^2}, \quad \xi_L(u) = j \cdot u = j \cdot (u^1 + iu^2) \tag{3.2}$$

and hence the equilateral hyperbolas are exactly the integral curves of ξ_L; also $(r_e^{\pm})''$ is a parametrization of the equilateral hyperbola $H\left(\frac{1}{R}\right)$. The Euclidean curvature for the plus (+) parametrization of $H_e(R)$ is still negative but non-constant:

$$k_E(t) = -\frac{1}{R(\cosh(2t))^{\frac{3}{2}}} < 0. \tag{3.3}$$

The elliptic curve Γ does not depend on t (so, it is *universal for* $H_e(R)$) but on R:

$$\Gamma_R : y^2 = x\left(x^2 - \frac{1}{R^2}\right). \tag{3.4}$$

Hence, the unit hyperbola $H_e(1)$ with $k_L^{\pm} = \mp 1$ has the associated elliptic curve:

$$\Gamma_1 : y^2 = x^3 - x \tag{3.5}$$

which is a remarkable one; for example, is a CM-elliptic curve, here CM means *complex multiplication*. Several of its properties and invariants are available

at:
https://www.lmfdb.org/EllipticCurve/Q/32/a/3. Also:

$$\int_1^{\infty}\frac{dx}{y}=\int_1^{\infty}\frac{dx}{\sqrt{x^3-x}}=\frac{2\sqrt{\pi}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\simeq 2.62206. \tag{3.6}$$

□

A main result of this section is as follows:

Proposition 3.2 *The unique spacelike curve for which the elliptic curve $\Gamma(t)$ is expressed in the normal form* ([6]):

$$\Gamma(t): y^2=x(x-1)(x-\lambda(t)),\quad \lambda(t)\notin\{0,1\} \tag{3.7}$$

is the unit equilateral hyperbola $H_e(1)$ with the parametrization r^+_e.

Proof From (2.9) with $x=1$ we have the differential equation:

$$1-\frac{k'_L}{k_L}-k^2_L=0 \tag{3.8}$$

which is a Bernoulli one, having the general solution depending of a real constant $\mathcal{C}$:

$$k^{\mathcal{C}}_L(t)=\frac{e^t}{\sqrt{\mathcal{C}+e^{2t}}},\quad K^{\mathcal{C}}_L(t)=\ln(e^t+\sqrt{\mathcal{C}+e^{2t}}) \tag{3.9}$$

and which is defined only for $t>\frac{\ln(-C)}{2}$ if $C<0$. The degenerate case $\mathcal{C}=0$ yields exactly ($k_L\in\{-1,+1\},\lambda\in\{+1,-1\}$) corresponding to the unit equilateral hyperbola; due to $\lambda\neq 1$ we restrict to the parametrization r^+_e. This parametrization has the total curvature $K_L(t)=-t$ which is not covered by the second part of the formula (3.9).

For the nondegenerate case $\mathcal{C}\neq 0$ it follows that:

$$\lambda(t)=-(k^{\mathcal{C}}_L(t))^2=-\frac{e^{2t}}{\mathcal{C}+e^{2t}}<0 \tag{3.10}$$

while the first relation (2.5) provide the expression of the derivative for the given spacelike curve $r^{\mathcal{C}}$:

$$(r^{\mathcal{C}})'(t)=\left(\frac{1-(e^t+\sqrt{\mathcal{C}+e^{2t}})^2}{2(e^t+\sqrt{\mathcal{C}+e^{2t}})},\frac{1+(e^t+\sqrt{\mathcal{C}+e^{2t}})^2}{2(e^t+\sqrt{\mathcal{C}+e^{2t}})}\right) \tag{3.11}$$

or equivalently:

$$(r^{\mathcal{C}})'(t)=\left(\frac{1-\mathcal{C}}{2\mathcal{C}}(\sqrt{\mathcal{C}+e^{2t}}-e^t)-e^t,\frac{1+\mathcal{C}}{2\mathcal{C}}(\sqrt{\mathcal{C}+e^{2t}}-e^t)+e^t\right). \tag{3.12}$$

The final expression of the 1-parameter curve $r^{\mathcal{C}}$ is:

$$(r^{\mathcal{C}})(t)=(\frac{1-\mathcal{C}}{2\mathcal{C}}[\sqrt{\mathcal{C}+e^{2t}}+\frac{\sqrt{\mathcal{C}}}{2}(\ln\left(1-\sqrt{\frac{\mathcal{C}+e^{2t}}{\mathcal{C}}}\right)-\ln\left(1+\sqrt{\frac{\mathcal{C}+e^{2t}}{\mathcal{C}}}\right))-e^t]-e^t,$$

$$\frac{1+\mathcal{C}}{2\mathcal{C}}[\sqrt{\mathcal{C}+e^{2t}}+\frac{\sqrt{\mathcal{C}}}{2}(\ln\left(1-\sqrt{\frac{\mathcal{C}+e^{2t}}{\mathcal{C}}}\right)-\ln\left(1+\sqrt{\frac{\mathcal{C}+e^{2t}}{\mathcal{C}}}\right))-e^t]+e^t). \tag{3.13}$$

but the simultaneous terms $\sqrt{\mathcal{C}}$ and $\ln\left(1-\sqrt{\frac{\mathcal{C}+e^{2t}}{\mathcal{C}}}\right)$ imply a complex logarithm. So, we have the claimed conclusion. □

Remark 3.3 An inequality can be derived for the function $K_L^{\mathcal{C}}$. Since for two real numbers $a, b > 0$ we have $a+b \leq \sqrt{2(a^2+b^2)}$ it results:

$$e^t+\sqrt{\mathcal{C}+e^{2t}} \leq \sqrt{2(\mathcal{C}+2e^{2t})} \tag{3.14}$$

giving that:

$$K_L^{\mathcal{C}}(t) \leq \frac{1}{2}\left(\ln(\mathcal{C}+2e^{2t})+\ln 2\right). \tag{3.15}$$

□

Example 3.4 The spacelike analogue of the (Euclidean) logarithmic spiral is the curve (without vertices):

$$hS : r(t) = \frac{1}{2}\left(\ln t - \frac{t^2}{2}, \ln t + \frac{t^2}{2}\right), \quad t \in (0, +\infty) \tag{3.16}$$

since the radius of curvature (i.e. $\frac{1}{k}$) is a linear function of the natural parameter t:

$$k_L(t) = \frac{1}{t} > 0, \quad K_L(t) = \ln t, \quad k_E(t) = \frac{2t^5}{(t^4+1)^3} > 0. \tag{3.17}$$

We denote it hS as being *the hyperbolic spiral*; its implicit equation is:

$$hS : 2(y-x) = e^{2(y+x)} \tag{3.18}$$

and we recall that $u := y+x$, $v := y-x$ are *the null coordinates* in $\mathbb{L}^2$ since in terms of semi-Riemannian metrics we have:

$$g_{\mathbb{L}^2} := dy^2 - dx^2 = du \cdot dv. \tag{3.19}$$

It follows the time-dependent elliptic curve:

$$\Gamma(t) : y^2 = x\left(x^2+\frac{x}{t}-\frac{1}{t^2}\right), \quad \tilde{\Delta}(t) = \frac{5}{t^2} > 0. \tag{3.20}$$

The usual transformation $x := X - \frac{1}{3t}$ yields its *Weierstrass form*:

$$\Gamma(t) : y^2 = X^3 - \frac{4}{3t^2}X + \frac{11}{27t^3} = X^3 + pX + q \tag{3.21}$$

which gives its *discriminant* as elliptic curve:

$$\Delta(\Gamma(t)) := 4p^3 + 27q^2 = \frac{11^2-16^2}{27t^6} = -\frac{5}{t^6} < 0. \tag{3.22}$$

Fix $\alpha \in (0, +\infty)$. The *α-homothetical transformation* of $\Gamma(t)$ is the cubic curve:

$$H_\alpha(\Gamma(t)) : y^2 = X^3 + (\alpha^4 p)X + (\alpha^6 q), \quad \Delta(H_\alpha(\Gamma(t))) = \alpha^{12}\Delta(\Gamma(t)). \quad (3.23)$$

For $\alpha = \sqrt{3t}$ we obtain another universal (i.e. not depending on t) elliptic curve:

$$H_\alpha(\Gamma(t)) : y^2 = X^3 - 12X + 11 \quad (3.24)$$

which is https://www.lmfdb.org/EllipticCurve/Q/720/h/3.

Returning to the initial spacelike curve (3.16) we compute its *evolute*:

$$Ev(hS) : r_{Ev}(t) := r(t) + \frac{1}{k_L(t)}N(t) = r(t) + \frac{t}{2}\left(\frac{1}{t} + t, \frac{1}{t} - t\right) =$$

$$= \frac{1}{2}\left(\ln t + \frac{t^2}{2} + 1, \ln t - \frac{t^2}{2} + 1\right), r'_{Ev}(t) = (\cosh K_L(t), -\sinh K_L(t)) \quad (3.25)$$

which means the equality $Ev(hS) = j(hS) + \left(\frac{1}{2}, \frac{1}{2}\right)$. □

Example 3.5 The adjoint equation of $\mathcal{E}^3$ is obtained in [4, p. 3]:

$$\mathcal{E}^3_a : U'''_a + \frac{k'_L}{k_L}U''_a + \left[2\left(\frac{k'_L}{k_L}\right)' - k_L^2\right]U'_a + \left[\left(\frac{k'_L}{k_L}\right)'' - 2k'_L k_L\right]U_a = 0. \quad (3.26)$$

A straightforward computation of it for the previous curve gives:

$$\mathcal{E}^3_a : U'''_a - \frac{1}{t}U''_a + \frac{1}{t^2}U'_a = 0 \quad (3.27)$$

and hence we can define *the adjoint* of the elliptic curve (3.20) as being the elliptic curve:

$$\Gamma^a(t) : y^2 = x\left(x^2 - \frac{x}{t} + \frac{1}{t^2}\right), \quad \tilde{\Delta}^a(t) = -\frac{3}{t^2} < 0 \quad (3.28)$$

and the negativity of $\tilde{\Delta}^a$ means that this elliptic curve does not correspond to a spacelike curve in our approach. With the substitution $x = X + \frac{1}{3t}$ we derive:

$$\Gamma^a(t) : y^2 = X^3 + \frac{2}{3t^2}X + \frac{7}{27t^3}, \quad \Delta(\Gamma^a(t)) = \frac{3}{t^6} > 0. \quad (3.29)$$

The same homothetical transformation as in the previous example yields:

$$H_{\alpha=\sqrt{3t}}(\Gamma^a(t)) : y^2 = X^3 + 6X + 7 \quad (3.30)$$

which is https://www.lmfdb.org/EllipticCurve/Q/144/b/5. Remark that these last two elliptic curves, (3.24) and (3.30), are not CM but have the same torsion: $\mathbb{Z}/2\mathbb{Z}$. □

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References

[1] I. Castro Ildefonso, I. Castro-Infantes and J. Castro-Infantes, *Curves in the Lorentz-Minkowski plane: elasticae, catenaries and grim-reapers*, Open Math. **16** (2018), 747-766. Zbl 1426.53019

[2] I. Castro, I. Castro-Infantes and J. Castro-Infantes, *Curves in the Lorentz-Minkowski plane with curvature depending on their position*, Open Math. **18** (2020), 749-770. Zbl 1483.53013

[3] M. Crasmareanu, *The flow-curvature of spacelike parametrized curves in the Lorentz plane*, Proc. Int. Geom. Cent. **15** (2022), no. 2, 101-109. Zbl 1523.53024

[4] M. Crasmareanu, *Differential equations of spacelike parametrized curves in the Lorentz plane*, Turk. J. Math. Comput. Sci. **16** (2024), no. 1, 1-5.

[5] M. Crasmareanu, *Cubics associated to an Euclidean plane curve via curvature*, submitted.

[6] M. Crasmareanu, C.-L. Pripoae and G.-T. Pripoae, *CR-selfdual cubic curves*, Mathematics **13** (2025), no. 2, paper no. 317.

[7] E. H. Goins, *The ubiquity of elliptic curves*, Notices Am. Math. Soc. **66** (2019), no. 2, 169-174. Zbl 1444.11117

[8] A. Saloom and F. Tari, *Curves in the Minkowski plane and their contact with pseudo-circles*, Geom. Dedicata **159** (2012), 109-124. Zbl 1267.53011

[9] E. Woolgar and R. Xie, *Self-similar curve shortening flow in hyperbolic* 2*-space*, Proc. Am. Math. Soc. **150** (2022), no. 3, 1301-1319. Zbl 1491.53100

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