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A COMBINATORIAL APPROACH TO THE FOURIER EXPANSIONS OF POWERS OF COS AND SIN

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ABSTRACT. We present a combinatorial approach to the computation of the (real) Fourier expansions of $\cos^n(t)$ and $\sin^n(t)$, where $n \geq 1$ is an integer. As an application, we compute the Fourier expansions of $f(t) = \frac{1}{a-\cos t}$ and $g(t) = \frac{1}{a-\sin t}$, where $a \in \mathbb{R}$ with $|a| > 1$.

1. INTRODUCTION

Fourier series plays a very important role, both in pure mathematics and in various branches of applied mathematics and engineering, like, for instance, in the study of periodic signals. For a friendly introduction in the topic of Fourier series and Fourier analysis, we refer the reader to [6] and [1, Chapter 15].

It is well known that if $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable periodic function, with the period $T = 2\pi$, then $f(t)$ has the (real) Fourier expansion:

$$\begin{aligned}
&f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt), \text{ where} \\
&a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\,\mathrm{dt}, \text{ for all } n \geq 0, \text{ and} \\
&b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,\mathrm{dt}, \text{ for all } n \geq 1.
\end{aligned}$$

Moreover, the Fourier series is uniformly convergent; see [6, Corollary 2.2.4].

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Our aim is to provide a combinatorial approach to the computation of the Fourier series of $\cos^n t$ and $\sin^n t$. Our paper is structured as follows:

In Section 2 we prove several combinatorial formulas which will be used later on. For instance, in Lemma 2.1, we prove that

$$\sum_{j\geq 0}\binom{n}{2j}\binom{j}{s} = 2^{n-2s-1}\left(\binom{n-s}{s}+\binom{n-s-1}{s-1}\right),$$
$$\sum_{j\geq 0}\binom{n}{2j+1}\binom{j}{s} = 2^{n-2s-1}\binom{n-s-1}{s},\text{ for all } n\geq 1, s\geq 0.$$

Section 3 is dedicated to our main results. In Proposition 3.1 we proved that for any integer $n \geq 1$ and any $t \in \mathbb{R}$, we have that

$$\cos(nt) = \sum_{s=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^s 2^{n-2s-1}\left(\binom{n-s}{s}+\binom{n-s-1}{s-1}\right)\cdot \cos^{n-2s}(t).$$

In Remark 3.2(2), we note the connection with the Chebyshev polynomials of the first kind. In Proposition 3.4, we prove similar formulas for $\sin(nt)$, where $n \geq 1$.

In Theorem 3.6 we proved that for any integer $k \geq 0$, we have that

$$\cos^{2k}(t) = \frac{1}{2^{2k-1}}\sum_{j=0}^{k-1}\binom{2k}{j}\cos((2k-2j)t) + \frac{1}{2^{2k}}\binom{2k}{k} \text{ and}$$
$$\cos^{2k+1}(t) = \frac{1}{2^{2k}}\sum_{j=0}^{k}\binom{2k+1}{j}\cos((2k+1-2j)t),\text{ for all } t\in\mathbb{R}.$$

See also [2]. Using the fact that $\sin(t) = \cos(t-\frac{\pi}{2})$, we derive similar formulas for $\sin^{2k}(t)$ and $\sin^{2k+1}(t)$; see Corollary 3.8. See also [3].

As an application, in Section 4, we compute the Fourier expansions of $f(t) = \frac{1}{a-\cos t}$ and $g(t) = \frac{1}{a-\sin t}$, where $a \in \mathbb{R}$ with $|a| > 1$; see Theorem 4.2 and Corollary 4.3.

2. Combinatorial lemmata

For any integers $n \geq 0$ and $s \geq 0$, we denote

(2.1)
$$\alpha_{n,s} = \sum_{j\geq 0}\binom{n}{2j}\binom{j}{s},\ \alpha'_{n,s} = \sum_{j\geq 0}\binom{n}{2j-1}\binom{j}{s} \text{ and } \alpha''_{n,s} = \sum_{j\geq 0}\binom{n}{2j+1}\binom{j}{s}.$$

Note that $\alpha_{n,s} = 0$, for $s > \lfloor\frac{n}{2}\rfloor$, $\alpha'_{n,s} = 0$, for $s > \lfloor\frac{n+1}{2}\rfloor$ and $\alpha''_{n,s} = 0$, for $s > \lfloor\frac{n-1}{2}\rfloor$.

Lemma 2.1. *With the above notations, we have:*

(1) $\alpha_{n,s} = 2^{n-2s-1}\left(\binom{n-s}{s} + \binom{n-s-1}{s-1}\right)$,

(2) $\alpha'_{n,s} = 2^{n-2s-1}\left(\binom{n+1-s}{s} + 2\binom{n-s}{s-1} + \binom{n-s-1}{s-2}\right)$,

(3) $\alpha''_{n,s} = 2^{n-2s-1}\binom{n-s-1}{s}$.

for all $n \geq 1, s \geq 0$.

Proof. First, note that from (2.1), we have $\alpha_{1,0} = 1$ and $\alpha_{1,s} = 0$ for $s \geq 1$. Also, we have $\alpha'_{1,0} = \alpha'_{1,1} = 1$ and $\alpha'_{1,s} = 0$ for $s \geq 1$. It follows that the required formulas hold for $n = 1$ and $s \geq 0$. On the other hand, since $\sum_{j=0}^{n}\binom{n}{2j} = \sum_{j=0}^{n}\binom{n}{2j-1} = 2^{n-1}$, it follows that $\alpha_{n,0} = \alpha'_{n,0} = 2^{n-1}$ and, thus, the required formulas hold for $n \geq 1$ and $s = 0$.

Let $n \geq 1$. From (2.1) it follows that

(2.2)
$$\alpha_{n+1,s} = \sum_{j\geq 0}\binom{n+1}{2j}\binom{j}{s} = \sum_{j\geq 0}\binom{n}{2j}\binom{j}{s} + \sum_{j\geq 0}\binom{n}{2j-1}\binom{j}{s} = \alpha_{n,s} + \alpha'_{n,s}.$$

Assume $s \geq 1$. From (2.1) it follows also that

(2.3)
$$\alpha'_{n+1,s} = \sum_{j\geq 0}\binom{n}{2j-1}\binom{j}{s} + \sum_{j\geq 0}\binom{n}{2j-2}\binom{j}{s} = \alpha'_{n,s} + \sum_{j\geq 0}\binom{n}{2j}\binom{j+1}{s}.$$

On the other hand, we have

(2.4)
$$\sum_{j\geq 0}\binom{n}{2j}\binom{j+1}{s} = \sum_{j\geq 0}\binom{n}{2j}\binom{j}{s} + \sum_{j\geq 0}\binom{n}{2j}\binom{j+1}{s-1} = \alpha_{n,s} + \alpha_{n,s-1}.$$

From (2.2), (2.3) and (2.4) it follows that $\alpha_{n,s}$'s and $\alpha'_{n,s}$'s satisfy the recurrence relations:

$$\begin{cases} \alpha_{n+1,s} = \alpha_{n,s} + \alpha'_{n,s}, \text{ for all } n \geq 1, s \geq 0, \\ \alpha'_{n+1,s} = \alpha'_{n,s} + \alpha_{n,s} + \alpha_{n,s-1} \text{ for all } n \geq 1, s \geq 1 \end{cases}. \tag{2.5}$$

Hence, we can apply an inductive argument in order to complete the proof, as the required formulas hold for $n = 1$ and $s \geq 0$, and, also, for $n \geq 1$ and $s = 0$. Assume $n \geq 1$ and $s \geq 1$ and the formulas hold for the pairs (n, s) and $(n, s-1)$. From (2.5) it follows that

$$\begin{aligned} \alpha_{n+1,s} =& 2^{n-2s-1}\left(\binom{n-s}{s} + \binom{n-s-1}{s-1} + \binom{n+1-s}{s} + \right. \\ & \left. +2\binom{n-s}{s-1} + \binom{n-s-1}{s-2}\right). \end{aligned}$$

Since $\binom{n-s}{s}+\binom{n-s}{s-1}=\binom{n+1-s}{s}$ and $\binom{n-s-1}{s-1}+\binom{n-s-1}{s-2}=\binom{n-s}{s-1}$, it follows that

$$\begin{aligned}\alpha_{n+1,s} &= 2^{n-2s-1}\left(2\binom{n+1-s}{s}+2\binom{n-s}{s-1}\right)\\ &= 2^{n+1-2s-1}\left(\binom{n+1-s}{s}+\binom{n-s}{s-1}\right),\end{aligned}$$

as required. Also, from (2.5) it follows that

$$\begin{aligned}\alpha'_{n+1,s} &= \alpha_{n+1,s}+\alpha_{n,s-1}\\ &= 2^{n-2s-1}\left(2\binom{n+1-s}{s}+2\binom{n-s}{s-1}+4\binom{n+1-s}{s-1}+4\binom{n-s}{s-2}\right).\end{aligned}$$

Since $\binom{n+1-s}{s}+\binom{n+1-s}{s-1}=\binom{n+2-s}{s}$ and $\binom{n-s}{s-1}+\binom{n-s}{s-2}=\binom{n+1-s}{s-1}$, we therefore obtain

$$\alpha'_{n+1,s}=2^{n-2s}\left(\binom{n+2-s}{s}+2\binom{n+1-s}{s-1}+\binom{n-s}{s-2}\right),$$

as required. Note that

(2.6)
$$\alpha''_{0,s}=0,\ \text{for all } s\geq 0, \text{and },\alpha''_{n,0}=\sum_{j\geq 1}\binom{n}{2j+1}=2^{n-1},\ \text{for all } n\geq 0.$$

Assume $n,s\geq 1$. Since $\binom{n}{2j+1}=\binom{n-1}{2j}+\binom{n-1}{2j+1}$, from (2.1) it follows that

$$\alpha''_{n,s}=\alpha''_{n-1,s}+\alpha_{n-1,s},\ \text{where } \alpha_{0,s}=0. \tag{2.7}$$

We claim that

$$\alpha''_{n,s}=2^{n-2s-1}\binom{n-s-1}{s}\ \text{for all } n,s\geq 0. \tag{2.8}$$

From (2.6) it follows that (2.8) holds for $n=0$ or $s=0$. Assume $n,s\geq 1$. In order to prove (2.8), using (2.7), it suffices to show that

$$\alpha_{n-1,s}=2^{n-2s-1}\binom{n-s-1}{s}-2^{n-2s-2}\binom{n-s-2}{s},$$

which follows immediately from (1) and the identity $\binom{n-s-1}{s}-\binom{n-s-2}{s}=\binom{n-s-2}{s-1}$. Hence, the proof is complete. ∎

Lemma 2.2. *For all integers $0\leq s\leq k$, we have that:*

$$\sum_{j=1}^{s}\binom{2k}{2j-1}\binom{k-j}{s-j}=2^{2s-1}\binom{k+s-1}{2s-1}.$$

Proof. We denote $u=k-j$. It follows that

$$\sum_{j=1}^{s}\binom{2k}{2j-1}\binom{k-j}{s-j}=\sum_{u\geq 0}\binom{2k}{2u+1}\binom{u}{k-s}.$$

Hence, the conclusion follows from Lemma 2.1(3). □

LEMMA 2.3. *Let $\ell \geq 0$ be an integer. For any integer $t \geq 1$, we have that:*

$$\sum_{s=0}^{t}(-1)^s\left(\binom{2\ell+s}{s}+\binom{2\ell+s-1}{s-1}\right)\binom{2t+2\ell}{t-s}=0.$$

PROOF. Using the identity $(-1)^k\binom{x}{k}=\binom{-x+k-1}{k}$ and the Chu-Vandermonde summation (see [4]), we get:

$$\sum_{s=0}^{t}(-1)^s\binom{2\ell+s}{s}\binom{2t+2\ell}{t-s}=\sum_{s=0}^{t}\binom{-2\ell-1}{s}\binom{2t+2\ell}{t-s}=\binom{2t-1}{t}. \tag{2.9}$$

Similarly, by denoting $u=s-1$, we have:

$$\sum_{s=0}^{t}(-1)^s\binom{2\ell+s-1}{s-1}\binom{2t+2\ell}{t-s}=-\sum_{u=0}^{t-1}\binom{2\ell+u}{u}\binom{2t+2\ell}{t-1-u}=\binom{2t-1}{t-1}. \tag{2.10}$$

Since $t \geq 1$, the conclusion follows from (2.9) and (2.10). □

LEMMA 2.4. *For all integers $n, \ell \geq 0$, we have that:*

$$\sum_{k=0}^{\ell}\left(\binom{n+2k-1}{k}-\binom{n+2k-1}{k-1}\right)\binom{2(\ell-k)}{\ell-k}=\binom{n+2\ell}{\ell}.$$

PROOF. We use induction on $n, \ell \geq 0$.

Assume $n=0$ and $\ell \geq 0$. Since $\binom{-1}{0}=1$, $\binom{-1}{-1}=0$ and $\binom{2k-1}{k}=\binom{2k-1}{k-1}$ for all $k \geq 1$, it follows that

$$\sum_{k=0}^{\ell}\left(\binom{2k-1}{k}-\binom{2k-1}{k-1}\right)\binom{2(\ell-k)}{\ell-k}=\binom{2\ell}{\ell},$$

as required. Assume $n \geq 1$ and $\ell = 0$. Since $\binom{n}{0}=\binom{n-1}{0}=\binom{0}{0}=1$ and $\binom{n-1}{-1}=0$, the assertion is trivial.

Now, assume $n,\ell \geq 1$. Since $\binom{n+2k-1}{k} = \binom{n+2k-2}{k} + \binom{n+2k-1}{k-1}$ and $\binom{n+2k-1}{k-1} = \binom{n+2k-2}{k-1} + \binom{n+2k-1}{k-2}$, it follows that

$$\begin{aligned} S_{n,\ell} := & \sum_{k=0}^{\ell}\left(\binom{n+2k-1}{k} - \binom{n+2k-1}{k-1}\right)\binom{2(\ell-k)}{\ell-k} = \\ = & \sum_{k=0}^{\ell}\left(\binom{n+2k-2}{k} - \binom{n+2k-2}{k-1}\right)\binom{2(\ell-k)}{\ell-k} + \\ + & \sum_{k=0}^{\ell}\left(\binom{n+2k-2}{k-1} - \binom{n+2k-2}{k-2}\right)\binom{2(\ell-k)}{\ell-k} = \\ = & S_{n-1,\ell} + \sum_{k=1}^{\ell}\left(\binom{n+2k-2}{k-1} - \binom{n+2k-2}{k-2}\right)\binom{2(\ell-k)}{\ell-k}. \end{aligned}$$

Denoting $s = k-1$, it follows that

$$\begin{aligned} S_{n,\ell} = & S_{n-1,\ell} + \sum_{s=0}^{\ell-1}\left(\binom{n+2s}{s} - \binom{n+2s}{s-1}\right)\binom{2(\ell-1-s)}{\ell-1-s} \\ = & S_{n-1,\ell} + S_{n+1,\ell-1}. \end{aligned}$$

Using the induction hypothesis on n and ℓ, it follows that

$$S_{n,\ell} = \binom{n-1+2\ell}{\ell} + \binom{n+1+2\ell-2}{\ell-1} = \binom{n+2\ell}{\ell},$$

as required. Hence, the proof is complete. □

3. Main results

Proposition 3.1. *Let $n \geq 1$ be an integer. We have that:*

$$\cos(nt) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^s 2^{n-2s-1}\left(\binom{n-s}{s} + \binom{n-s-1}{s-1}\right)\cdot \cos^{n-2s}(t),$$

for all $t \in \mathbb{R}$.

Proof. We consider the Moivre identity

$$e^{int} = \cos(nt) + i\sin(nt) = (\cos t + i\sin t)^n = (e^{it})^n. \tag{3.11}$$

From (3.11) it follows that

$$\cos(nt) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j} \cos^{n-2j}(t)\sin^{2j}(t). \tag{3.12}$$

Since $\sin^2(t) = 1 - \cos^2(t)$, from (3.12) it follows that

$$\cos(nt) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j} \cos^{n-2j}(t) \sum_{\ell=0}^{j} (-1)^\ell \binom{j}{\ell} \cos^{2\ell}(t). \tag{3.13}$$

Denoting $s = j - \ell$ in (3.12), we obtain

$$\cos(nt) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left((-1)^s \sum_{j=s}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \binom{j}{j-s} \right) \cos^{n-2s}(t). \tag{3.14}$$

On the other hand, we have:

$$\sum_{j=s}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \binom{j}{j-s} = \sum_{j=0}^{n} \binom{n}{2j} \binom{j}{s}. \tag{3.15}$$

Now, the conclusion follows from Lemma 2.1, (3.14) and (3.15). □

Remark 3.2. (1) Note that Proposition 3.1 can be also proved by using the well known recurrence

$$\cos((n+1)t) = 2\cos(t)\cos(nt) - \cos((n-1)t).$$

(2) We recall that the *Chebyshev polynomial of the first kind of order n*, denoted by T_n, is the unique polynomial satisfying $T_n(\cos\theta) = \cos(n\theta)$; see [5, Chapter 2] for further details. From Proposition 3.1, it easily follows that

$$T_n(x) = \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^s 2^{n-2s-1} \left(\binom{n-s}{s} + \binom{n-s-1}{s-1} \right) x^{n-2s}.$$

Example 3.3. (1) From Proposition 3.1, we have $\cos(2t) = 2\cos^2 t - 1$.
(2) From Proposition 3.1, it follows that $\cos(4t) = 8\cos^4(t) - 8\cos^2(t) + 1$.

Proposition 3.4. (1) *Let $k \geq 0$ be an integer. Then:*

$$\sin((2k+1)t) = \sum_{s=0}^{k} (-4)^{k-s} \left(\binom{2k+1-s}{s} + \binom{2k-s}{s-1} \right) \cdot \sin^{2k+1-2s}(t),$$

for all $t \in \mathbb{R}$.

(2) *Let $k \geq 0$ be an integer. Then:*

$$\sin(2kt) = \cos(t) \sum_{s=1}^{k} (-1)^{s-1} \cdot 2^{2s-1} \binom{k+s-1}{2s-1} \cdot \sin^{2s-1}(t), \text{ for all } t \in \mathbb{R}.$$

Proof. (1) Since $\sin(t) = \cos(t - \frac{\pi}{2})$, the result follows from Proposition 3.1 and the identities:

$$\begin{aligned}\cos((2k+1)(t-\frac{\pi}{2})) &= \cos((2k+1)t - k\pi - \frac{\pi}{2}) \\ &= (-1)^k \cos((2k+1)t - \frac{\pi}{2}) = (-1)^k \sin((2k+1)t).\end{aligned}$$

(2) From Moivre's identity (see (3.11)) it follows that

$$\sin(2kt) = \sum_{j=1}^{k} (-1)^{j-1} \binom{2k}{2j-1} \cos^{2k+1-2j}(t) \sin^{2j-1}(t). \tag{3.16}$$

Since $\cos^{2(k-j)}(t) = (1-\sin^2(t))^{k-j}$, from (3.16) we deduce that

$$\sin(2kt) = \cos(t) \sum_{j=1}^{k} (-1)^{j-1} \binom{2k}{2j-1} \sum_{\ell=0}^{k-j} (-1)^{\ell} \binom{k-j}{\ell} \sin^{2j+2\ell-1}(t). \tag{3.17}$$

Denoting $s = j + \ell$, from (3.17) it follows that

$$\sin(2kt) = \cos(t) \sum_{s=1}^{k} (-1)^{s-1} \sin^{2s-1}(t) \sum_{j=1}^{s} \binom{2k}{2j-1} \binom{k-j}{s-j}.$$

Hence, the required conclusion follows from Lemma 2.2. □

Example 3.5. (1) According to Proposition 3.4(1), we have

$$\sin(3t) = -4\binom{3}{0}\sin^3(t) + \left(\binom{2}{1} + \binom{1}{0}\right)\sin(t) = 3\sin(t) - 4\sin^3(t).$$

(2) According to Proposition 3.4(2), we have

$$\sin(4t) = \cos(t)\left(2\binom{2}{1}\sin(t) - 8\binom{3}{3}\sin^3(t)\right) = -4\cos(t)\sin(t) + 8\cos(t)\sin^3(t).$$

Theorem 3.6. (1) *Let $k \geq 0$ be an integer. Then:*

$$\cos^{2k}(t) = \frac{1}{2^{2k-1}} \sum_{j=0}^{k-1} \binom{2k}{j} \cos((2k-2j)t) + \frac{1}{2^{2k}} \binom{2k}{k}.$$

(2) *Let $k \geq 0$ be an integer. Then:*

$$\cos^{2k+1}(t) = \frac{1}{2^{2k}} \sum_{j=0}^{k} \binom{2k+1}{j} \cos((2k+1-2j)t).$$

Proof. (1) We write $\cos^{2k}(t)=\frac{\beta_0}{2}+\sum_{j=1}^{k}\beta_j\cos(2jt)$. From Proposition 3.1, it follows that

(3.18)

$$\cos^{2k}(t)=\frac{\beta_0}{2}+\sum_{j=1}^{k}\beta_j\sum_{s=0}^{j}(-1)^s 2^{2j-2s-1}\left(\binom{2j-s}{s}+\binom{2j-s-1}{s-1}\right)\cos^{2j-2s}(t).$$

From (3.18), it follows that $1=2^{2k-1}\beta_k$ and thus $\beta_k=\frac{1}{2^{2k-1}}$. In order to prove (1), we have to show that

$$\beta_\ell=\frac{1}{2^{2k-1}}\binom{2k}{k-\ell},\text{ for all } 0\leq\ell\leq k. \tag{3.19}$$

We proceed by descending induction on ℓ. The assertion is true for $\ell=k$. Suppose $\ell<k$. From (3.18) it follows that

$$0=\sum_{s=0}^{k-\ell}(-1)^s 2^{2\ell-1}\left(\binom{2\ell+s}{s}+\binom{2\ell+s-1}{s-1}\right)\beta_{\ell+s}.$$

Therefore, using the induction hypothesis, it follows that:

$$\begin{aligned}\beta_\ell&=-\sum_{s=1}^{k-\ell}(-1)^s\left(\binom{2\ell+s}{s}+\binom{2\ell+s-1}{s-1}\right)\beta_{\ell+s}=\\&=-\frac{1}{2^{2k-1}}\sum_{s=1}^{k-\ell}(-1)^s\left(\binom{2\ell+s}{s}+\binom{2\ell+s-1}{s-1}\right)\binom{2k}{k-\ell-s}.\end{aligned}$$

Hence, in order to prove (3.19), it is enough to show that

$$\sum_{s=1}^{k-\ell}(-1)^s\left(\binom{2\ell+s}{s}+\binom{2\ell+s-1}{s-1}\right)\binom{2k}{k-\ell-s}=-\binom{2k}{k-\ell},$$

which is equivalent to

$$\sum_{s=0}^{k-\ell}(-1)^s\left(\binom{2\ell+s}{s}+\binom{2\ell+s-1}{s-1}\right)\binom{2k}{k-\ell-s}=0.$$

The last identity follows from Lemma 2.3, using the substitution $t=k-\ell$.

(2) The proof is similar. □

Example 3.7. (1) Let $k=2$. According to Theorem 3.6(1), we have:

$$\cos^4(t)=\frac{1}{2^3}\left(\binom{4}{0}\cos(4t)+\binom{4}{1}\cos(2t)\right)+\frac{1}{2^4}\binom{4}{2}=\frac{1}{8}(\cos(4t)+4\cos(2t)+3).$$

(2) Let $k=1$. According to Theorem 3.6(2), we have:

$$\cos^3(t)=\frac{1}{2^2}\left(\binom{3}{0}\cos(3t)+\binom{3}{1}\cos(t)\right)=\frac{1}{4}(\cos(3t)+3\cos(t)).$$

Also, from Theorem 3.6(1), we deduce the identity $2\cos^2(t)=\cos(2t)+1$.

Corollary 3.8. (1) *Let $k \geq 0$ be an integer. Then:*

$$\sin^{2k}(t) = \frac{1}{2^{2k-1}} \sum_{j=0}^{k-1} (-1)^{k-j} \binom{2k}{j} \cos((2k-2j)t) + \frac{1}{2^{2k}} \binom{2k}{k}.$$

(2) *Let $k \geq 0$ be an integer. Then:*

$$\sin^{2k+1}(t) = \frac{1}{2^{2k}} \sum_{j=0}^{k} (-1)^{k-j} \binom{2k+1}{j} \sin((2k+1-2j)t).$$

Proof. Since $\sin(t) = \cos(t - \frac{\pi}{2})$ and $\cos(t + n\pi) = (-1)^n \cos(t)$, from Theorem 3.6 it follows that

$$\sin^{2k}(t) = \frac{1}{2^{2k-1}} \sum_{j=0}^{k-1} (-1)^{k-j} \binom{2k}{j} \cos((2k-2j)t) + \frac{1}{2^{2k}} \binom{2k}{k}.$$

Hence (1) holds. The proof of (2) is similar. $\square$

4. An application

Let $a \in \mathbb{R}$ with $|a| > 1$. We consider the function

$$f(t) := \frac{1}{a - \cos t}, \text{ where } t \in \mathbb{R}.$$

Since $|\cos t| \leq 1$, we have the expansion

$$f(t) = \sum_{n=0}^{\infty} a^{-n-1} \cos^n t, \text{ for all } t \in \mathbb{R}. \tag{4.20}$$

From (4.20), using Theorem 3.6, we can write

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt), \text{ where } a_n = \frac{2}{a} \sum_{k \geq 0} (2a)^{-n-2k} \binom{n+2k}{k}, \tag{4.21}$$

for all $n \geq 0$. Using the binomial expansion, we have

$$(1-a^{-2})^{-\frac{1}{2}} = \sum_{k \geq 0} (-1)^n \binom{-\frac{1}{2}}{k} a^{-2n} = \sum_{k \geq 0} \frac{(2k-1)!!}{2^k \cdot k!} a^{-2k} = \sum_{k \geq 0} \binom{2k}{k} (2a)^{-2k}. \tag{4.22}$$

From (4.21) and (4.22) it follows that

$$a_0 = \frac{2}{a\sqrt{1-a^{-2}}}. \tag{4.23}$$

With the above notations, we have the following:

Lemma 4.1. *With the above notations, we have that* $(a_{n-1}-aa_n)a_0 = 2a_n$.

Proof. From (4.20), the required formula is equivalent to

$$\sum_{k\geq 0}(2a)^{-n-2k}\left(2\binom{n+2k-1}{k}-\binom{n+2k}{k}\right)\cdot\sum_{t\geq 0}(2a)^{-2t}\binom{2t}{t}$$
$$=\sum_{\ell=0}^{\infty}(2a)^{-n-2\ell}\binom{n+2\ell}{\ell}.$$

Since $2\binom{n+2k-1}{k}-\binom{n+2k}{k}=\binom{n+2k-1}{k}-\binom{n+2k-1}{k-1}$, the above identity is equivalent to

$$\sum_{k\geq 0}(2a)^{-2k}\left(\binom{n+2k-1}{k}-\binom{n+2k-1}{k-1}\right)\cdot\sum_{t\geq 0}(2a)^{-2t}\binom{2t}{t}$$
$$=\sum_{\ell=0}^{\infty}(2a)^{-2\ell}\binom{n+2\ell}{\ell}.$$

The last identity follows immediately from Lemma 2.4. ∎

Theorem 4.2. *With the above notations, we have that:*

$$f(t)=\frac{1}{a-\cos t}=\frac{1}{a\sqrt{1-a^{-2}}}+\sum_{n=1}^{\infty}\frac{2a^{n-1}(1-\sqrt{1-a^{-2}})^n}{\sqrt{1-a^{-2}}}\cos(nt).$$

Proof. Note that $f(t)$ has expansion given in (4.21), that is

$$\frac{1}{a-\cos t}=\frac{a_0}{2}+\sum_{n\geq 1}a_n\cos(nt).$$

Thus, in order to prove the theorem, we have to show that

$$a_n=\frac{2a^{n-1}(1-\sqrt{1-a^{-2}})^n}{\sqrt{1-a^{-2}}},\text{ for all } n\geq 0. \tag{4.24}$$

We use induction on $n \geq 0$. The case $n = 0$ follows from (4.23). In order to prove the induction step, it is enough show that the desired expression of a_n satisfies the recurrence relation given in Lemma 4.1. Indeed, if we replace a_n, a_{n-1} and a_0 with their required expressions, we get

$$(a_{n-1}-aa_n)a_0=\frac{2a^{n-2}(1-\sqrt{1-a^{-2}})^{n-1}-2a^n(1-\sqrt{1-a^{-2}})^n}{\sqrt{1-a^{-2}}}\cdot\frac{2}{a\sqrt{1-a^{-2}}}. \tag{4.25}$$

On the other hand, we have:

$$\frac{\frac{1}{a(1-\sqrt{1-a^{-2}})}-a}{a\sqrt{1-a^{-2}}}=\frac{a(1+\sqrt{1-a^{-2}})-a}{a\sqrt{1-a^{-2}}}=1. \tag{4.26}$$

From (4.25) and (4.26) it follows that $(a_{n-1}-aa_n)a_0 = 2a_n$, as required. Hence, the proof is complete. ∎

Corollary 4.3. *With the above notations, we have that:*

$$g(t)=\frac{1}{a-\sin t}=\frac{1}{a\sqrt{1-a^{-2}}}+\sum_{k=1}^{\infty}(-1)^k\frac{2a^{2k-1}(1-\sqrt{1-a^{-2}})^{2k}}{\sqrt{1-a^{-2}}}\cos(2kt)+$$
$$+\sum_{k=0}^{\infty}(-1)^k\frac{2a^{2k}(1-\sqrt{1-a^{-2}})^{2k+1}}{\sqrt{1-a^{-2}}}\sin((2k+1)t).$$

Proof. Note that $\sin(t)=\cos(t-\frac{\pi}{2})$ implies $g(t)=f(t-\frac{\pi}{2})$. Hence, the required formula follows from Theorem 4.2 and the identities:

$$\cos(nt-\frac{n\pi}{2})=\begin{cases}(-1)^k\cos(nt), & n=2k\\ (-1)^k\sin(nt), & n=2k+1\end{cases}.$$

□

Example 4.4. (1) According to Theorem 4.2 we have that

$$f(t)=\frac{1}{\sqrt{3}-\cos t}=\frac{1}{2\sqrt{3}}+\frac{1}{\sqrt{3}}\sum_{n=1}^{\infty}(-\sqrt{3})^n\cos(nt),\text{ for all } t\in\mathbb{R}.$$

(2) According to Corollary 4.3, for all $t\in\mathbb{R}$, we have that

$$g(t)=\frac{1}{\sqrt{3}-\sin t}=\frac{1}{2\sqrt{3}}+\frac{1}{\sqrt{3}}\sum_{k=1}^{\infty}(-3)^k\cos(2kt)-\sum_{k=0}^{\infty}(-3)^k\sin((2k+1)t).$$

References

[1] E. Maor, *Trigonometric delights*, Princeton University Press, 2002.
[2] Stack Exchange Network, *Fourier series for* $\cos^n(x)$, https://math.stackexchange.com/questions/4720800/fourier-series-for-cosxn.
[3] Stack Exchange Network, *Fourier series for* $\sin^n(x)$, https://math.stackexchange.com/questions/2422961/fourier-series-for-sinnx.
[4] J. Riordan, *Combinatorial identities*, J. Wiley, New York, 1968.
[5] T. J. Rivlin, *Pure and Applied Mathematics (1st ed.)*, New York-London-Sydney: Wiley-Interscience [John Wiley & Sons], 1974.
[6] E. M. Stein, R. Shakarchi, *Fourier analysis. An introduction*, Princeton University Press, 2003.

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