RAD HRVATSKE AKADEMIJE ZNANOSTI I UMJETNOSTI
MATEMATIČKE ZNANOSTI

A. Choudhry

Expressing three consecutive integers as sums of three cubes

Manuscript accepted for publication

EXPRESSING THREE CONSECUTIVE INTEGERS AS SUMS OF THREE CUBES

Ajai Choudhry

Abstract. This paper is concerned with the problem of expressing three consecutive integers as sums of three cubes. We give several parametric solutions of the problem. We also give some examples of five and seven consecutive integers that can all be expressed as sums of three cubes. We conclude the paper with an open problem regarding four or more consecutive integers expressible as sums of three cubes.

1. Introduction

Ever since Mordell's observation [9, p. 505] that he did "not know anything about the integer solutions of $X^3 + Y^3 + Z^3 = 3$ beyond the existence of the four sets $(1, 1, 1), (4, 4, -5)$ etc.; and it must be very difficult indeed to find out anything about any other solutions", there has been considerable interest in the representation of integers as a sum of three cubes of integers.

Remarkable progress has been made in recent years and the following new representation of the integer 3 as a sum of three cubes was discovered by Booker and Sutherland [2] in 2020:

$$569936821221962380720^3 + (-569936821113563493509)^3 + (-472715493453327032)^3 = 3. \tag{1.1}$$

Except for integers not expressible as a sum of three cubes because of congruence considerations, representations of all other integers ≤ 100 are now known with the representations for $74, 33$ and 42 having been obtained only in the last eight years (see [6], [1] and [2]).

2010 *Mathematics Subject Classification.* 11D25.
Key words and phrases. sums of three cubes, consecutive integers.

This paper is concerned with the representation of three consecutive integers as sums of three cubes, that is, we need to find integers n such that there is a solution of the three simultaneous equations,

$$n = x_1^3 + x_2^3 + x_3^3, \tag{1.2}$$

$$n+1 = y_1^3 + y_2^3 + y_3^3, \tag{1.3}$$

$$n+2 = z_1^3 + z_2^3 + z_3^3, \tag{1.4}$$

in which $x_i, y_i, z_i, i = 1, 2, 3$, are all integers.

On eliminating n between Eqs. (1.2) and (1.3), and again between Eqs. (1.3) and (1.4), we get the following two equations, respectively:

$$y_1^3 + y_2^3 + y_3^3 - x_1^3 - x_2^3 - x_3^3 = 1, \tag{1.5}$$

$$z_1^3 + z_2^3 + z_3^3 - y_1^3 - y_2^3 - y_3^3 = 1. \tag{1.6}$$

A solution of Eq. (1.5) will be considered *trivial* if two of the six integers $y_1, y_2, y_3, -x_1, -x_2, -x_3$, are 0 and 1, and the remaining four integers consist of two pairs such that the sum of the two integers in each pair is 0. If the six integers $z_1, z_2, z_3, -y_1, -y_2, -y_3$, satisfy similar conditions, we get a solution of Eq. (1.6) that will be considered *trivial*.

A solution of the simultaneous Eqs. (1.2), (1.3) and (1.4) will be considered *trivial* if both Eqs. (1.5) and (1.6) are trivially satisfied. If only one of the two equations (1.5) and (1.6) is trivially satisfied, the solution will be considered *semi-trivial*, and if neither of the two Eqs. (1.5) and (1.6) is trivially satisfied, the solution will be considered a *nontrivial* solution.

Examples of trivial solutions of the simultaneous Eqs. (1.2), (1.3) and (1.4) are given by

$$n = a^3 + 0^3 + 0^3, \quad n+1 = a^3 + 1^3 + 0^3, \quad n+2 = a^3 + 1^3 + 1^3,$$

and

$$n = a^3 + b^3 + (-1)^3, \quad n+1 = a^3 + b^3 + 0^3, \quad n+2 = a^3 + b^3 + 1^3,$$

where a and b are arbitrary integers.

In the next section we obtain semi-trivial solutions of our problem while in Section 3 we obtain several nontrivial parametric solutions of the simultaneous Eqs. (1.2), (1.3) and (1.4). We conclude the paper with some remarks about more than three consecutive integers that are expressible as sums of three cubes and a related open problem.

2. Semi-trivial examples of three consecutive integers expressible as sums of three cubes

To obtain semi-trivial solutions of Eqs. (1.2), (1.3) and (1.4), we start with trivial solutions of Eq. (1.5) and then obtain nontrivial solutions of the resulting equation obtained from Eq. (1.6). If we start with trivial solutions of Eq. (1.6) and then solve Eq. (1.5), we get similar results.

As is readily seen, trivial solutions of Eq. (1.5) may be written, without loss of generality, in one of the following six ways:

$$(x_1, x_2, x_3, y_1, y_2, y_3) = (0, u, v, 1, u, v), \tag{2.1}$$

$$(x_1, x_2, x_3, y_1, y_2, y_3) = (0, u, -u, 1, v, -v), \tag{2.2}$$

$$(x_1, x_2, x_3, y_1, y_2, y_3) = (0, -1, u, u, v, -v), \tag{2.3}$$

$$(x_1, x_2, x_3, y_1, y_2, y_3) = (-1, u, v, 0, u, v), \tag{2.4}$$

$$(x_1, x_2, x_3, y_1, y_2, y_3) = (-1, u, -u, 0, v, -v), \tag{2.5}$$

$$(x_1, x_2, x_3, y_1, y_2, y_3) = (u, v, -v, 0, 1, u), \tag{2.6}$$

where u and v are arbitrary integer parameters.

On substituting the values of x_i, y_i given by the six trivial solutions (2.1)–(2.6), in succession, in Eq. (1.6), we get the following six equations, respectively:

$$-u^3 - v^3 + z_1^3 + z_2^3 + z_3^3 = 2, \tag{2.7}$$

$$z_1^3 + z_2^3 + z_3^3 = 2, \tag{2.8}$$

$$-u^3 + z_1^3 + z_2^3 + z_3^3 = 1, \tag{2.9}$$

$$-u^3 - v^3 + z_1^3 + z_2^3 + z_3^3 = 1, \tag{2.10}$$

$$z_1^3 + z_2^3 + z_3^3 = 1, \tag{2.11}$$

$$-u^3 + z_1^3 + z_2^3 + z_3^3 = 2. \tag{2.12}$$

Nontrivial identities expressing the integers 1 and 2 as a sum of three cubes of polynomials have been given by Mahler [7] and by Werebrusow (as stated by Mordell [8]), respectively. Further, Choudhry [3] has given identities expressing the integers 1 and 2 as a sum of four cubes of polynomials and has also described a method of obtaining identities expressing any arbitrary integer as a sum of five cubes of polynomials, and thus the integers 1 and 2 can also be expressed in this manner.

Using the identities expressing 1 or 2 as a sum of cubes of three, four or five polynomials, we immediately get nontrivial parametric solutions of the six equations (2.7)–(2.12). We can thus obtain several parametric solutions of the simultaneous diophantine Eqs. (1.5) and (1.6) in which Eq. (1.5) is trivially satisfied, and we thus obtain semi-trivial parametric solutions of the three simultaneous Eqs. (1.2), (1.3) and (1.4).

As an example, using the identity

$$(t^2)^3 + (t^2)^3 + (-t^2 + t + 1)^3 + (-t^2 - t + 1)^3 = 2, \tag{2.13}$$

given by Choudhry [3, p. 4], we get the following solution of (2.12):

$$u = t^2 + t - 1, \quad z_1 = t^2, \quad z_2 = t^2, \quad z_3 = -t^2 + t + 1. \tag{2.14}$$

The solution (2.14) yields the following solution of the simultaneous Eqs. (1.5) and (1.6):

$$\begin{aligned} &x_1 = t^2 + t - 1, \quad x_2 = v, \quad x_3 = -v, \quad y_1 = 0, \quad y_2 = 1, \\ &y_3 = t^2 + t - 1, \quad z_1 = t^2, \quad z_2 = t^2, \quad z_3 = -t^2 + t + 1, \end{aligned}$$

where t and v are arbitrary integer parameters, and hence we obtain the integer $n = (t^2 + t - 1)^3$ such that the three consecutive integers $n, n + 1$, and $n + 2$ are expressible as $x_1^3 + x_2^3 + x_3^3, y_1^3 + y_2^3 + y_3^3, z_1^3 + z_2^3 + z_3^3$, respectively. Taking $(t, v) = (2, 1)$, yields the numerical example,

$$125 = 5^3 + 1^3 + (-1)^3, \quad 126 = 0^3 + 1^3 + 5^3, \quad 127 = 4^3 + 4^3 + (-1)^3.$$

3. Nontrivial examples of three consecutive integers expressible as sums of three cubes

We will describe, in the next two subsections, two different methods of obtaining nontrivial solutions of the simultaneous diophantine Eqs. (1.2), (1.3) and (1.4), or the equivalent pair of simultaneous Eqs. (1.5) and (1.6). The first method generates several multi-parameter solutions in polynomials of high degree and some of these solutions yield numerical solutions in positive integers. The second method generates several solutions in terms of linear and quadratic polynomials but all such solutions that we could obtain yield numerical examples that necessarily include negative integers.

3.1. *First method.* We will first solve the simultaneous Eqs. (1.5) and (1.6) by rewriting Eq.(1.5) as

$$y_1^3 + y_2^3 + y_3^3 - x_3^3 = x_1^3 + x_2^3 + 1. \tag{3.1}$$

and solving Eq.(3.1) together with the following equation obtained by eliminating y_1, y_2, y_3, from Eqs. (1.5) and (1.6):

$$z_1^3 + z_2^3 + z_3^3 - x_1^3 - x_2^3 - x_3^3 = 2. \tag{3.2}$$

We now impose the auxiliary condition $z_3 = x_3$ when Eq. (3.2) reduces to

$$z_1^3 + z_2^3 - x_1^3 - x_2^3 = 2. \tag{3.3}$$

While the auxiliary condition $z_3 = x_3$ facilitates the solution of our problem, it also ensures that the solution obtained by this method will never consist of distinct integers.

Now we have to solve the simultaneous Eqs. (3.1) and (3.3). Our strategy to solve these simultaneous equations is to reduce them, in two steps, to a single equation in which one of the unknown variables occurs only in degree 1 so that the equation can be solved. The first step is to use a known solution of Eq. (3.3), and substitute the values of x_1, x_2 given by this solution in the right-hand side of Eq. (3.1). For the second step, we will assign values to the four variables x_3, y_1, y_2 and y_3 (which occur only on the left-hand side of Eq. (3.1)) in terms of certain new variables in such a way that at least one of the new variables occurs only in the first degree on simplifying the left-hand side of Eq. (3.1).

For executing the first step, we give below two known solutions of Eq. (3.3). One such solution, that immediately follows from the identity (2.13), may be written, in terms of an arbitrary parameter t, as follows:

$$x_1 = -t^2, \quad x_2 = t^2 - t - 1, \quad z_1 = t^2, \quad z_2 = -t^2 - t + 1. \tag{3.4}$$

A second solution of Eq. (3.3) is as follows:

$$\begin{aligned} x_1 &= 6gt^2(g^3 + h^3), & x_2 &= 6t^3(g^3 + h^3)^2 - 1, \\ z_1 &= 6t^3(g^3 + h^3)^2 + 1, & z_2 &= -6ht^2(g^3 + h^3), \end{aligned} \tag{3.5}$$

where g, h and t are arbitrary parameters. The solution (3.5) follows from an identity, given by Choudhry [3, p. 4], expressing 2 as a sum of four cubes of polynomials in three variables. It may also be verified by direct computation.

We now describe two ways of executing the second step of our strategy. The first way is by writing

$$x_3 = m+p-q, \quad y_1 = m+p+q, \quad y_2 = -m+p-q, \quad y_3 = m-p-q, \tag{3.6}$$

where m, p and q are arbitrary parameters, and now the left-hand side of Eq. (3.1) reduces to $24mpq$.

The second way is to write

$$x_3 = um + p, \quad y_1 = v_1 m + p, \quad y_2 = v_2 m + q, \quad y_3 = v_3 m - q, \tag{3.7}$$

where u and $v_i, i = 1, 2, 3$, are chosen so as to satisfy the condition,

$$u^3 = v_1^3 + v_2^3 + v_3^3, \tag{3.8}$$

while m, p and q are arbitrary parameters. Now the left-hand side of Eq. (3.1) reduces to

$$3((v_1^2 - u^2)p + (v_2^2 - v_3^2)q)m^2 - 3((u - v_1)p^2 - (v_2 + v_3)q^2)m,$$

and, on choosing

$$p = k(v_2^2 - v_3^2), \quad q = k(u^2 - v_1^2), \tag{3.9}$$

where k is a suitably chosen rational number such that p, q are integers, it further reduces to

$$3k^2 m(u - v_1)(v_2 + v_3)(u^3 + u^2 v_1 - u v_1^2 - v_1^3 - v_2^3 + v_2^2 v_3 + v_2 v_3^2 - v_3^3),$$

where m occurs only in degree 1 as we desired.

We can use either of the two solutions of Eq. (3.3) with either of the two ways given above of reducing the left-hand side of Eq. (3.1) to a linear polynomial in the independent parameter m and thus obtain several parametric solutions of the simultaneous diophantine Eqs. (1.5) and (1.6).

As an example, taking the values of x_1, x_2 given by the solution (3.4) of Eq. (3.3), and the values of x_3, y_1, y_2, y_3, given by (3.6), Eq. (3.1) reduces to

$$24mpq = -t(3t^4 - 5t^2 + 3), \tag{3.10}$$

and we can readily find several parametric solutions by a suitable choice of t. For instance, we may solve Eq. (3.10) for m, and we get $m = -t(3t^4 - 5t^2 + 3)/(24pq)$. Since we want a solution in integers, we now write

$$t = -24pqr, \tag{3.11}$$

where r is an arbitrary integer parameter, and we thus obtain

$$m = 3r(331776p^4q^4r^4 - 960p^2q^2r^2 + 1). \tag{3.12}$$

On substituting the values of t and m given by (3.11) and (3.12), respectively, in the relations (3.4) and (3.6), and noting that we have already imposed the condition $z_3 = x_3$, we get the following nontrivial solution of the simultaneous Eqs. (1.5) and (1.6):

$$\begin{aligned}
x_1 &= -576p^2q^2r^2,\\
x_2 &= 576p^2q^2r^2 + 24pqr - 1,\\
x_3 &= 995328p^4q^4r^5 - 2880p^2q^2r^3 + p - q + 3r,\\
y_1 &= 995328p^4q^4r^5 - 2880p^2q^2r^3 + p + q + 3r,\\
y_2 &= -995328p^4q^4r^5 + 2880p^2q^2r^3 + p - q - 3r,\\
y_3 &= 995328p^4q^4r^5 - 2880p^2q^2r^3 - p - q + 3r,\\
z_1 &= 576p^2q^2r^2,\\
z_2 &= -576p^2q^2r^2 + 24pqr + 1,\\
z_3 &= x_3,
\end{aligned} \tag{3.13}$$

where p, q and r are arbitrary integer parameters.

We can obtain another solution of the simultaneous Eqs. (1.5) and (1.6) by following a similar procedure using the values of x_1, x_2 given by the second solution (3.5) of Eq. (3.3). As it is cumbersome to write this solution, we do not give it explicitly.

We will now obtain solutions of the simultaneous Eqs. (1.5) and (1.6) by following the second way mentioned above using the values of x_3, y_1, y_2, y_3, given by (3.7). We can use the values of x_1, x_2 given by either of the two solutions (3.4) and (3.5) of Eq. (3.3), but we restrict ourselves only to the second solution (3.5) of Eq. (3.3).

We need values of u, v_1, v_2, v_3, satisfying the condition (3.8). Several parametric solutions of Eq. (3.8) are already known ([4, p. 257–260], [5, p. 290–291], and starting with such solutions of Eq. (3.8), we can obtain multi-parameter solutions of the simultaneous Eqs. (1.5) and (1.6).

As the more general multi-parameter solutions obtained by this method are too cumbersome to write, we give a simpler example taking $(u, v_1, v_2, v_3) = (9, 6, 1, 8)$ when the condition (3.8) is satisfied. We use the relations (3.7) and following the procedure already described, we get, on using the relations (3.9), $(p, q) = (-63k, 45k)$ and on taking $k = -1/9$, we get take $(p, q) = (7, -5)$. We now take $t = 13r$ in the solution (3.5), when Eq. (3.1) simply reduces to

$$\begin{aligned}(3.14)\quad m = {} & 169r^3(g+h)^2(g^2-gh+h^2)^2(57921708g^{12}r^6 \\ & + 231686832g^9h^3r^6 + 347530248g^6h^6r^6 + 231686832g^3h^9r^6 \\ & + 57921708h^{12}r^6 + 13182g^6r^3 - 13182h^6r^3 + 1).\end{aligned}$$

Thus a solution of the simultaneous Eqs. (1.5) and (1.6) is given by (3.7) where $(u, v_1, v_2, v_3, p, q) = (9, 6, 1, 8, 7, -5)$ and m is given by (3.14) in terms of three arbitrary integer parameters g, h and r. As a special case we take $(g, h) = (2, -1)$, when we get the following solution of the simultaneous Eqs. (1.5) and (1.6):

$$\begin{aligned} x_1 &= 14196r^2, \\ x_2 &= 645918r^3 - 1, \\ x_3 &= 10364749588252332r^9 + 61893800514r^6 + 74529r^3 + 7, \\ y_1 &= 6909833058834888r^9 + 41262533676r^6 + 49686r^3 + 7, \\ (3.15)\qquad y_2 &= 1151638843139148r^9 + 6877088946r^6 + 8281r^3 - 5, \\ y_3 &= 9213110745113184r^9 + 55016711568r^6 + 66248r^3 + 5, \\ z_1 &= 645918r^3 + 1, \\ z_2 &= 7098r^2, \\ z_3 &= x_3, \end{aligned}$$

where r is an arbitrary integer parameter.

We now have two parametric solutions (3.13) and (3.15) of the simultaneous diophantine Eqs. (1.5) and (1.6). With the values of $x_i, y_i, z_i, i = 1, 2, 3$, given by (3.13) and by (3.15), we may take $n = x_1^3 + x_2^3 + x_3^3$, and obtain two nontrivial examples of three consecutive integers $n, n+1, n+2$ that may be expressed as $x_1^3 + x_2^3 + x_3^3$, $y_1^3 + y_2^3 + y_3^3$, and $z_1^3 + z_2^3 + z_3^3$, respectively.

As a numerical example, taking $(p, q, r) = (2, 1, 1)$ in the solution (3.13), we get $n = 4030102758035382018255$, and the three consecutive integers beginning with n may be written as sums of three cubes as follows:

$$\begin{aligned} n &= (-2304)^3 + 2351^3 + 15913732^3, \\ n+1 &= 15913734^3 + (-15913730)^3 + 15913728^3, \\ n+2 &= 2304^3 + (-2255)^3 + 15913732^3. \end{aligned}$$

The second solution (3.15) yields, on taking $r \geq 1$, infinitely many nontrivial solutions of our problem in positive integers. For instance, taking $r = 1$ in the solution (3.15), we get

$$n = 111348461898100166854345162800473206860712609871\,7,$$

and the representations of the three consecutive integers as sums of three cubes of positive integers are as follows:

$$\begin{aligned} n &= 14196^3 + 645917^3 + 10364811482127382^3, \\ n+1 &= 6909874321418257^3 + 1151645720236370^3 \\ &\quad + 9213165761891005^3, \\ n+2 &= 7098^3 + 645919^3 + 10364811482127382^3. \end{aligned}$$

3.2. *Second method.* We will now obtain nontrivial solutions of the simultaneous Eqs. (1.5) and (1.6) by first obtaining parametric solutions of the corresponding simultaneous homogeneous equations namely,

$$y_1^3 + y_2^3 + y_3^3 - x_1^3 - x_2^3 - x_3^3 = t^3, \tag{3.16}$$

$$z_1^3 + z_2^3 + z_3^3 - y_1^3 - y_2^3 - y_3^3 = t^3, \tag{3.17}$$

and then choosing the parameters such that we get $t = 1$.

On writing

$$\begin{aligned} x_1 &= a_1u + b_1v, & x_2 &= -a_1u - b_4v, & x_3 &= -a_4u - b_1v, \\ y_1 &= a_2u + b_2v, & y_2 &= -a_2u, & y_3 &= -b_2v, \\ z_1 &= a_3u + b_3v, & z_2 &= -a_3u + b_4v, & z_3 &= a_4u - b_3v, \\ t &= a_4u + b_4v, \end{aligned} \tag{3.18}$$

where $u, v, a_i, b_i, i = 1, \ldots, 4$, are all arbitrary parameters, Eqs. (3.16) and (3.17) reduce, after transposing all terms to one side and removing the factor $3uv$ in both cases, to the following two equations, respectively:

$$((a_1^2-a_4^2)b_1-a_2^2b_2-(a_1^2-a_4^2)b_4)u+((a_1-a_4)b_1^2-a_2b_2^2-(a_1-a_4)b_4^2)v = 0, \tag{3.19}$$

and

$$(a_2^2b_2-(a_3^2-a_4^2)b_3-(a_3^2-a_4^2)b_4)u+(a_2b_2^2-(a_3+a_4)b_3^2+(a_3+a_4)b_4^2)v = 0. \tag{3.20}$$

We now equate to 0 the coefficients of u and v in Eq. (3.19) and the coefficient of u in Eq. (3.20) and solve for $b_i, i = 1, \ldots, 4$ excluding solutions in which $b_2 = 0$ since that would make $y_3 = 0$. We thus get the following solution for $b_i, i = 1, \ldots, 4$:

$$
\begin{aligned}
b_1 &= k(a_3^2 - a_4^2)(a_1^3 + a_1^2 a_4 - a_1 a_4^2 + a_2^3 - a_4^3),\\
b_2 &= 2ka_2(a_1^2 - a_4^2)(a_3^2 - a_4^2),\\
b_3 &= -k((a_3^2 - a_4^2)a_1^3 - (2a_2^3 - a_3^2 a_4 + a_4^3)a_1^2 - a_4^2(a_3^2 - a_4^2)a_1\\
&\qquad - a_2^3 a_3^2 + 3a_2^3 a_4^2 - a_3^2 a_4^3 + a_4^5),\\
b_4 &= k(a_3^2 - a_4^2)(a_1^3 + a_1^2 a_4 - a_1 a_4^2 - a_2^3 - a_4^3),
\end{aligned} \tag{3.21}
$$

where k is an arbitrary parameter.

With the values of $b_i, i = 1, \ldots, 4$, given by (3.21), Eq. (3.19) is identically satisfied for all u and v while Eq. (3.20) reduces to the following equation:

$$
\begin{aligned}
(a_3^2 - a_4^2)a_1^3 - (a_2^3 - a_3^3 + a_3 a_4^2)a_1^2 - (a_3^2 - a_4^2)a_4^2 a_1&\\
- (a_3^2 - 2a_4^2)a_2^3 - a_3^3 a_4^2 + a_3 a_4^4 &= 0.
\end{aligned} \tag{3.22}
$$

Eq. (3.22) has a parametric solution given by

$$
a_1 = -p^2 - pq + q^2, \quad a_2 = 2pq, \quad a_3 = p^2 - pq - q^2, \quad a_4 = p^2 + pq + q^2, \tag{3.23}
$$

where p and q are arbitrary parameters. It is also possible to find infinitely many integer solutions of Eq. (3.22) that are not given by the parametric solution (3.23). While these solutions of Eq. (3.22) will yield solutions of the simultaneous diophantine Eqs. (3.16) and (3.17) in terms of two linear parameters u and v, in order to obtain solutions of the simultaneous diophantine Eqs. (1.5) and (1.6), we need to find such solutions in which we can find u, v so as to satisfy the condition $t = a_4 u + b_4 v = 1$. For this purpose, it suffices to find integer solutions of Eq. (3.22) such that the resulting value of b_4, obtained from (3.21), is coprime with a_4.

The parametric solution (3.23) of Eq. (3.22) does not lead to coprime values of a_4 and b_4. Accordingly, we performed trials over the range $|a_1| + |a_2| + |a_3| + |a_4| \le 100$, and found several solutions of Eq. (3.22) such that $\gcd(a_4, b_4) = 1$. However, only two of these solutions yielded independent solutions of the simultaneous diophantine Eqs. (1.5) and (1.6).

One of the aforementioned two solutions of Eq. (3.22) namely, $(a_1, a_2, a_3, a_4) = (5, 2, -2, 1)$, yields on using the relations (3.21) and taking $k = 1/24$, $(b_1, b_2, b_3, b_4) = (19, 12, -1, 17)$, and now, on using the relations (3.18), we get the following solution of the simultaneous diophantine Eqs. (3.16) and (3.17):

$$
\begin{aligned}
x_1 &= 5u + 19v, & x_2 &= -5u - 17v, & x_3 &= -u - 19v,\\
y_1 &= 2u + 12v, & y_2 &= -2u, & y_3 &= -12v,\\
z_1 &= -2u - v, & z_2 &= 2u + 17v, & z_3 &= u + v,\\
t &= u + 17v,
\end{aligned} \tag{3.24}
$$

where u and v are arbitrary parameters. Now, on writing $u = 1 - 17v$, we get $t = 1$, and hence we get the following solution of the simultaneous diophantine Eqs. (1.5) and (1.6):

$$
\begin{aligned}
&x_1 = -66v + 5, \quad && x_2 = 68v - 5, \quad && x_3 = -2v - 1,\\
&y_1 = -22v + 2, \quad && y_2 = 34v - 2, \quad && y_3 = -12v,\\
&z_1 = 33v - 2, \quad && z_2 = -17v + 2, \quad && z_3 = -16v + 1,
\end{aligned} \tag{3.25}
$$

where v is an arbitrary integer parameter.

With the values of $x_i, y_i, z_i, i = 1, 2, 3$, given by (3.25), we get

$$n = x_1^3 + x_2^3 + x_3^3 = 26928v^3 - 4032v^2 + 144v - 1,$$

and the three consecutive integers $n, n + 1, n + 2$ may be expressed as $x_1^3 + x_2^3 + x_3^3$, $y_1^3 + y_2^3 + y_3^3$, and $z_1^3 + z_2^3 + z_3^3$, respectively.

On taking $v = 2$ in the solution (3.25), we get the following numerical example of three consecutive integers expresible as sums of three cubes:

$$
\begin{aligned}
199583 &= 131^3 + (-5)^3 + (-127)^3,\\
199584 &= 66^3 + (-24)^3 + (-42)^3,\\
199585 &= 64^3 + (-31)^3 + (-32)^3.
\end{aligned}
$$

A second solution of Eq. (3.22) namely, $(a_1, a_2, a_3, a_4) = (-4, 6,\ 19, 1)$, yields, on following a similar procedure as above, the following solution of the simultaneous diophantine Eqs. (1.5) and (1.6):

$$
\begin{aligned}
&x_1 = -97v - 4, \quad && x_2 = 145v + 4, \quad && x_3 = -48v - 1,\\
&y_1 = 194v + 6, \quad && y_2 = -174v - 6, \quad && y_3 = -20v,\\
&z_1 = 582v + 19, \quad && z_2 = -580v - 19, \quad && z_3 = -2v + 1,
\end{aligned} \tag{3.26}
$$

where, as before, v is an arbitrary integer parameter. This gives

$$n = x_1^3 + x_2^3 + x_3^3 = 2025360v^3 + 132480v^2 + 2160v - 1,$$

and, as before, the three consecutive integers $n, n+1, n+2$ may be expressed as $x_1^3 + x_2^3 + x_3^3$, $y_1^3 + y_2^3 + y_3^3$, and $z_1^3 + z_2^3 + z_3^3$, respectively.

While the two linear parametric solutions (3.25) and (3.26) of the simultaneous diophantine Eqs. (1.5) and (1.6) give numerical solutions of these equations in distinct integers, it is easily seen that some of these integers will necessarily be negative. We, accordingly, explored the existence of parametric solutions of degree 2 and obtained one such solution which is as follows:

$$
\begin{aligned}
&x_1 = -147m^2 - 42m - 1, \quad && x_2 = 294m^2 + 77m + 3,\\
&x_3 = -147m^2 - 35m - 3, \quad && y_1 = -147m^2 - 56m - 4,\\
&y_2 = 294m^2 + 77m + 4, \quad && y_3 = -147m^2 - 21m,\\
&z_1 = -147m^2 - 14m + 1, \quad && z_2 = 294m^2 + 77m + 5,\\
&z_3 = -147m^2 - 63m - 5,
\end{aligned} \tag{3.27}
$$

where m is an arbitrary integer parameter.

The procedure for obtaining the solution (3.27) is similar to that used for obtaining the two linear solutions but we omit the tedious details. The solution (3.27) may be readily verified by direct computation.

As in the case of earlier solutions, it follows from (1.2) that

$$n = 19059138m^6+14975037m^5+4429845m^4+617400m^3+40572m^2+1008m-1,$$

and, in view of the relations (1.2), (1.3) and (1.4), the three consecutive integers $n, n+1$ and $n+2$ are all expressible as sums of three cubes of integers.

As a numerical example, taking $m = 1$ in the solution (3.27), we get $n = 39122999$, and the three consecutive integers beginning with n may be written as sums of three cubes as follows:

$$
\begin{aligned}
39122999 &= 374^3 + (-185)^3 + (-190)^3,\\
39123000 &= 375^3 + (-168)^3 + (-207)^3,\\
39123001 &= 376^3 + (-160)^3 + (-215)^3.
\end{aligned}
$$

While the solution (3.27) yields numerical solutions of Eqs. (1.5) and (1.6) in distinct integers, here also some of the integers are necessarily negative.

4. Four or more consecutive integers expressible as sums of three cubes and a related open problem

If a is any arbitrary integer, the representations $a^3, a^3+0^3\pm1^3, a^3\pm1^3\pm1^3$ furnish a trivial example of five consecutive integers $a^3-2, \ldots, a^3+2$, that can be expressed as a sum of three cubes. Less trivial examples are obtained from integer solutions of the diophantine Eq. (3.3) and taking $n = x_1^3+x_2^3-1$ when the five integers $n, n+1, \ldots, n+4$ are all expressible as sums of three cubes since we have,

$$
\begin{aligned}
&n = x_1^3 + x_2^3 + (-1)^3, \quad && n+1 = x_1^3 + x_2^3 + 0^3,\\
&n+2 = x_1^3 + x_2^3 + 1^3, \quad && n+3 = z_1^3 + z_2^3 + 0^3,\\
&n+4 = z_1^3 + z_2^3 + 1^3.
\end{aligned}
$$

Thus the parametric solutions (3.4) and (3.5) of Eq. (3.3) yield infinitely many examples of five consecutive integers expressible as sums of three cubes.

Further, if the integers $x_i, i = 1, 2, 3$, satisfy the relation $x_1^3 + x_2^3 - x_3^3 = 3$ and we take $n = x_3^3 - 2$, then the 7 consecutive integers $n + j, j = 0, \ldots, 6$, can all be represented as a sum of three cubes since we have

$$\begin{aligned} n &= x_3^3 + (-1)^3 + (-1)^3, & n+1 &= x_3^3 + (-1)^3 + 0^3, \\ n+2 &= x_3^3 + 0^3 + 0^3, & n+3 &= x_1^3 + 1^3 + 0^3, \\ n+4 &= x_1^3 + 1^3 + 1^3, & n+5 &= x_1^3 + x_2^3 + 0^3, \\ n+6 &= x_1^3 + x_2^3 + 1^3. \end{aligned}$$

Since $4^3+4^3-5^3 = 3$, taking $n = 5^3-2 = 123$ gives 7 consecutive integers, commencing with 123, that can all be represented as a sum of three cubes. Similarly, in view of the identity (1.1), the 7 consecutive integers starting with $472715493453327032^3 - 2$ can all be expressed as sums of three cubes.

None of the above examples of 5 or 7 consecutive integers is really nontrivial since the representations of several of the integers as a sum of three cubes include the cubes of 0, 1 and -1. It is an open problem of considerable interest to find infinitely many nontrivial examples of four or more consecutive integers that can all be expressed as a sum of three cubes without using the cubes of the integers $0, -1$ and 1.

Acknowledgements.

I am grateful to the referee for carefully perusing the manuscript and giving valuable comments which have led to improvements in the paper.

References

[1] A. R. Booker, *Cracking the problem with 33*, Res. Number Theory 5 (2019), Paper No. 26.
[2] A. R. Booker and A. V. Sutherland, *On a question of Mordell*, arXiv:2007.01209 (2020).
[3] A. Choudhry, *Expressing an integer as a sum of cubes of polynomials*, arXiv:2311.07325 (2023).
[4] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers, Sixth edition*, Oxford University Press, 2008
[5] L. K. Hua, *Introduction to number theory*, Springer-Verlag, 1982.
[6] S. G. Huisman, *Newer sums of three cubes*, arXiv:1604.07746 (2016).
[7] K. Mahler, *Note on hypothesis K of Hardy and Littlewood*, J. Lond. Math. Soc. 11 (1936), 136–138.
[8] L. J. Mordell, *On sums of three cubes*, J. Lond. Math. Soc. 17, (1942) 139–144 .
[9] L. J. Mordell, *On the integer solutions of the equation $x^2 + y^2 + z^2 = 2xyz$*, J. London Math. Soc. 28 (1953), 500–510.

Naslov

Prvi autor, drugi autor i treći autor

Sažetak. Hrvatski prijevod sažetka.

13/4 A Clay Square, Lucknow - 226001, India
E-mail: ajaic203@yahoo.com