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MODIFIED HELICAL EXTENSION CURVE OF A SPACE CURVE

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Abstract. This study advances the theory of the helical extension curve. We provide its construction using Sasai's modified frame, which remains well-defined even at points where the Frenet frame becomes discontinuous. For this class of curves, explicit equations are obtained for the Frenet apparatus. The study also focuses on the helical extension curve derived from the Salkowski curve, the general helix, and the cylindrical helix and provides their characterisations. The results presented herein hold potential for applications in computer-aided geometric design, robotics, and mathematical modelling.

1. Introduction

The theory of curves and their moving frames has an important role in various branches of science, particularly in differential geometry [5, 6, 14, 16, 19]. It provides useful tools for analysing the geometric properties of curves, e.g., curvature, torsion and arc length. The Frenet frame is the most commonly used frame to study a space curve. However, it cannot be defined at the points where the curvature vanishes, such as the singular point of order 1 ($\boldsymbol{\beta}'' = 0$) [1]. At such points, the normal and binormal vectors become discontinuous [1, 17]. To overcome this problem, T. Sasai introduced an orthogonal frame for a curve whose curvature doesn't vanish identically [17]. Furthermore, Bucku and Karacan extended the idea of modified frame to Minkowski 3-space [2], and several curves and surfaces are characterised using this frame [1, 7, 9, 10]. A significant number of results concerning special curves are described in

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terms of the relationship between their curvature and torsion. For example, "Lancret's theorem states that a necessary and sufficient condition for a curve to be a general helix is that the ratio of curvature to torsion remains constant" [3, 9]. Similarly, a curve is classified as circular helix if both its curvature and torsion are non-zero constants [3]. Furthermore, the Salkowski curves are characterised by their constant curvature and variable torsion [11, 12]. Special curves like helices and Salkowski curves are not only mathematically significant but also have important real-life applications. Helices, for instance, appear naturally in the structure of DNA molecules, springs, helical staircases, etc. [3, 8, 18]. Salkowski curves, due to their unique geometric properties, have been studied in the context of computer graphics, animation, and robotics, where smooth yet flexible motion paths are required [12]. Recently, researchers have used existing knowledge and classical differential geometry to define new types of curves with interesting properties. These curves have been created by using basic ideas like the Frenet frame, curvature, and torsion to describe more complex shapes. One such example is the helical extension curve, introduced by M. Dede [3], which is constructed from a given space curve [3]. Dede investigated the helical extensions of several well-known curves, including general helices, cylindrical helices, and rectifying curves [3]. Motivated by recent developments in moving frames for curves and by the new class of curve introduced by M. Dede, we focus on studying the helical extension curve derived from Sasai's modified frame. This approach helps us analyse the curves more effectively, especially in situations where the classical Frenet frame is not suitable.

The rest of the paper is structured as follows: Section 2 outlines the basic concepts and results. In Section 3, firstly, the helical extension curve is constructed, and its Frenet apparatus is derived. Furthermore, for the specific class of cylindrical helix, the helical extension and its associated distance function are computed. Finally, the approach is extended to Salkowski curves with constant curvature, for which the corresponding helical extension is constructed. To illustrate the theoretical results, a few examples are presented along with their graphical representations. Section 4 illustrates the difference between the helical extension curves derived from the Frenet frame and Sasai's frame. Section 5 summarises our key findings and suggests directions for future research on this topic.

2. Preliminaries

We begin by summarising some essential theorems and the results needed for the remaining sections.

Let $\boldsymbol{\beta} = \boldsymbol{\beta}(u)$ be an analytic space curve, then the Frenet frame associated to every point u is given by ([4], [13], [15])

$$\mathbf{e}_1(u) = \frac{\boldsymbol{\beta}'}{\|\boldsymbol{\beta}'\|}, \quad \mathbf{e}_2(u) = \mathbf{e}_3(u) \times \mathbf{e}_1(u), \quad \mathbf{e}_3(u) = \frac{\boldsymbol{\beta}' \times \boldsymbol{\beta}''}{\|\boldsymbol{\beta}' \times \boldsymbol{\beta}''\|}. \tag{2.1}$$

For the curve $\boldsymbol{\beta}$, curvature κ and torsion τ are given by

$$\kappa = \frac{\|\boldsymbol{\beta}' \times \boldsymbol{\beta}''\|}{\|\boldsymbol{\beta}'\|^3}, \quad \tau = \frac{det(\boldsymbol{\beta}', \boldsymbol{\beta}'', \boldsymbol{\beta}''')}{\|\boldsymbol{\beta}' \times \boldsymbol{\beta}''\|^2}. \tag{2.2}$$

We consider that the curve $\boldsymbol{\beta}(s)$ is parameterised by its arc length s. If the curvature function satisfies $\kappa(s) \not\equiv 0$, an orthogonal frame $\{\mathbf{E}_1(s), \mathbf{E}_2(s), \mathbf{E}_3(s)\}$ may be defined along the curve as

$$\mathbf{E_1}(s) = \frac{d\boldsymbol{\beta}(s)}{ds}, \quad \mathbf{E_2} = \frac{d\mathbf{E_1}(s)}{ds}, \quad \mathbf{E_3}(s) = \mathbf{E_1}(s) \times \mathbf{E_2}(s).$$

The relationship between the Frenet frame and the Sasai's modified frame is given by

$$\mathbf{E_1}(s) = \mathbf{e_1}(s), \quad \mathbf{E_2}(s) = \kappa \mathbf{e_2}(s), \quad \mathbf{E_3}(s) = \kappa \mathbf{e_3}(s). \tag{2.3}$$

Differentiating (2.3) with respect to s yields

$$\frac{d\mathbf{E_1}}{ds} = \mathbf{E_2}, \quad \frac{d\mathbf{E_2}}{ds} = -\kappa^2 \mathbf{E_1} + \frac{\kappa'}{\kappa}\mathbf{E_2} + \tau \mathbf{E_3}, \quad \frac{d\mathbf{E_3}}{ds} = -\tau \mathbf{E_2} + \frac{\kappa'}{\kappa}\mathbf{E_3}.$$

Let $\langle\ ,\ \rangle$ represents the standard inner product in $\mathbb{R}^3$. Then we obtain

$$\langle \mathbf{E_1}, \mathbf{E_1} \rangle = 1,\ \langle \mathbf{E_2}, \mathbf{E_2} \rangle = \kappa^2 = \langle \mathbf{E_3}, \mathbf{E_3} \rangle,$$
$$\langle \mathbf{E_1}, \mathbf{E_2} \rangle = \langle \mathbf{E_2}, \mathbf{E_3} \rangle = \langle \mathbf{E_1}, \mathbf{E_3} \rangle = 0.$$

The frame $\{\mathbf{E_1}(s), \mathbf{E_2}(s), \mathbf{E_3}(s)\}$ is the Sasai's modified frame at non-zero curvature points [17].

Definition 2.1. *([3]) Let $\boldsymbol{\beta}(u)$ be an analytic curve with curvature and torsion κ and τ, respectively. Then the corresponding helical extension curve $\boldsymbol{\alpha}(u)$ is given by*

$$\boldsymbol{\alpha}(u) = \boldsymbol{\beta}(u) - \int \frac{\tau}{\kappa} \|\beta'(u)\| du\, \mathbf{e_3}, \tag{2.4}$$

where $\mathbf{e_3}$ is the binormal vector according to Frenet frame.

3. Modified Helical Extension Curve

This section is devoted to the study of geometric properties of the modified helical extension curve. To simplify the computations, the constants of integration are omitted.

Definition 3.1. *Let $\boldsymbol{\beta}(u)$ be an analytic curve with curvature and torsion κ and τ, respectively. Then the corresponding modified helical extension curve $\boldsymbol{\alpha}(u)$ is defined as*

$$\boldsymbol{\alpha}(u) = \boldsymbol{\beta}(u) - \int \frac{\tau}{\kappa} \|\beta'(u)\| du\, \mathbf{E_3}, \tag{3.5}$$

where $\mathbf{E_3}$ *is the binormal vector of* $\boldsymbol{\beta}(u)$ *according to the Sasai's modified frame.*
For a curve $\boldsymbol{\beta}(s)$ *parameterised by arc length* s*, equation (3.5) becomes*

$$\boldsymbol{\alpha}(s) = \boldsymbol{\beta}(s) - \int \frac{\tau}{\kappa} ds\, \mathbf{E_3}. \tag{3.6}$$

THEOREM 3.2. *Let* $\boldsymbol{\beta}(s)$ *be an analytic space curve parameterised by the arc length* s*. Then the curvature* κ_α *and torsion* τ_α *of the modified helical extension curve* $\boldsymbol{\alpha}(s)$ *is given by*

$$\kappa_\alpha = \frac{\left\{\left[p\tau a_2 + \left(\frac{\tau}{\kappa} + \frac{p\kappa'}{\kappa}\right) a_1\right]^2 + \left[p\tau\kappa(\tau + p\kappa') - a_2\right]^2 \kappa^2 + \left[p^2\kappa^2\tau^2 + a_1\right]^2 \kappa^2\right\}^{\frac{1}{2}}}{\left[1 + p^2\kappa^2\tau^2 + (\tau + p\kappa')^2\right]^{3/2}}$$

where

$$p = \int \frac{\tau}{\kappa} ds$$
$$a_1 = 1 + p\tau' + \frac{2\tau}{\kappa}(\tau + p\kappa')$$
$$a_2 = p\left(\tau^2 - \frac{\kappa''}{\kappa}\right) - \frac{1}{\kappa}\left(\tau' + \frac{\kappa'\tau}{\kappa}\right)$$

and

$$\tau_\alpha = \frac{a_1(b_4 + b_5) - a_2(b_2 + b_3) + p\tau[p\kappa^2\tau(b_4 + b_5) + a_2 b_1] + \left(\frac{\tau}{\kappa} + \frac{p\kappa'}{\kappa}\right)[p\kappa^2\tau(b_2 + b_3) + a_1 b_1]}{\left[p\tau a_2 + \left(\frac{\tau}{\kappa} + \frac{p\kappa'}{\kappa}\right) a_1\right]^2 + [p\kappa\tau(\tau + p\kappa') - a_2]^2\kappa^2 + (p^2\kappa^2\tau^2 + a_1)^2\kappa^2}$$

where

$$b_1 = 2p\kappa[2\kappa'\tau + \kappa\tau' + \kappa(\kappa + 3\tau^2)]$$
$$b_2 = p\left(\tau'' + \frac{3\kappa''\tau}{\kappa} + \frac{3\kappa'\tau'}{\kappa} - \tau^3\right)$$
$$b_3 = \frac{1}{\kappa}\left[3\tau\left(2\tau' + \frac{\kappa'\tau}{\kappa}\right) + \kappa'\right]$$
$$b_4 = p\left[3\tau\left(\tau' + \frac{\kappa'\tau}{\kappa}\right) - \frac{\kappa'''}{\kappa}\right]$$
$$b_5 = \tau\left(1 + \frac{3\tau^2}{\kappa}\right) - \frac{\tau''}{\kappa} - \frac{1}{\kappa^2}\left[2\kappa''\tau + \kappa'\left(\tau' - \frac{\kappa'}{\kappa}\right)\right].$$

PROOF. Differentiating (3.6), we have

$$\boldsymbol{\alpha}' = \mathbf{E_1} + p\tau\mathbf{E_2} - \left(\frac{\tau}{\kappa} + \frac{p\kappa'}{\kappa}\right)\mathbf{E_3} \tag{3.7}$$

where $$p = \int \frac{\tau}{\kappa} ds$$
and

$$\boldsymbol{\alpha}'' = -p\kappa^2\tau\mathbf{E_1} + a_1\mathbf{E_2} + a_2\mathbf{E_3} \tag{3.8}$$

where

$$a_1 = 1 + p\tau' + \frac{2\tau}{\kappa}(\tau + p\kappa')$$

$$a_2 = p\left(\tau^2 - \frac{\kappa''}{\kappa}\right) - \frac{1}{\kappa}\left(\tau' + \frac{\kappa'\tau}{\kappa}\right)$$

Substituting (3.7) and (3.8) in (2.2), the curvature of curve $\boldsymbol{\alpha}(s)$ is given by

$$\kappa_\alpha = \frac{\left\{\left[p\tau a_2 + \left(\frac{\tau}{\kappa} + \frac{p\kappa'}{\kappa}\right)a_1\right]^2 + \left[p\tau\kappa(\tau + p\kappa') - a_2\right]^2\kappa^2 + \left[p^2\kappa^2\tau^2 + a_1\right]^2\kappa^2\right\}^{\frac{1}{2}}}{\left[1 + p^2\kappa^2\tau^2 + (\tau + p\kappa')^2\right]^{3/2}}$$

Differentiating (3.8), we get

$$\boldsymbol{\alpha}''' = b_1\mathbf{E_1} + (b_2 + b_3)\mathbf{E_2} + (b_4 + b_5)\mathbf{E_3} \tag{3.9}$$

where

$$b_1 = 2p\kappa[2\kappa'\tau + \kappa\tau' + \kappa(\kappa + 3\tau^2)]$$

$$b_2 = p\left(\tau'' + \frac{3\kappa''\tau}{\kappa} + \frac{3\kappa'\tau'}{\kappa} - \tau^3\right)$$

$$b_3 = \frac{1}{\kappa}\left[3\tau\left(2\tau' + \frac{\kappa'\tau}{\kappa}\right) + \kappa'\right]$$

$$b_4 = p\left[3\tau\left(\tau' + \frac{\kappa'\tau}{\kappa}\right) - \frac{\kappa'''}{\kappa}\right]$$

$$b_5 = \tau\left(1 + \frac{3\tau^2}{\kappa}\right) - \frac{\tau''}{\kappa} - \frac{1}{\kappa^2}\left[2\kappa''\tau + \kappa'\left(\tau' - \frac{\kappa'}{\kappa}\right)\right]$$

Substituting (3.7), (3.8), and (3.9) in (2.2), the torsion of curve $\boldsymbol{\alpha}(s)$ is given by

$$\tau_\alpha = \frac{a_1(b_4 + b_5) - a_2(b_2 + b_3) + p\tau[p\kappa^2\tau(b_4 + b_5) + a_2b_1] + \left(\frac{\tau}{\kappa} + \frac{p\kappa'}{\kappa}\right)[p\kappa^2\tau(b_2 + b_3) + a_1b_1]}{\left[p\tau a_2 + \left(\frac{\tau}{\kappa} + \frac{p\kappa'}{\kappa}\right)a_1\right]^2 + [p\kappa\tau(\tau + p\kappa') - a_2]^2\kappa^2 + (p^2\kappa^2\tau^2 + a_1)^2\kappa^2}.$$

□

THEOREM 3.3. *Let $\boldsymbol{\beta}(s)$ be an analytic space curve parameterised by the arc length s. Then the Frenet frame of the modified helical extension curve $\boldsymbol{\alpha}(s)$ is given by*

$$\mathbf{e_1}_\alpha = \frac{1}{h_1}\left[\mathbf{E_1} + p\tau\mathbf{E_2} - \left(\frac{\tau}{\kappa} + \frac{p\kappa'}{\kappa}\right)\mathbf{E_3}\right]$$

$$\mathbf{e_2}_\alpha = \frac{1}{h_1 h_2}\Bigg\{ \left[\left(\frac{\tau}{\kappa}+\frac{p\kappa'}{\kappa}\right)(a_2 - p\kappa\tau(\tau+p\kappa')) - p\tau(p^2\kappa^2\tau^2+a_1)\right]\mathbf{E_1} + \left[\left(\frac{\tau}{\kappa}+\frac{p\kappa'}{\kappa}\right)\left(p\tau a_2 + \left(\frac{\tau}{\kappa}+\frac{p\kappa'}{\kappa}\right)a_1\right) + p^2\kappa^2\tau^2 + a_1\right]\mathbf{E_2} + \left[p\tau\left(p\tau a_2 + \left(\frac{\tau}{\kappa}+\frac{p\kappa'}{\kappa}\right)a_1\right) - (p\kappa\tau(\tau+p\kappa')-a_2)\right]\mathbf{E_3}\Bigg\}$$

$$\mathbf{e_3}_\alpha = \frac{1}{h_2}\left[p\tau a_2 + \left(\frac{\tau}{\kappa}+\frac{p\kappa'}{\kappa}\right)a_1\right]\mathbf{E_1} + [p\kappa\tau(\tau+p\kappa')-a_2]\mathbf{E_2} + (p^2\kappa^2\tau^2+a_1)\mathbf{E_3}$$

where

$$h_1 = \left[1+p^2\kappa^2\tau^2-(\tau+p\kappa')^2\right]^{\frac{1}{2}}$$

$$h_2 = \left\{\left[p\tau a_2 + \left(\frac{\tau}{\kappa}+\frac{p\kappa'}{\kappa}\right)a_1\right]^2 + [p\kappa\tau(\tau+p\kappa')-a_2]^2\kappa^2 + \left(p^2\kappa^2\tau^2+a_1\right)^2\kappa^2\right\}^{\frac{1}{2}}.$$

PROOF. The proof follows from (2.1), (3.7), and (3.8). □

EXAMPLE 3.4. Let $\boldsymbol{\beta}(s) = \left(\tan^{-1}s,\ \frac{1}{\sqrt{2}}\log(1+s^2),\ s-\tan^{-1}s\right)$, $s\in[-2,2]$ be a curve with unit speed and $\kappa=\tau=\frac{\sqrt{2}}{1+s^2}$. Then its helical extension is given by (Figure 1).

$$\boldsymbol{\alpha}(s) = \left(\tan^{-1}s - \frac{\sqrt{2}s^3}{(1+s^2)^2},\ \frac{1}{\sqrt{2}}\log(1+s^2)+\frac{\sqrt{2}s^2}{(1+s^2)^2},\ s\left(1-\frac{\sqrt{2}}{(1+s^2)^2}\right)-\tan^{-1}s\right).$$

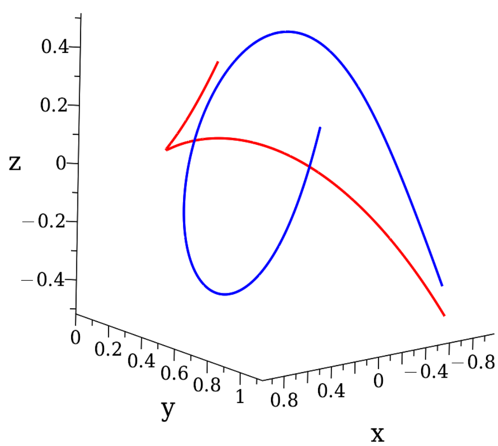

FIGURE 1. Helical extension curve $\boldsymbol{\alpha}(s)$ (Blue) constructed by the curve $\boldsymbol{\beta}(s)$ (Red).

COROLLARY 3.5. *The helical extension curve of a plane curve $\boldsymbol{\beta}(s)$ is the curve iteself.*

THEOREM 3.6. *Let $\boldsymbol{\beta}(s)$ be a general helix and $\boldsymbol{\alpha}(s)$ be its helical extension curve, then*

$$\mathbf{e_1}_\alpha = \frac{1}{h_3}\left[\mathbf{E_1} + cs\tau\mathbf{E_2} - c\left(1+\frac{s\kappa'}{\kappa}\right)\mathbf{E_3}\right]$$

$$\begin{aligned}\mathbf{e_2}_\alpha = \frac{1}{h_3h_4}\Bigg\{&\left[\left(c+\frac{cs\kappa'}{\kappa}\right)(d_2 - cs\kappa\tau(\tau+cs\kappa')) - cs\tau(c^2s^2\kappa^2\tau^2+d_1)\right]\mathbf{E_1}\\ &+\left[c^2\left(1+\frac{s\kappa'}{\kappa}\right)\left(s\tau d_2 + 1 + \frac{s\kappa'}{\kappa}\right)d_1 + (c^2s^2\kappa^2\tau^2+d_1)\right]\mathbf{E_2}\\ &+\left[c^2s\tau\left(s\tau d_2 + \left(1+\frac{s\kappa'}{\kappa}\right)d_1\right) - (cs\kappa\tau(\tau+cs\kappa') - d_2)\right]\mathbf{E_3}\Bigg\}\end{aligned}$$

$$\mathbf{e_3}_\alpha = \frac{1}{h_4}\left\{c\left[s\tau d_2 + \left(1+\frac{s\kappa'}{\kappa}\right)d_1\right]\mathbf{E_1} + [cs\kappa\tau(\tau+cs\kappa') - d_2]\mathbf{E_2} + (c^2s^2\kappa^2\tau^2+d_1)\mathbf{E_3}\right\}$$

where
$$h_3 = \{1 + (cs\kappa\tau)^2 + c^2(\kappa+s\kappa')^2\}^{\frac{1}{2}}$$

$$h_4 = \left\{c^2\left[s\tau d_2 + \left(1+\frac{s\kappa'}{\kappa}\right)d_1\right]^2 + [cs\kappa\tau(\tau+cs\kappa') - d_2]^2\kappa^2 + [(c^2s^2\kappa^2\tau^2+d_1)]^2\kappa^2\right\}^{\frac{1}{2}}$$

and
$$d_1 = 1 + cs\tau' + 2c(\tau + cs\kappa')$$
$$d_2 = cs\left(\tau^2 - \frac{\kappa''}{\kappa}\right) - \frac{1}{\kappa}\left(\frac{\kappa'\tau}{\kappa} + \tau'\right).$$

PROOF. For a general helix $\frac{\tau}{\kappa} = c$, this implies $p = cs$. Substituting these values into (Theorem 3.3) completes the proof. Also the relation between κ_α, h_3 and h_4 is given by

$$\kappa_\alpha = \frac{h_4}{h_3^3}.$$

□

EXAMPLE 3.7. Consider a general helix $\boldsymbol{\beta}(s) = \left(\cos\frac{s}{\sqrt{2}},\ \sin\frac{s}{\sqrt{2}},\ \frac{s}{\sqrt{2}}\right)$, $s \in [-8\pi, 8\pi]$. The binormal vector according to Sasai's modified frame, the curvature and torsion of $\boldsymbol{\alpha}(s)$ is given by

$$\mathbf{E}_3 = \frac{1}{2\sqrt{2}}\left(\sin\frac{s}{\sqrt{2}},\ -\cos\frac{s}{\sqrt{2}}, 1\right)$$

and

$$\kappa = \frac{1}{2}, \qquad \tau = \frac{1}{2}.$$

The helical extension $\boldsymbol{\alpha}(s)$ is given by (Figure 2).

$$\boldsymbol{\alpha}(s) = \left(\cos\frac{s}{\sqrt{2}} - \frac{s}{2\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \sin\frac{s}{\sqrt{2}} + \frac{s}{2\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{s}{2\sqrt{2}}\right).$$

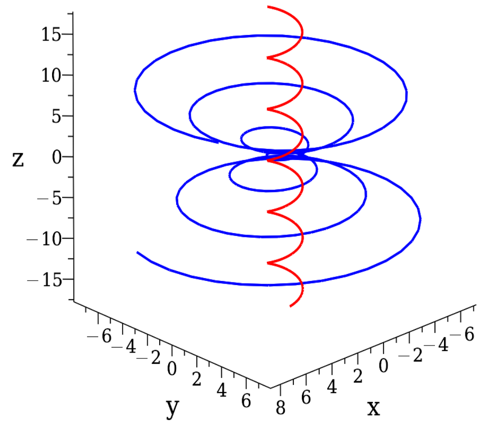

FIGURE 2. Helical extension curve $\boldsymbol{\alpha}(s)$ (Blue) constructed by the general helix $\boldsymbol{\beta}(s)$ (Red).

THEOREM 3.8. *Let $\boldsymbol{\beta}(u) = (r\cos u,\ r\sin u,\ hu)$ be a cylindrical helix with $r > 0$, if $\boldsymbol{\alpha}(u)$ be its helical extension curve, then*

1. *The curve $\boldsymbol{\alpha}(u)$ is parameterised as*

$$\boldsymbol{\alpha}(u) = \left(r\cos u - \frac{h^2\kappa u}{r}\sin u,\ r\sin u + \frac{h^2\kappa u}{r}\cos u,\ hu(1-\kappa)\right)$$

or

$$\boldsymbol{\alpha}(u) = (r\cos u - hu\tau\sin u,\ r\sin u + hu\tau\cos u,\ u(h - r\tau))$$

2. *The distance function $\delta_\alpha = \|\alpha(u)\|$ of $\alpha(u)$ satisfies*

$$\delta_\alpha^2 = r^2 + h^2u^2\left[\frac{h^2\kappa^2}{r^2} + (1-\kappa)^2\right]$$

or

$$\delta_\alpha^2 = r^2 + u^2[h^2\tau^2 + (h - r\tau)^2].$$

PROOF. Given $\boldsymbol{\beta}(u) = (r\cos u,\ r\sin u,\ hu)$, then its curvature and torsion is given by

$$\kappa = \frac{r}{r^2+h^2}, \qquad \tau = \frac{h}{r^2+h^2}$$

The helical extension is given by

$$\boldsymbol{\alpha}(u) = \boldsymbol{\beta}(u) - \int \frac{\tau}{\kappa}\|\beta'(u)\| ds\,\mathbf{E_3}, \tag{3.10}$$

On computing the Sasai's modified binormal vector using (2.1), (2.2), and (2.3), and substituting in (3.10), we have

$$\boldsymbol{\alpha}(u) = \boldsymbol{\beta}(u) - \frac{hu}{r^2+h^2}(h\sin u,\ -h\cos u,\ r)$$

Since, $\kappa = \dfrac{r}{r^2+h^2}$ and $\tau = \dfrac{h}{r^2+h^2}$, we get

$$\boldsymbol{\alpha}(u) = \left(r\cos u - \frac{h^2\kappa u}{r}\sin u,\ r\sin u + \frac{h^2\kappa u}{r}\cos u,\ hu(1-\kappa) \right)$$

or

$$\boldsymbol{\alpha}(u) = \left(r\cos u - hu\tau\sin u,\ r\sin u + hu\tau\cos u,\ u(h-r\tau) \right).$$

A simple calculation further yields

$$\delta_\alpha^2 = r^2 + h^2u^2\left[\frac{h^2\kappa^2}{r^2} + (1-\kappa)^2\right]$$

or

$$\delta_\alpha^2 = r^2 + u^2[h^2\tau^2 + (h-r\tau)^2].$$

□

EXAMPLE 3.9. Let the equation of cylindrical helix is given as follows

$$\boldsymbol{\beta}(s) = \frac{1}{\sqrt{2}}\left(\cos s,\ \sin s,\ s\right), \quad s \in [-2\pi, 2\pi]$$

then the helical extension curve is given by (Figure 3).

$$\boldsymbol{\alpha}(s) = \left(\frac{1}{\sqrt{2}}\cos s - \frac{s}{2}\sin s,\ \frac{1}{\sqrt{2}}\sin s + \frac{s}{2}\sin s,\ \frac{(\sqrt{2}-1)s}{2} \right).$$

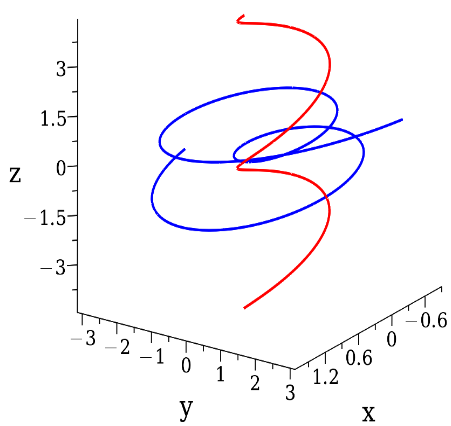

FIGURE 3. Helical extension curve $\boldsymbol{\alpha}(s)$ (Blue) constructed by the cylindrical helix $\boldsymbol{\beta}(s)$ (Red).

COROLLARY 3.10. *The helical extension curve* $\boldsymbol{\alpha}(u)$ *of a cylindrical helix* $\boldsymbol{\beta}(u)$ *is a plane curve if* $\kappa = 1$.

PROOF. On substituting $\kappa = 1$ in (Theorem 3.8), yields

$$\boldsymbol{\alpha}(u) = \left(r\cos u - \frac{h^2 u}{r}\sin u,\ r\sin u + \frac{h^2 u}{r}\cos u,\ 0 \right).$$

□

EXAMPLE 3.11. Let cylindrical helix is given by parameterised equation $\boldsymbol{\beta}(u) = \left(\frac{1}{4}\cos u,\ \frac{1}{4}\sin u,\ \frac{\sqrt{3}}{4}u\right)$, $s \in [-2\pi, 2\pi]$. Then helical extension curve $\boldsymbol{\alpha}(u) = \frac{1}{4}\left(\cos u - 3u\sin u,\ \sin u + 3\cos u,\ 0\right)$ as shown in (Figure 4).

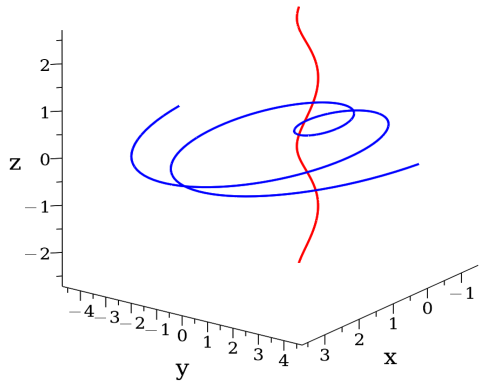

FIGURE 4. $\boldsymbol{\alpha}(u)$ (Blue) is a plane curve for $\kappa = 1$.

THEOREM 3.12. *Let $\boldsymbol{\beta}_m(u)$ is Salkowski curve given by the parameteric equation*

(3.11)

$$\boldsymbol{\beta}_m = \frac{1}{\sqrt{1+m^2}}\Bigg(-\frac{1-n}{4(1+2n)}\sin((1+2n)u) - \frac{1+n}{4(1-2n)}\sin((1-2n)u) - \frac{1}{2}\sin u,$$
$$\frac{1-n}{4(1+2n)}\cos((1+2n)u) + \frac{1+n}{4(1-2n)}\cos((1-2n)u) + \frac{1}{2}\cos u,\ \frac{1}{4m}\cos(2n)u\Bigg)$$

with $u \in (-\frac{\pi}{2n}, \frac{\pi}{2n})$, $m \in \mathbb{R} - \{\pm\frac{1}{\sqrt{3}}, 0\}$, and $n = \frac{m}{\sqrt{1+m^2}}$, the helical extension curve is given by

$$\boldsymbol{\alpha}_m = \boldsymbol{\beta}_m + \frac{\cos(nu)}{n\sqrt{1+m^2}}\Big(-n\cos(nu)\sin u + \cos u\sin(nu),$$
$$n\cos u\cos(nu) + \sin u\sin(nu),\ -\frac{n}{m}\cos(nu)\Big).$$

PROOF. For the curve $\boldsymbol{\beta}_m(u)$, we have

$$\mathbf{E_3} = -\left(-n\cos(nu)\sin u + \cos u\sin(nu),\ n\cos u\cos(nu) + \sin u\sin(nu),\ -\frac{n}{m}\cos(nu)\right)$$

and $\quad \|\beta'(u)\| = \dfrac{\cos(nu)}{\sqrt{1+m^2}}, \qquad \kappa = 1, \qquad \tau = -\tan(nu).$

Substituting the above values in (3.5). □

EXAMPLE 3.13. For the Salkowski curve given by (3.11), the following helical extensions are obtained:

1. For $m = \frac{1}{5}$, $s \in [-3\pi, 3\pi]$ (Figure 5).

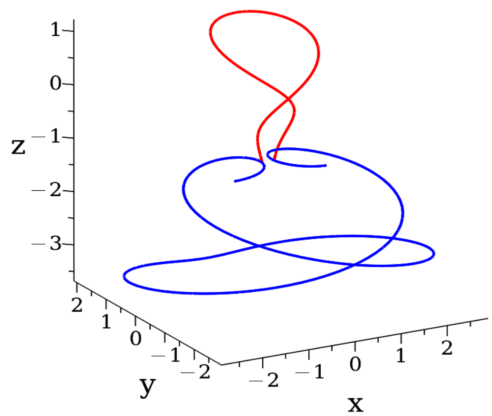

FIGURE 5. Salkowski curve $\boldsymbol{\beta}_m$ (Red) and the corresponding helical extension curve $\boldsymbol{\alpha}_m$ (Blue).

2. For $m = \frac{1}{24}$, with $s \in [-3\pi, 3\pi]$, the Salkowski curve $\boldsymbol{\beta}_m$ is shown in (Figure 6A), and its helical extension is shown in (Figure 6B). Their combined graph is presented in (Figure 6).

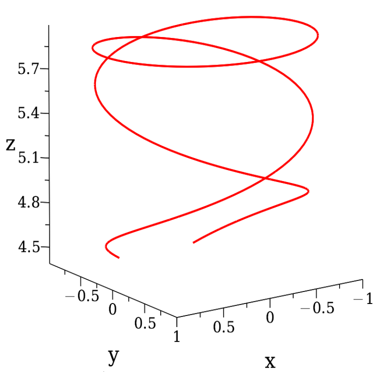

FIGURE 6A. Salkowski curve $\boldsymbol{\beta}_m$ for $m = \frac{1}{24}$.

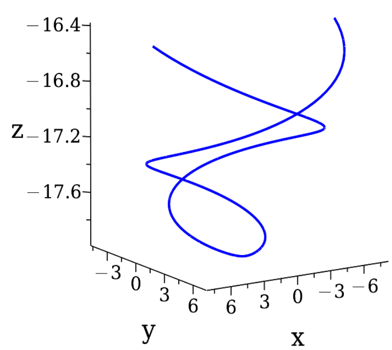

FIGURE 6B. Helical extension curve $\boldsymbol{\alpha}_m$ for $m = \frac{1}{24}$.

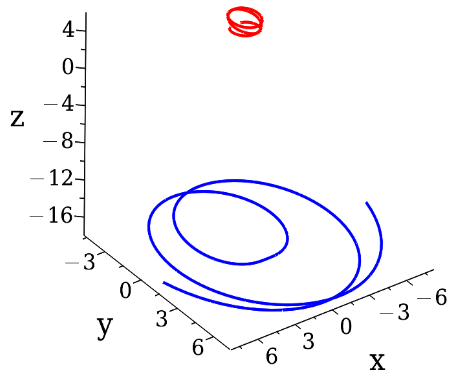

FIGURE 6. Salkowski curve $\boldsymbol{\beta}_m$ (Red) and the corresponding helical extension curve $\boldsymbol{\alpha}_m$ (Blue) for $m = \frac{1}{24}$.

4. EXAMPLE

In this section, we consider an example to illustrate the difference between the helical extension curves obtained using the Frenet frame and Sasai's frame.

EXAMPLE 4.1. Consider the curve $\boldsymbol{\beta}(s) = \left(\frac{3}{5}\sin s,\ \frac{3}{5}\cos s,\ \frac{4}{5}s\right)$. The Frenet frame of $\boldsymbol{\beta}(s)$ is given by

$$\mathbf{e}_1 = \left(\frac{3}{5}\cos s,\ -\frac{3}{5}\sin s,\ \frac{4}{5}\right),$$
$$\mathbf{e}_2 = \left(-\sin s,\ -\cos s,\ 0\right),$$
$$\mathbf{e}_3 = \left(\frac{4}{5}\cos s,\ -\frac{4}{5}\sin s,\ -\frac{3}{5}\right),$$

also,
$$\kappa = \frac{3}{5} \qquad \text{and} \qquad \tau = -\frac{4}{5}.$$

Therefore, Sasai's modified frame of $\boldsymbol{\beta}(s)$ is given by

$$\mathbf{E}_1 = \left(\frac{3}{5}\cos s,\ -\frac{3}{5}\sin s,\ \frac{4}{5}\right),$$
$$\mathbf{E}_2 = \left(-\frac{3}{5}\sin s,\ -\frac{3}{5}\cos s,\ 0\right),$$
$$\mathbf{E}_3 = \left(\frac{12}{25}\cos s,\ -\frac{12}{25}\sin s,\ -\frac{9}{25}\right).$$

We obtain following helical extension curves:

1. According to the Frenet frame

$$\boldsymbol{\gamma}(s) = \left(\frac{3}{5}\sin s + \frac{16}{15}s\cos s,\ \frac{3}{5}\cos s - \frac{16}{15}s\sin s,\ 0\right).$$

The curve $\boldsymbol{\gamma}(s)$ is a plane curve and it is plotted in green in (Figure 7).

2. According to Sasai's modified frame

$$\boldsymbol{\alpha}(s) = \left(\frac{3}{5}\sin s + \frac{16}{25}s\cos s,\ \frac{3}{5}\cos s - \frac{16}{25}s\sin s,\ \frac{8}{25}s\right).$$

The curve $\boldsymbol{\alpha}(s)$ is plotted in blue in (Figure 7).

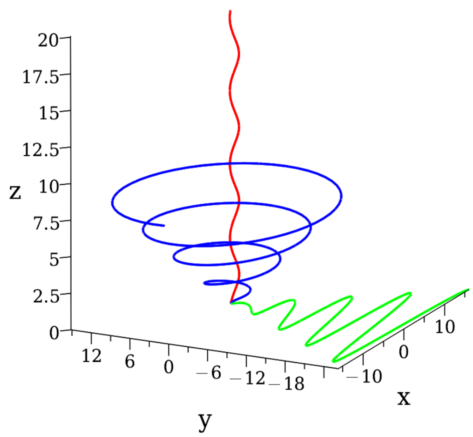

FIGURE 7. Helical extension curves according to Frenet frame $\boldsymbol{\gamma}(s)$ (Green) and Sasai's frame $\boldsymbol{\alpha}(s)$ (Blue)

5. CONCLUSION

This work investigates the construction of a new class of curve through Sasai's modified frame. Using the condition $\frac{\tau}{\kappa} = c$, we examined the helical extension curve of general helices and found their Frenet apparatus. For cylindrical helices, we gave exact formulae for both the extension curve and its distance function. We also constructed the helical extension curve for Salkowski curves with constant curvature $\kappa \equiv 1$. Some examples are provided with their illustrative figures. This approach presents a clear way to build new curves from existing ones, adding useful tools to differential geometry. Such ideas can be applied in real-world areas like computer-aided geometric design (CAGD) and computer-aided design (CAD), where controlling curvature and torsion is important for engineering, manufacturing, and digital modelling. To the best of our knowledge (Google Scholar), helical extension curves have not yet been constructed or studied using Sasai's modified frame; therefore, our study provides a new perspective on this topic. Future work may focus on constructing and analysing helical extension curves using different frames, such as the quasi frame and the Bishop frame, or on generalising the concept to Minkowski space.

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