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On a variant and extension of Gabler inequality

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ON A VARIANT AND EXTENSION OF GABLER INEQUALITY

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ABSTRACT. We propose a Jensen-Mercer type variant and a Niezgoda type extension of Gabler inequality along with applications.

1. Introduction and Preliminaries

It would not be an exaggeration if we say that "Jensen inequality is among the most celebrated inequalities of all time". That is why most of the researcher are continuously working on this inequality for long time. In recent past we can find number of different variants, extensions, generalizations and refinements of this renowned inequality, for reference see [1, 2, 3, 8, 10, 11, 12, 15, 16, 18, 19, 20, 23, 32, 33, 34, 35] and the references given therein. We also adduce to [26] and [30] for detailed discussion on Jensen's inequality and for some remarks on literature and history of the topic. Throughout the article we assume that J is an interval in $\mathbb{R}$ and for weights $w_1, \ldots, w_n$ we define the notation

$$W_i = \sum_{j=1}^{i} w_j, \ i \in \{1, \ldots, n\} \quad \text{and clearly} \quad W_n = \sum_{j=1}^{n} w_j.$$

Now we start with Jensen's inequality [30].

Proposition 1. *Let $\mathbf{x}$ be a n-tuple with $x_i \in J$ for $i \in \{1, \ldots, n\}$ and let $\mathbf{w}$ is a nonnegative n-tuple with $W_n > 0$, then for a convex function f on J following inequality holds*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i). \tag{1}$$

The following variant of the Jensen's inequality was introduced by Mercer in [25], which is usually referred as "Jensen-Mercer inequality".

Proposition 2. *Let all the assumptions of Proposition 1 be true, the following inequality holds*

$$f\left(a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq f(a) + f(b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i), \tag{2}$$

where

$$a = \min_{x_i \in J}\{x_i\} \quad \text{and} \quad b = \max_{x_i \in J}\{x_i\}.$$

Before going on to our next preliminary, let us recall a prerequisite concept of majorization from [24].
Let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ denote two m-tuples and $x_{[1]} \geq \cdots \geq x_{[m]}$, $y_{[1]} \geq \cdots \geq y_{[m]}$, be their ordered components.

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Definition 1. For $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$,

$$\mathbf{x} \prec \mathbf{y} \quad \text{if} \quad \begin{cases} \sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} \quad , \quad k \in \{1, \ldots, m-1\}, \\ \sum_{i=1}^{m} x_{[i]} = \sum_{i=1}^{m} y_{[i]} \end{cases}$$

when $\mathbf{x} \prec \mathbf{y}$, $\mathbf{x}$ *is said to be majorized by* $\mathbf{y}$ or $\mathbf{y}$ *majorizes* $\mathbf{x}$.

This notion and notation of majorization was first introduced by Hardy et al. in [14]. In the same book [14] we find a very power result namely majorization theorem (see also [24]).

Theorem 1. *Let* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$. *Then following inequality is true for all continuous convex functions* $f : \mathbb{R} \to \mathbb{R}$,

$$\sum_{i=1}^{n} f(x_i) \le \sum_{i=1}^{n} f(y_i)$$

if and only if $\mathbf{x} \prec \mathbf{y}$.

We now state Niezgoda's inequality which is actually an extension of (2) by Niezgoda [28].

Proposition 3. *Suppose that* $\boldsymbol{a}$ *be an* m*-tuple such that* $a_i \in J$ *and let* $\boldsymbol{X} = (\mathbf{x}_j) = (x_{ij})$ $n \times m$ *be a matrix with* $x_{ij} \in J$ *for all* $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, m\}$.

If $\boldsymbol{a}$ *majorizes each row of* $\boldsymbol{X}$, *that is,*

$$\mathbf{x}_{i.} = (x_{i1}, \ldots, x_{im}) \prec (a_1, \ldots, a_m) = \boldsymbol{a} \text{ for each } i \in \{1, \ldots, n\},$$

then for a continuous convex function f *on* J *following inequality holds:*

$$f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i x_{ij}\right) \le \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i f(x_{ij}), \tag{3}$$

with $w_i \ge 0$.

In [13] Gabler defined a special case of convex functions namely sequentially convex functions by employing the following double index function.

Definition 2. For $\mathbf{x} = (x_1, \ldots, x_n) \in J^n$ and a real-valued function $f : J \to \mathbb{R}$, define

$$f_{k,n} = f_{k,n}(\mathbf{x}) = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} f\left(\frac{1}{k}(x_{i_1} + \cdots + x_{i_k})\right). \tag{4}$$

Gabler termed this double index function as an arithmetic mean of all possible convex functions generated by arithmetic means of any k values chosen from $(x_1, \ldots, x_n)$. Gabler then defined sequentially convex functions as follows:

Definition 3. Let $f : J \to \mathbb{R}$ and let $f_{k,n}$ be defined as in (4), then f is said to be sequentially convex if $(f_{k,n})$ is a convex sequence in k for all $n > 2$ and all $x_1, \ldots, x_n \in J$.

While investigating sequentially convex functions Gabler also made the following important observation which we shall call as Gabler inequality.

Proposition 4. *For a sequentially convex function* f *of type* (4) *the following inequality holds where* $k \in \{1, \ldots, n-1\}$

$$f_{k,n} \ge f_{k+1,n}. \tag{5}$$

Through the proof in [13] it is interesting to notice that (5) is also true for midconvex functions see [27, 31].
It was 1994 when Pečarić upgraded the double index function (4) and came up with the following weighted version see [29].

Definition 4. For $\mathbf{x} = (x_1, \ldots, x_n) \in J^n$ and a real-valued function $f : J \to \mathbb{R}$, define

$$f_{k,n} = f_{k,n}(\mathbf{x}, w) = \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \le i_1 < \cdots < i_k \le n} (w_{i_1} + \cdots + w_{i_k}) f\left(\frac{w_{i_1} x_{i_1} + \cdots + w_{i_k} x_{i_k}}{w_{i_1} + \cdots + w_{i_k}}\right) \tag{6}$$

where w_i's are positive weights for $i \in \{1, \ldots, n\}$.

In the same article the Gabler result was further strengthen by defining it for convex functions in the following way.

Proposition 5. *For a convex function f of type* (6) *the following inequality holds where $k \in \{1, \ldots, n-1\}$*

$$f_{k,n} \ge f_{k+1,n} \tag{7}$$

Furthermore, it was proved that the inequality (7) is an interpolating inequality for Jensen's inequality. i.e;

$$f\left(\frac{1}{W_n} \sum_{i=1}^{n} w_i x_i\right) = f_{n,n} \le \cdots \le f_{k+1,n} \le f_{k,n} \le \cdots \le f_{1,n} = \frac{1}{W_n} \sum_{i=1}^{n} w_i f(x_i). \tag{8}$$

For recent work on Gabler inequality we refer the reader to [6].

In the present article a variant of Gabler inequality in terms of Jensen-Mercer inequality and its extension via Niezgoda's inequality will be stated along with some refinements similar to (8).

This article is organized in the following manner. The first section states preliminaries and introduction. In second section we give a variant of Gabler inequality through Jensen-Mercer inequality. Third section is devoted to an extension of Gabler inequality via Niezgoda's inequality. While the forth section is devoted to applications of our obtained results in terms of generalized means.

2. Jensen-Mercer Type Variant of Gabler Inequality

Theorem 2. *Under the assumptions of Proposition* 2*, if we define*

$$f_{k,n} = f_{k,n}(\mathbf{x}, w, a, b) = \frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \le i_1 < \cdots < i_k \le n} (w_{i_1} + \cdots + w_{i_k}) f\left(a + b - \frac{w_{i_1} x_{i_1} + \cdots + w_{i_k} x_{i_k}}{w_{i_1} + \cdots + w_{i_k}}\right), \tag{9}$$

then the inequality (7) *holds.*

Proof. By using the definition of convex functions and rearrangements we have

$$\begin{aligned}
&\left(w_{i_1} + \cdots + w_{i_{k+1}}\right) f\left(a + b - \frac{w_{i_1} x_{i_1} + \cdots + w_{i_{k+1}} x_{i_{k+1}}}{w_{i_1} + \cdots + w_{i_{k+1}}}\right) \\
&= \left(w_{i_1} + \cdots + w_{i_{k+1}}\right) f\left[\frac{1}{\sum_{l=1}^{k+1} \left(w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}\right)}\right. \\
&\times \left.\sum_{l=1}^{k+1} \left(w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}\right) \left(a + b - \frac{w_{i_1} x_{i_1} + \cdots + w_{i_{k+1}} x_{i_{k+1}} - w_{i_l} x_{i_l}}{w_{i_1} + \cdots + w_{i_{k+1}} - w_{i_l}}\right)\right]
\end{aligned}$$

$$\leq \left(w_{i_1}+\cdots+w_{i_{k+1}}\right)\left[\frac{1}{\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)}\right.$$
$$\left.\times\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)f\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_{k+1}}x_{i_{k+1}}-w_{i_l}x_{i_l}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right)\right]$$
$$=\frac{1}{k}\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)f\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_{k+1}}x_{i_{k+1}}-w_{i_l}x_{i_l}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right).$$

In order to use above result we consider

$$f_{k+1,n}=\frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n}\left(w_{i_1}+\cdots+w_{i_{k+1}}\right)f\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_{k+1}}x_{i_{k+1}}}{w_{i_1}+\cdots+w_{i_{k+1}}}\right)$$
$$\leq\frac{1}{k\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n}\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)$$
$$\times f\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_{k+1}}x_{i_{k+1}}-w_{i_l}x_{i_l}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right)$$
$$=\frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n}\left(w_{i_1}+\cdots+w_{i_k}\right)f\left(a+b-\frac{w_{i_1}x_{i_1}+\cdots+w_{i_k}x_{i_k}}{w_{i_1}+\cdots+w_{i_k}}\right)=f_{k,n}.$$

Which concludes our proof. □

Corollary 1. *Similar to* (8)*, the following refinement for* (2) *using* (9) *can be defined.*

$$f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right)=f_{n,n}$$
$$\leq\cdots\leq f_{k+1,n}\leq f_{k,n}\leq\cdots\leq f_{1,n}\leq f(a)+f(b)-\frac{1}{W_n}\sum_{i=1}^{n}w_if(x_i).$$

Proof. For $k=n$, double index function (9) yields to

$$f_{n,n}=\frac{1}{\binom{n-1}{n-1}W_n}\left(w_1+\cdots+w_n\right)f\left(a+b-\frac{w_1x_1+\cdots+w_nx_n}{w_1+\cdots+w_n}\right)$$
$$=f\left(a+b-\frac{1}{W_n}\sum_{i=1}^{n}w_ix_i\right).$$

Similarly for $k=1$, double index function (9) and Lemma 1.3 of [25] gives us

$$f_{1,n}=\frac{1}{\binom{n-1}{1-1}W_n}\sum_{i_1=1}^{n}w_{i_1}f\left(a+b-\frac{w_{i_1}x_{i_1}}{w_{i_1}}\right)$$
$$=\frac{1}{W_n}\sum_{i=1}^{n}w_if\left(a+b-x_i\right).$$
$$\leq\frac{1}{W_n}\sum_{i=1}^{n}w_i\left[f(a)+f(b)-f(x_i)\right]$$

$$= f(a) + f(b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i).$$

Above calculations in accordance to inequality (7) concludes our proof. □

3. Niezgoda Type Extension of Gabler Inequality

Theorem 3. *Under the assumptions of Proposition* 3*, if we define*

$$\begin{aligned} f_{k,n} &= f_{k,n}(\mathbf{x}, \mathbf{a}, w) \\ &= \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\le i_1<\cdots<i_k\le n}(w_{i_1}+\cdots+w_{i_k})\, f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1}+\cdots+w_{i_k}x_{i_k}}{w_{i_1}+\cdots+w_{i_k}}\right) \end{aligned} \tag{10}$$

then the inequality (7) *holds.*

Proof. By using the definition of convex functions and rearrangements we have

$$\begin{aligned} &\left(w_{i_1}+\cdots+w_{i_{k+1}}\right) f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_{k+1}}x_{i_{k+1} j}}{w_{i_1}+\cdots+w_{i_{k+1}}}\right) \\ &= \left(w_{i_1}+\cdots+w_{i_{k+1}}\right) f\Bigg[\frac{1}{\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)} \\ &\times \sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_{k+1}}x_{i_{k+1} j}-w_{i_l}x_{i_l j}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right)\Bigg] \\ &\le \left(w_{i_1}+\cdots+w_{i_{k+1}}\right)\Bigg[\frac{1}{\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right)} \\ &\times \sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right) f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_{k+1}}x_{i_{k+1} j}-w_{i_l}x_{i_l j}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right)\Bigg] \\ &= \frac{1}{k}\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right) f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_{k+1}}x_{i_{k+1} j}-w_{i_l}x_{i_l j}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right). \end{aligned}$$

In order to use the above result we consider

$$\begin{aligned} f_{k+1,n} &= \frac{1}{\binom{n-1}{k}W_n}\sum_{1\le i_1<\cdots<i_{k+1}\le n}\left(w_{i_1}+\cdots+w_{i_{k+1}}\right) \\ &\times f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_{k+1}}x_{i_{k+1} j}}{w_{i_1}+\cdots+w_{i_{k+1}}}\right) \\ &\le \frac{1}{k\binom{n-1}{k}W_n}\sum_{1\le i_1<\cdots<i_{k+1}\le n}\sum_{l=1}^{k+1}\left(w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}\right) \\ &\times f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\frac{w_{i_1}x_{i_1 j}+\cdots+w_{i_{k+1}}x_{i_{k+1} j}-w_{i_l}x_{i_l j}}{w_{i_1}+\cdots+w_{i_{k+1}}-w_{i_l}}\right) \end{aligned}$$

$$= \frac{1}{\binom{n-1}{k-1}W_n} \sum_{1\leq i_1<\cdots<i_k\leq n} (w_{i_1} + \cdots + w_{i_k})\, f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \frac{w_{i_1}x_{i_1 j} + \cdots + w_{i_k}x_{i_k j}}{w_{i_1} + \cdots + w_{i_k}}\right) = f_{k,n}.$$

Which concludes our proof. □

Corollary 2. *Similar to* (8)*, the following refinement for* (3) *using* (10) *can be defined.*

$$f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) = f_{n,n} \leq \cdots \leq f_{k+1,n}$$

$$\leq f_{k,n} \leq \cdots \leq f_{1,n} \leq \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f(x_{ij}).$$

Proof. For $k = n$, double index function (10) yields to

$$\begin{aligned} f_{n,n} &= \frac{1}{\binom{n-1}{n-1}W_n}(w_1 + \cdots + w_n)\, f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \frac{w_1 x_{1j} + \cdots + w_n x_{nj}}{w_1 + \cdots + w_n}\right) \\ &= f\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) \end{aligned}$$

Now by using similar technique for majorization theorem as in Theorem 2.1 of [28] and double index function (10) for $k = 1$, we have

$$\begin{aligned} f_{1,n} &= \frac{1}{\binom{n-1}{1-1}W_n}\sum_{i_1=1}^{n} w_{i_1} f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \frac{w_{i_1}x_{i_1 j}}{w_{i_1}}\right) \\ &= \sum_{i=1}^{n} w_i f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij}\right) \\ &\leq \sum_{j=1}^{m} f(a_j) - \frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f(x_{ij}). \end{aligned}$$

Above calculations in accordance to inequality (7) concludes our proof. □

Remark 1. If we set $k = m = 2$, $a_1 = a$, $a_2 = b$ and $x_{i1} = x_i$ for $i \in \{1, \ldots, n\}$, then Theorem 2 and related results will become special cases of Theorem 3.

4. **Applications**

4.1. **For Jensen-Mercer Type Variant of Gabler Inequality.** For $[a, b] \subset J$, $0 < a < b$ and positive weights w_i for $i \in \{1, \ldots, \alpha\}$ where $\alpha \in \{1, \ldots, n\}$, we define the following (modified) arithmetic, geometric and harmonic means along with power mean of order $r \in \mathbb{R}$ for all $x_i \in [a,b]$.

$$A(x_1,\dots,x_\alpha;w_1,\dots,w_\alpha) = a+b-\frac{1}{W_\alpha}\sum_{i=1}^{\alpha} w_i x_i,$$

$$G(x_1,\dots,x_\alpha;w_1,\dots,w_\alpha) = \frac{ab}{(\prod_{i=1}^{\alpha} x_i^{w_i})^{\frac{1}{W_\alpha}}},$$

$$H(x_1,\dots,x_\alpha;w_1,\dots,w_\alpha) = \left(a^{-1}+b^{-1}-\frac{1}{W_\alpha}\sum_{i=1}^{\alpha} w_i \frac{1}{x_i}\right)^{-1}$$

$$M^{[r]} = (x_1,\dots,x_\alpha;w_1,\dots,w_\alpha) = \begin{cases} \left(a^r+b^r-\frac{1}{W_\alpha}\sum_{1=1}^{\alpha} w_i x_i^r\right)^{\frac{1}{r}}, & r\neq 0,\\ G(x_1,\dots,x_\alpha;w_1,\dots,w_\alpha), & r=0.\end{cases}$$

Also for $i \in \{1,\dots,n\}$ and $p \in \{1,\dots k, k+1\dots,n\}$ we define

$$\begin{aligned} A_{p,n} &= A(x_{i_1},\dots,x_{i_p};w_{i_1},\dots,w_{i_p}),\\ G_{p,n} &= G(x_{i_1},\dots,x_{i_p};w_{i_1},\dots,w_{i_p}),\\ H_{p,n} &= H(x_{i_1},\dots,x_{i_p};w_{i_1},\dots,w_{i_p}),\\ M_{p,n}^{[r]} &= M_{[r]}\left(x_{i_1},\dots,x_{i_p};w_{i_1},\dots,w_{i_p}\right).\end{aligned}$$

For $x_{i_p} \mapsto \left(1-x_{i_p}\right)$, $a \mapsto (1-a)$ and $b \mapsto (1-b)$ we propose

$$\begin{aligned} A'_{p,n} &= A(1-x_{i_1},\dots,1-x_{i_p};w_{i_1},\dots,w_{i_p}),\\ G'_{p,n} &= G(1-x_{i_1},\dots,1-x_{i_p};w_{i_1},\dots,w_{i_p}),\\ H'_{p,n} &= H(1-x_{i_1},\dots,1-x_{i_p};w_{i_1},\dots,w_{i_p}).\end{aligned}$$

Clearly for $n=p$,

$$A_{n,n} = a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i = A_n,$$

$$G_{n,n} = \frac{ab}{(\prod_{i=1}^{n} x_i^{w_i})^{\frac{1}{W_n}}} = G_n,$$

$$H_{n,n} = \left(a^{-1}+b^{-1}-\frac{1}{W_n}\sum_{i=1}^{n} w_i \frac{1}{x_i}\right)^{-1} = H_n,$$

$$M_{n,n}^{[r]} = \begin{cases} \left(a^r+b^r-\frac{1}{W_n}\sum_{1=1}^{n} w_i x_i^r\right)^{\frac{1}{r}}, & r\neq 0,\\ G_{n,n}, & r=0,\end{cases} = M^{[n]}$$

$$A'_{n,n} = (1-a)+(1-b)-\frac{1}{W_n}\sum_{i=1}^{n} w_i\left(1-x_i\right) = A'_n,$$

$$G'_{n,n} = \frac{(1-a)(1-b)}{(\prod_{i=1}^{n}\left(1-x_i\right)^{w_i})^{\frac{1}{W_n}}} = G'_n,$$

$$H'_{n,n} = \left((1-a)^{-1}+(1-b)^{-1}-\frac{1}{W_n}\sum_{i=1}^{l} w_i\frac{1}{(1-x_i)}\right)^{-1} = H'_n.$$

We now introduce mixed symmetric means as follows:

$$M_{k,n}^{[s,r]} = \begin{cases} \left(\frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \le i_1 < \cdots < i_k \le n} \left(\sum_{l=1}^{k} w_{i_l} \right) \left(M_{k,n}^{[r]} \right)^s \right)^{\frac{1}{s}}, & s \neq 0, \\ \left(\prod_{1 \le i_1 < \cdots < i_k \le n} M_{k,n}^{[r]} \right)^{\frac{1}{\binom{n-1}{k-1} W_n}}, & s = 0, \end{cases}$$

Under the assumptions of Theorem 2 here we state a couple of applications starting with the following refinement series of the arithmetic-geometric and Ky Fan inequalities (see [7] and reference therein):

Theorem 4. *(i)*

$$A_n \le \cdots \le \left(\prod_{1 \le i_1 < \cdots < i_{k+1} \le n} (A_{k+1,n})^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k} W_n}}$$

$$\le \left(\prod_{1 \le i_1 < \cdots < i_k \le n} (A_{k,n})^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1} W_n}}$$

$$\le \cdots \le \left(\prod_{i=1}^{n} (A_{1,n})^{w_i} \right)^{\frac{1}{W_n}} \le G_n.$$

(ii)

$$\frac{A'_n}{A_n} \le \cdots \le \left(\prod_{1 \le i_1 < \cdots < i_{k+1} \le n} \left(\frac{A'_{k+1,n}}{A_{k+1,n}} \right)^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k} W_n}}$$

$$\le \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(\frac{A'_{k,n}}{A_{k,n}} \right)^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1} W_n}}$$

$$\le \cdots \le \left(\prod_{i=1}^{n} \left(\frac{A'_{1,n}}{A_{1,n}} \right)^{w_i} \right)^{\frac{1}{W_n}} \le \frac{G'_n}{G_n},$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$.

Proof.
(i) By apply convex function $f(x) = -\ln(x)$, to Corollary 1 we obtain required result.

(ii) For $x \in (0, \frac{1}{2}]$, applying convex function $f(x) = \ln\left(\frac{1-x}{x}\right)$ to the Corollary 1 we get,

$$\ln\left(\frac{1-a-b+\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i}{a+b-\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i}\right) \leq \cdots \leq \frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n} W_{i_{k+1}} \ln\left(\frac{1-a-b+\frac{1}{W_{i_{k+1}}}\sum_{l=1}^{k+1} w_{i_l}x_{i_l}}{a+b-\frac{1}{W_{i_{k+1}}}\sum_{l=1}^{k+1} w_{i_l}x_{i_l}}\right)$$

$$\leq \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n} W_{i_k} \ln\left(\frac{1-a-b+\frac{1}{W_{i_k}}\sum_{l=1}^{k} w_{i_l}x_{i_l}}{a+b-\frac{1}{W_{i_k}}\sum_{l=1}^{k} w_{i_l}x_{i_l}}\right)$$

$$\leq \cdots \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i \ln\left(\frac{1-a-b+x_i}{a+b-x_i}\right)$$

$$\leq \ln\left(\frac{1-a}{a}\right) + \ln\left(\frac{1-b}{b}\right) - \frac{1}{W_n}\sum_{i=1}^{n} w_i \ln\left(\frac{1-x_i}{x_i}\right).$$

Consequently,

$$\ln\left(\frac{A'_n}{A_n}\right) \leq \cdots \leq \ln\left(\prod_{1\leq i_{k+1}<\cdots<i_1\leq n}\left(\frac{A'_{k+1}}{A_{k+1}}\right)^{W_{i_{k+1}}}\right)^{\frac{1}{\binom{n-1}{k}W_n}}$$

$$\leq \ln\left(\prod_{1\leq i_k<\cdots<i_1\leq n}\left(\frac{A'_k}{A_k}\right)^{W_{i_k}}\right)^{\frac{1}{\binom{n-1}{k-1}W_n}}$$

$$\leq \cdots \leq \ln\left(\prod_{i=1}^{n}\left(\frac{A'_{1,n}}{A_{1,n}}\right)^{w_i}\right)^{\frac{1}{W_n}} \leq \ln\left(\frac{G'_n}{G_n}\right).$$

Finally, by taking exponential we deduced our result. □

The refinements series of the variant of arithmetic-geometric inequality is given as:

Theorem 5.

$$\frac{G_n}{G_n+G'_n} \leq \cdots \leq \frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n} W_{i_{k+1}}\left(\frac{G_{k+1,n}}{G_{k+1,n}+G'_{k+1,n}}\right)$$

$$\leq \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n} W_{i_k}\left(\frac{G_{k,n}}{G_{k,n}+G'_{k,n}}\right)$$

$$\leq \cdots \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\frac{G_{1,n}}{G_{1,n}+G'_{1,n}}\right) \leq A_n,$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}.$

Proof. By applying the convex function $f(x) = \frac{1}{1+\exp x}$ to Corollary 1 and replacing a by $\ln\frac{1-a}{a}$, b by $\ln\frac{1-b}{b}$ and x_{i_l} by $\ln\frac{1-x_{i_l}}{x_{i_l}}$ we obtain the result. □

Now, we present refinement series of the arithmetic and harmonic mean as follow:

Theorem 6.
(i)

$$\frac{1}{A_n} \leq \cdots \leq \frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n} W_{i_{k+1}}\left(\frac{1}{A_{k+1,n}}\right) \leq \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n} W_{i_k}\left(\frac{1}{A_{k,n}}\right) \leq \cdots \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\frac{1}{A_{1,n}}\right) \leq \frac{1}{H_n}.$$

(ii)

$$\frac{1}{A'_n} \leq \cdots \leq \frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n} W_{i_{k+1}}\left(\frac{1}{A'_{k+1,n}}\right) \leq \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n} W_{i_k}\left(\frac{1}{A'_{k,n}}\right) \leq \cdots \leq \frac{1}{W_n}\sum_{1}^{n} w_i\left(\frac{1}{A'_{1,n}}\right) \leq \frac{1}{H'_n},$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$.

Proof.
(i) By applying convex function $f(x) = \frac{1}{x}$ to Corollary 1 we obtain required result.
(ii) By applying convex function $f(x) = \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Corollary 1 we obtain required result. □

In the following theorem we establish a refinement series of the difference of the arithmetic and harmonic mean.

Theorem 7.

$$\frac{1}{A_n} - \frac{1}{A'_n} \leq \cdots \leq \frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n} W_{i_{k+1}}\left(\frac{1}{A_{k+1,n}} - \frac{1}{A'_{k+1,n}}\right) \leq \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n} W_{i_k}\left(\frac{1}{A_{k,n}} - \frac{1}{A'_{k,n}}\right) \leq \cdots \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\frac{1}{A_{1,n}} - \frac{1}{A'_{1,n}}\right) \leq \frac{1}{H_n} - \frac{1}{H'_n},$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$.

Proof. By applying convex function $f(x) = \frac{1}{x} - \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Corollary 1 we obtain required result. □

We now prove some results in terms of mixed symmetric means.

Theorem 8. *Let $r, s \in \mathbb{R}$ such that $s \leq r$. Then we have*

$$M_s = M_{n,n}^{[r,s]} \leq \cdots \leq M_{k+1,n}^{[r,s]} \leq M_{k,n}^{[r,s]} \leq \cdots \leq M_{1,n}^{[r,s]} \leq M_r. \tag{11}$$

$$M_s \leq M_{1,n}^{[s,r]} \leq \cdots \leq M_{k,n}^{[s,r]} M_{k+1,n}^{[s,r]} \leq \cdots \leq M_{n,n}^{[s,r]} = M_r. \tag{12}$$

Proof. Let $r, s \neq 0$. By applying the function $\phi(x) = x^{\frac{s}{r}}$, in the Corollary 1 and replacing a, b and x_{i_l} by a^r, b^r and $(x_{i_l})^r$ respectively, and then raising the power $\frac{1}{s}$, we get (11). Similarly, using the function $\phi(x) = x^{\frac{r}{s}}$ in the Corollary 1 and replacing a, b and x_{i_l} by a^s, b^s and $(x_{i_l})^s$ respectively then raising the power $\frac{1}{r}$, we get (12). For $s = 0$ or $r = 0$, we obtain the required result by taking limit. □

Let $\phi, \psi : J \to \mathbb{R}$ be continuous strictly monotonic functions. We define the quasi-arithmetic means with respect to Theorem 2, as follow:

$$\hat{M}_{k,n}^{[\phi,\psi]} = \phi^{-1}\left[\frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n}\left(\sum_{l=1}^{k} w_{i_l}\right)\right.$$
$$\left.\times\left(\phi o\psi^{-1}\right)\left(\psi(a) + \psi(b) - \frac{\sum_{l=1}^{k} w_{i_l}\psi(x_{i_l})}{\sum_{l=1}^{k} w_{i_l}}\right)\right], \tag{13}$$

where $\phi o\psi^{-1}$ is convex function.

Corollary 3. *If we define a continuous and strictly monotonic function $\varphi : J \to \mathbb{R}$ as*

$$\hat{M}^{[\varphi]} = \varphi^{-1}\left[\varphi(a) + \varphi(b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i\varphi(x_i)\right],$$

then the following monotonicity of generalized quasi-arithmetic means holds

$$\hat{M}^{[\phi]} \geq \hat{M}_{1,n}^{[\phi,\psi]} \geq \cdots \geq \hat{M}_{k,n}^{[\phi,\psi]} \geq \hat{M}_{k+1,n}^{[\phi,\psi]} \geq \cdots \geq \hat{M}_{n,n}^{[\phi,\psi]} = \hat{M}^{[\psi]}. \tag{14}$$

Proof. Setting $f \mapsto \phi o\psi^{-1}$ and replacing a, b and x_{i_l} by $\psi(a), \psi(b)$ and $\psi(x_{i_l})$ respectively in Corollary 1 and then applying ϕ^{-1} we get (14). □

4.2. **For Niezgoda Type Extension of Gabler Inequality.** For $[a, b] \subset J$, $0 < a < b$ and positive weights w_i for $i \in \{1, \ldots, \alpha\}$, where $\alpha \in \{1, \ldots, n\}$, we define the following (generalized) arithmetic, geometric and harmonic means along with power mean of the order $r \in \mathbb{R}$ for all $x_{ij} \in [a, b]$, $j \in \{1, \ldots, m\}$.

$$A(x_{1j}, \ldots, x_{\alpha j}; w_1, \ldots, w_\alpha) = \sum_{j=1}^{m} a_j - \frac{1}{W_\alpha}\sum_{j=1}^{m-1}\sum_{i=1}^{\alpha} w_i x_{ij},$$

$$G(x_{1j}, \ldots, x_{\alpha j}; w_1, \ldots, w_\alpha) = \frac{\prod_{j=1}^{m} a_j}{\left(\prod_{j=1}^{m-1}\prod_{i=1}^{\alpha} x_{ij}^{w_i}\right)^{\frac{1}{W_\alpha}}},$$

$$H(x_{1j}, \ldots, x_{\alpha j}; w_1, \ldots, w_\alpha) = \left(\sum_{j=1}^{m} a_j^{-1} - \frac{1}{W_\alpha}\sum_{j=1}^{m-1}\sum_{i=1}^{\alpha} w_i\frac{1}{x_{ij}}\right)^{-1}.$$

$$M^{[r]} = (x_{1j}, \ldots, x_{\alpha j}; w_1, \ldots, w_\alpha) = \begin{cases} \left(\sum_{j=1}^{m} (a_j)^r - \frac{1}{W_\alpha} \sum_{j=1}^{m-1} \sum_{1=1}^{\alpha} w_i (x_{ij})^r\right)^{\frac{1}{r}}, & r \neq 0, \\ G(x_{1j}, \ldots, x_{\alpha j}; w_1, \ldots, w_\alpha), & r = 0, \end{cases}$$

Also for $i \in \{1, \ldots, n\}$ and $p \in \{1, \ldots k+1, k \ldots, n\}$ we define

$$\begin{aligned} A_{p,n} &= A(x_{i_1 j}, \ldots, x_{i_p j}; w_{i_1}, \ldots, w_{i_p}), \\ G_{p,n} &= G(x_{i_1 j}, \ldots, x_{i_p j}; w_{i_1}, \ldots, w_{i_p}), \\ H_{p,n} &= H(x_{i_1 j}, \ldots, x_{i_p j}; w_{i_1}, \ldots, w_{i_p}), \\ M_{p,n}^{[r]} &= M_{[r]}\left(x_{i_1 j}, \ldots, x_{i_p j}; w_{i_1}, \ldots, w_{i_p}\right). \end{aligned}$$

For $x_{i_p j} \mapsto \left(1 - x_{i_p j}\right)$ and $a_j \mapsto (1 - a_j)$ we define following means

$$\begin{aligned} A'_{p,n} &= A(1 - x_{i_1 j}, \ldots, 1 - x_{i_p j}; w_{i_1}, \ldots, w_{i_p}), \\ G'_{p,n} &= G(1 - x_{i_1 j}, \ldots, 1 - x_{i_p j}; w_{i_1}, \ldots, w_{i_p}), \\ H'_{p,n} &= H(1 - x_{i_1 j}, \ldots, 1 - x_{i_p j}; w_{i_1}, \ldots, w_{i_p}). \end{aligned}$$

Clearly for $n = p$,

$$\begin{aligned} A_{n,n} &= \sum_{j=1}^{m} a_j - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i x_{ij} = A_n, \\ G_{n,n} &= \frac{\prod_{j=1}^{m} a_j}{\left(\prod_{j=1}^{m-1} \prod_{i=1}^{n} x_{ij}^{w_i}\right)^{\frac{1}{W_n}}} = G_n, \\ H_{n,n} &= \left(\sum_{j=1}^{m} a_j^{-1} - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i \frac{1}{x_{ij}}\right)^{-1} = H_n, \\ M_{n,n}^{[r]} &= \begin{cases} \left(\sum_{j=1}^{m} (a_j)^r - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{1=1}^{n} w_i (x_{ij})^r\right)^{\frac{1}{r}}, & r \neq 0, \\ G_{n,n}, & r = 0, \end{cases} \\ &= M^{[n]} \\ A'_{n,n} &= \sum_{j=1}^{m} (1 - a_j) - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i \left(1 - x_{ij}\right) = A'_n, \\ G'_{n,n} &= \frac{\prod_{j=1}^{m} (1 - a_j)}{\left(\prod_{j=1}^{m-1} \prod_{i=1}^{n} (1 - x_{ij})^{w_i}\right)^{\frac{1}{W_n}}} = G'_n, \\ H'_{n,n} &= \left(\sum_{j=1}^{m} (1 - a_j)^{-1} - \frac{1}{W_n} \sum_{j=1}^{m-1} \sum_{i=1}^{l} w_i \frac{1}{(1 - x_{ij})}\right)^{-1} = H'_n. \end{aligned}$$

We now introduce mixed symmetric means as follows:

$$M_{k,n}^{[s,r]} = \begin{cases} \left(\frac{1}{\binom{n-1}{k-1} W_n} \sum_{1 \le i_1 < \cdots < i_k \le n} \left(\sum_{l=1}^{k} w_{i_l} \right) \left(M_{k,n}^{[r]} \right)^s \right)^{\frac{1}{s}}, & s \neq 0, \\ \left(\prod_{1 \le i_1 < \cdots < i_k \le n} M_{k,n}^{[r]} \right)^{\frac{1}{\binom{n-1}{k-1} W_n}}, & s = 0, \end{cases}$$

Under the assumptions of Theorem 3 here we state a couple of applications starting with the following refinement series of the arithmetic-geometric and Ky Fan inequalities:

Theorem 9. (i)

$$A_n \le \cdots \le \left(\prod_{1 \le i_1 < \cdots < i_{k+1} \le n} (A_{k+1,n})^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k} W_n}}$$

$$\le \left(\prod_{1 \le i_1 < \cdots < i_k \le n} (A_{k,n})^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1} W_n}}$$

$$\le \cdots \le \left(\prod_{i=1}^{n} (A_{1,n})^{w_i} \right)^{\frac{1}{W_n}} \le G_n.$$

(ii)

$$\frac{A'_n}{A_n} \le \cdots \le \left(\prod_{1 \le i_1 < \cdots < i_{k+1} \le n} \left(\frac{A'_{k+1,n}}{A_{k+1,n}} \right)^{W_{i_{k+1}}} \right)^{\frac{1}{\binom{n-1}{k} W_n}}$$

$$\le \left(\prod_{1 \le i_1 < \cdots < i_k \le n} \left(\frac{A'_{k,n}}{A_{k,n}} \right)^{W_{i_k}} \right)^{\frac{1}{\binom{n-1}{k-1} W_n}}$$

$$\le \cdots \le \left(\prod_{i=1}^{n} \left(\frac{A'_{1,n}}{A_{1,n}} \right)^{w_i} \right)^{\frac{1}{W_n}} \le \frac{G'_n}{G_n},$$

where $W_{i_{k+1}} = w_{i_1} + \cdots + w_{i_{k+1}}$.

Proof.
(i) By apply convex function $f(x) = -\ln(x)$ to Corollary 2 we obtain required result.
(ii) For $x \in (0, \frac{1}{2}]$ and $a_j < 1$ for all $j \in \{1, \ldots, m\}$, applying convex function $f(x) = \ln\left(\frac{1-x}{x}\right)$ and adopting the technique of Theorem 4 we get the proof. □

The refinements series of the variant of arithmetic-geometric inequality is given as:

Theorem 10.

$$\frac{G_n}{G_n+G'_n} \leq \cdots \leq \frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n} W_{i_{k+1}}\left(\frac{G_{k+1,n}}{G_{k+1,n}+G'_{k+1,n}}\right)$$
$$\leq \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n} W_{i_k}\left(\frac{G_{k,n}}{G_{k,n}+G'_{k,n}}\right)$$
$$\leq \cdots \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\frac{G_{1,n}}{G_{1,n}+G'_{1,n}}\right) \leq A_n,$$

where $\quad W_{i_{k+1}} = w_{i_1}+\cdots+w_{i_{k+1}}.$

Proof. By applying the convex function $f(x)=\frac{1}{1+\exp x}$ with $a_j<1$ for all $j\in\{1,\ldots,m\}$, to Corollary 2 and replacing a_j by $\ln\frac{1-a_j}{a_j}$ and $x_{i_l j}$ by $\ln\frac{1-x_{i_l j}}{x_{i_l j}}$ we obtain the result. □

Now, we present refinement series of the arithmetic and harmonic mean as follow:

Theorem 11.
(*i*)

$$\frac{1}{A_n} \leq \cdots \leq \frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n} W_{i_{k+1}}\left(\frac{1}{A_{k+1,n}}\right)$$
$$\leq \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n} W_{i_k}\left(\frac{1}{A_{k,n}}\right)$$
$$\leq \cdots \leq \frac{1}{W_n}\sum_{i=1}^{n} w_i\left(\frac{1}{A_{1,n}}\right) \leq \frac{1}{H_n}.$$

(*ii*)

$$\frac{1}{A'_n} \leq \cdots \leq \frac{1}{\binom{n-1}{k}W_n}\sum_{1\leq i_1<\cdots<i_{k+1}\leq n} W_{i_{k+1}}\left(\frac{1}{A'_{k+1,n}}\right)$$
$$\leq \frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\leq i_1<\cdots<i_k\leq n} W_{i_k}\left(\frac{1}{A'_{k,n}}\right)$$
$$\leq \cdots \leq \frac{1}{W_n}\sum_{1}^{n} w_i\left(\frac{1}{A'_{1,n}}\right) \leq \frac{1}{H'_n},$$

where $\quad W_{i_{k+1}} = w_{i_1}+\cdots+w_{i_{k+1}}.$

Proof.
(i) By applying convex function $f(x)=\frac{1}{x}$ to Corollary 2 we obtain required result.
(ii) By applying convex function $f(x)=\frac{1}{1-x}$ with $x\in(0,\frac{1}{2}]$ and $a_j<1$ for all $j\in\{1,\ldots,m\}$, to Corollary 2 we obtain required result. □

In the following theorem we establish a refinement series of the difference of the arithmetic and harmonic mean.

Theorem 12.

$$\frac{1}{A_n}-\frac{1}{A'_n}\le\cdots\le\frac{1}{\binom{n-1}{k}W_n}\sum_{1\le i_1<\cdots<i_{k+1}\le n}W_{i_{k+1}}\left(\frac{1}{A_{k+1,n}}-\frac{1}{A'_{k+1,n}}\right)$$
$$\le\frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\le i_1<\cdots<i_k\le n}W_{i_k}\left(\frac{1}{A_{k,n}}-\frac{1}{A'_{k,n}}\right)$$
$$\le\cdots\le\frac{1}{W_n}\sum_{i=1}^{n}w_i\left(\frac{1}{A_{1,n}}-\frac{1}{A'_{1,n}}\right)\le\frac{1}{H_n}-\frac{1}{H'_n},$$

where $W_{i_{k+1}}=w_{i_1}+\cdots+w_{i_{k+1}}$.

Proof. By applying convex function $f(x)=\frac{1}{x}-\frac{1}{1-x}$ with $x\in(0,\frac{1}{2}]$ and $a_j<1$ for all $j\in\{1,\ldots,m\}$, to Corollary 2 we obtain required result. □

We now prove some results in terms of mixed symmetric means:

Theorem 13. *Let $r,s\in\mathbb{R}$ such that $s\le r$. Then we have*

$$M_s=M_{n,n}^{[r,s]}\le\cdots\le M_{k+1,n}^{[r,s]}\le M_{k,n}^{[r,s]}\le\cdots\le M_{1,n}^{[r,s]}\le M_r. \tag{15}$$

$$M_s\le M_{1,n}^{[s,r]}\le\cdots\le M_{k,n}^{[s,r]}M_{k+1,n}^{[s,r]}\le\cdots\le M_{n,n}^{[s,r]}=M_r. \tag{16}$$

Proof. Let $r,s\neq 0$. By applying the function $\phi(x)=x^{\frac{s}{r}}$, in the Corollary 2 and replacing a_j and x_{i_lj} by $(a_j)^r$ and $(x_{i_lj})^r$ respectively, and then raising the power $\frac{1}{s}$, we get (15). Similarly, using the function $\phi(x)=x^{\frac{r}{s}}$ in the Corollary 2 and replacing a_j and x_{i_lj} by $(a_j)^s$ and $(x_{i_lj})^s$ respectively then raising the power $\frac{1}{r}$, we get (16). For $s=0$ or $r=0$, we obtain the required result by taking limit. □

Let $\phi,\psi:J\to\mathbb{R}$ be continuous strictly monotonic functions. We define the quasi-arithmetic means with respect to Theorem 3, as follow:

$$\hat{M}_{k,n}^{[\phi,\psi]}=\phi^{-1}\left[\frac{1}{\binom{n-1}{k-1}W_n}\sum_{1\le i_1<\cdots<i_k\le n}\left(\sum_{l=1}^{k}w_{i_l}\right)\right.$$
$$\left.\times\left(\phi o\psi^{-1}\right)\left(\psi\left(\sum_{j=1}^{m}a_j\right)-\frac{\sum_{j=1}^{m-1}\sum_{l=1}^{k}w_{i_l}\psi\left(x_{i_lj}\right)}{\sum_{l=1}^{k}w_{i_l}}\right)\right],$$

where $\phi o\psi^{-1}$ is convex function.

Corollary 4. *If we define a continuous and strictly monotonic function $\varphi:J\to\mathbb{R}$ as*

$$\hat{M}^{[\varphi]}=\varphi^{-1}\left[\varphi\left(\sum_{j=1}^{m}a_j\right)-\frac{1}{W_n}\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i\varphi(x_{ij})\right],$$

then following monotonicity of generalized quasi-arithmetic means holds

$$\hat{M}^{[\phi]}\ge\hat{M}_{1,n}^{[\phi,\psi]}\ge\cdots\ge\hat{M}_{k,n}^{[\phi,\psi]}\ge\hat{M}_{k+1,n}^{[\phi,\psi]}\ge\cdots\ge\hat{M}_{n,n}^{[\phi,\psi]}=\hat{M}^{[\psi]}. \tag{17}$$

Proof. Setting $f\mapsto\phi o\psi^{-1}$ and replacing a_j and x_{i_lj} by $\psi(a_j)$ and $\psi(x_{i_lj})$ respectively in Corollary 2 and then applying ϕ^{-1} we get (17). □

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