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**EXTENSIONS OF $D(4)$ -PAIRS $\{a, ka\}$ WITH
 $k \in \{7, 8, 10, 11, 12, 13\}$**

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ABSTRACT. We study the extensibility of $D(4)$ -pairs $\{a, b\}$, where $b = ka$ and $k \in \{7, 8, 10, 11, 12, 13\}$. Firstly, we show that it can be extended to a $D(4)$ -triple with an element c , which is a member of a family of positive integers depending on a . Then, we prove that such a triple has a unique extension to a $D(4)$ -quadruple.

1. INTRODUCTION

DEFINITION 1.1. *Let $n \neq 0$ be an integer. We call a set of m distinct positive integers a $D(n)$ - m -tuple, or m -tuple with the property $D(n)$, if the product of any two of its distinct elements increased by n is a perfect square.*

We investigate the $n = 4$ case, which has many similarities to the classical $n = 1$ case. The first author and Filipin proved in [5] the nonexistence of $D(4)$ -quintuples.

For a $D(4)$ -triple $\{a, b, c\}$, $a < b < c$, we define

$$d_{\pm}(a, b, c) = a + b + c + \frac{1}{2} \left(abc \pm \sqrt{(ab + 4)(ac + 4)(bc + 4)} \right).$$

$D(4)$ -quadruple $\{a, b, c, d_{\pm}\}$ is called a **regular quadruple**. If $d_{-} \neq 0$, then $\{a, b, c, d_{-}\}$ is a regular $D(4)$ -quadruple with $d_{-} < c$. It is easy to verify that $c = d_{+}(a, b, d_{-})$.

In both the classical case, $n = 1$, and $n = 4$, conjectures about the uniqueness of an extension of a triple to a quadruple with a larger element are still open.

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CONJECTURE 1.2. *Any $D(4)$ -quadruple is regular.*

In this paper, following the methods from [1], we study the extensibility of $D(4)$ -pairs $\{a, b\}$, where $b = ka$ and $k \in \{7, 8, 10, 11, 12, 13\}$. The main theorem we prove is the following:

THEOREM 1.3. *Let k be a positive integer such that $k \in \{7, 8, 10, 11, 12, 13\}$. If $\{a, b, c, d\}$ is a $D(4)$ -quadruple with $b = ka$, then it is regular. In other words, we have $d = d_{\pm}$.*

The cases $k \in \{2, 3, 5, 6\}$ have already been considered in [1] and [10], and it is easy to verify that pairs $\{a, 4a\}$ and $\{a, 9a\}$ cannot satisfy the $D(4)$ property.

Firstly, in Section 2 we show that $D(4)$ -pairs $\{a, ka\}$, $k \in \{7, 8, 10, 11, 12, 13\}$ can be extended to a $D(4)$ -triple only with an element c , which is a member of a family of positive integers depending on a . Then in Sections 3–5, we prove that these triples can only be extended to regular $D(4)$ -quadruples $\{a, ka, c, d\}$. These examples support Conjecture 1.2.

2. EXTENSIONS OF PAIRS TO TRIPLES

If $\{a, ka\}$ is a $D(4)$ -pair, then there exists $r \in \mathbb{N}$ such that

$$(2.1) \quad ka^2 + 4 = r^2.$$

Rewriting (2.1) as a Pellian equation yields

$$(2.2) \quad r^2 - ka^2 = 4.$$

It is easy to verify that all positive solutions (r_p, a_p) of the equation (2.2) are given by

$$(2.3) \quad \frac{r_p + a_p\sqrt{k}}{2} = \left(\frac{r_1 + a_1\sqrt{k}}{2} \right)^p, \quad p \in \mathbb{N},$$

where $(r_1, a_1) \in \{(16, 6), (6, 2), (38, 12), (20, 6), (4, 1), (11, 3)\}$ for $k = 7, 8, 10, 11, 12, 13$ respectively. From this relation, if we denote the p -th element of the sequence for each k with $a_p^{(k)}$, we easily obtain the following relations:

$$(2.4) \quad a_p^{(7)} = \frac{1}{\sqrt{7}} \left((8 + 3\sqrt{7})^p - (8 - 3\sqrt{7})^p \right),$$

$$(2.5) \quad a_p^{(8)} = \frac{1}{\sqrt{8}} \left((3 + \sqrt{8})^p - (3 - \sqrt{8})^p \right),$$

$$(2.6) \quad a_p^{(10)} = \frac{1}{\sqrt{10}} \left((19 + 6\sqrt{10})^p - (19 - 6\sqrt{10})^p \right),$$

$$(2.7) \quad a_p^{(11)} = \frac{1}{\sqrt{11}} \left((10 + 3\sqrt{11})^p - (10 - 3\sqrt{11})^p \right),$$

$$(2.8) \quad a_p^{(12)} = \frac{1}{\sqrt{12}} \left(\left(\frac{4 + \sqrt{12}}{2} \right)^p - \left(\frac{4 - \sqrt{12}}{2} \right)^p \right),$$

$$(2.9) \quad a_p^{(13)} = \frac{1}{\sqrt{13}} \left(\left(\frac{11 + 3\sqrt{13}}{2} \right)^p - \left(\frac{11 - 3\sqrt{13}}{2} \right)^p \right).$$

As we will present, many proofs of the lemmas in this article require the following fact, which is easy to verify.

$$\gcd(a_p, r_p) = \begin{cases} 2, & k = 7, 8, 10, 11, \\ 2, & k = 12, p \equiv 0 \pmod{2}, \\ 1, & k = 12, p \equiv 1 \pmod{2}, \\ 2, & k = 13, p \equiv 0 \pmod{3}, \\ 1, & k = 13, p \equiv 1, 2 \pmod{3}. \end{cases}$$

If we assume that an irregular $D(4)$ -quadruple exists, from [4] we know a numerical lower bound on element b :

LEMMA 2.1. [4, Lemma 2.2] *Let $\{a, b, c, d\}$ be a $D(4)$ -quadruple such that $a < b < c < d_+ < d$. Then $b > 10^5$.*

It is straightforward to use this bound to determine the lower bound for a in an irregular $D(4)$ -quadruple $\{a, ka, c, d\}$. This is presented in Table 1.

TABLE 1. Lower bounds for the element a

k	7	8	10	11	12	13
p	4	6	3	4	8	5
$a_p^{(k)}$	24384	13860	17316	47760	10864	42837

Let $\{a, b\}$ be a $D(4)$ -pair. Then there exists a positive integer r such that $ab + 4 = r^2$. Extending this pair to a $D(4)$ -triple with an element c means finding $s, t \in \mathbb{N}$ such that

$$ac + 4 = s^2, \quad bc + 4 = t^2.$$

These two equations yield a Pellian equation

$$(2.10) \quad at^2 - bs^2 = 4(a - b).$$

Its solutions (t, s) are given by

$$(2.11) \quad (t_\nu \sqrt{a} + s_\nu \sqrt{b}) = (t_0 \sqrt{a} + s_0 \sqrt{b}) \left(\frac{r + \sqrt{ab}}{2} \right)^\nu, \quad \nu \geq 0,$$

where (t_0, s_0) is a fundamental solution of the equation (2.10) and ν is a nonnegative integer.

It has been proved in [4, Lemma 6.1] that for $b \leq 6.85a$ the only fundamental solutions of this equation from which we obtain the third element c , which is an integer, are $(t_0, s_0) = (\pm 2, 2)$. From [3] we know that all solutions (t_ν, s_ν) generated with $(t_0, s_0) = (\pm 2, 2)$ can be represented by recursively defined sequences:

$$(2.12) \quad t_0 = \pm 2, \quad t_1 = b \pm r, \quad t_{\nu+2} = rt_{\nu+1} - t_\nu,$$

$$(2.13) \quad s_0 = 2, \quad s_1 = r \pm a, \quad s_{\nu+2} = rs_{\nu+1} - s_\nu, \quad \nu \geq 0.$$

Since $c = \frac{s^2-4}{a}$, an explicit expression for the third element c in terms of a and b is given by

$$(2.14) \quad c = c_\nu^\pm = \frac{4}{ab} \left\{ \left(\frac{\sqrt{b} \pm \sqrt{a}}{2} \right)^2 \left(\frac{r + \sqrt{ab}}{2} \right)^{2\nu} + \left(\frac{\sqrt{b} \mp \sqrt{a}}{2} \right)^2 \left(\frac{r - \sqrt{ab}}{2} \right)^{2\nu} - \frac{a+b}{2} \right\},$$

where $\nu \geq 0$ is an integer. The first few elements of this sequence are:

$$c_1^\pm = a + b \pm 2r,$$

$$c_2^\pm = (ab + 4)(a + b \pm 2r) \mp 4r,$$

$$c_3^\pm = (a^2b^2 + 6ab + 9)(a + b \pm 2r) \mp 4r(ab + 3),$$

$$c_4^\pm = (a^3b^3 + 8a^2b^2 + 20ab + 16)(a + b \pm 2r) \mp 4r(a^2b^2 + 5ab + 6).$$

LEMMA 2.2. *Let $\{a, b\}$ be a $D(4)$ -pair. Then $d_+(a, b, c_\nu^\pm) = c_{\nu+1}^\pm$ and $d_-(a, b, c_\nu^\pm) = c_{\nu-1}^\pm$.*

PROOF. From (2.11) we get $s_{\nu+1} = \frac{r}{2}s_\nu + \frac{a}{2}t_\nu$. Since $c_\nu^\pm = \frac{(s_\nu)^2-4}{a} = \frac{(t_\nu)^2-4}{b}$ it follows that $c_{\nu+1}^\pm = \frac{(s_{\nu+1})^2-4}{a} = a + b + c_\nu^\pm + \frac{1}{2}(abc_\nu^\pm + rs_\nu t_\nu) = d_+(a, b, c_\nu^\pm)$. \square

The main goal of this section is to improve this upper bound for b in terms of a under an additional restriction.

LEMMA 2.3. *Let $\{a, b, c\}$ be a $D(4)$ -triple and $a < b \leq 13.92a$. Suppose that $\{1, 5, a, b\}$ is not a $D(4)$ -quadruple. Then $c = c_\nu^\pm$ for some positive integer ν .*

PROOF. We follow the idea of [4, Lemma 6.1] and [3, Lemma 1]. Define $s' = \frac{rs-at}{2}$, $t' = \frac{rt-bs}{2}$ and $c' = \frac{(s')^2-4}{a}$. The cases $c' > b$, $c' = b$ and $c' = 0$ are the same as in [3, Lemma 1] and yield $c = c_\nu^\pm$. It is only left to consider the

case $0 < c' < b$. Here we define $r' = \frac{s'r-at'}{2}$ and $b' = \frac{(r')^2-4}{a}$. If $b' = 0$ then it can be shown that $c' = c_1^-$ and $c = c_\nu^-$ for some positive integer ν . Notice that $b' = d_-(a, b, c')$, hence,

$$b' < \frac{b}{ac'} \leq \frac{13.92a}{ac'} = \frac{13.92}{c'} \implies b'c' < 13.92.$$

Since $b' > 0$ and $b'c' + 4$ is a square, we consider the following cases for b' , depending on the term c' :

c'	1	2	3	4	5	6	12
b'	5, 12	6	4	3	1	2	1

The cases $c' \in \{7, 8, 9, 10, 11\}$ and $c' \geq 13$ imply there is no $b' > 0$ satisfying our conditions. So, we obtain that a and b extend pairs $\{1, 5\}, \{3, 4\}, \{2, 6\}, \{1, 12\}$. As our assumption is that $\{1, 5, a, b\}$ is not a $D(4)$ -quadruple, we only need to show that the three remaining cases also cannot be $D(4)$ -quadruples when $b \leq 13.92a$. Let's suppose that $\{3, 4, a, b\}$ is a $D(4)$ -quadruple. From [4, Lemma 6.1], it follows that

$$a = a_\nu^\pm = \frac{1}{3} \left\{ \left(\frac{2 \pm \sqrt{3}}{2} \right)^2 \left(\frac{4 + \sqrt{12}}{2} \right)^{2\nu} + \left(\frac{\sqrt{2} \mp \sqrt{3}}{2} \right)^2 \left(\frac{4 - \sqrt{12}}{2} \right)^{2\nu} - \frac{7}{2} \right\},$$

and from Lemma 2.1 it follows that $b = d_+(3, 4, a)$. Lemma 2.2 implies $d_+(3, 4, a) = a_{\nu+1}^\pm$ for the same choice of \pm . Define $k := \frac{b}{a} = \frac{a_{\nu+1}^\pm}{a_\nu^\pm}$. It is easy to see that $k \leq 15.24$ and that it is decreasing as ν increases, and

$$\lim_{\nu \rightarrow \infty} \frac{a_{\nu+1}^\pm}{a_\nu^\pm} = \left(\frac{4 + \sqrt{12}}{2} \right)^2 > 13.92,$$

which gives us a contradiction to the assumption that $b \leq 13.92a$. We apply the same approach to the other two cases and arrive at the same conclusion. The only difference arises in the case $\{1, 12, a, b\}$, since here we cannot use [4, Lemma 6.1] to show that $a = a_\nu^\pm$. Instead, we use bounds on the fundamental solutions of the corresponding Pellian equation from [6, Theorem 10.21]. \square

COROLLARY 2.4. *Let $\{a, ka, c\}$ be a $D(4)$ -triple, $k \in \{10, 11, 12, 13\}$. Then $c = c_\nu^\pm$.*

PROOF. Let's show that $\{1, 5, a, ka\}$ is not a $D(4)$ -quadruple for $k \in \{10, 11, 12, 13\}$. If $\{1, 5, a, ka\}$ is a $D(4)$ -quadruple, by using [4, Lemma 6.1], Lemma 2.1, and Lemma 2.2, we obtain $a = a_\nu^\pm$, $ka = d_+(1, 5, a)$, and $d_+(1, 5, a) = a_{\nu+1}^\pm$ for the same choice of \pm . We divide both sides of

$$(2.15) \quad ka = d_+(1, 5, a) = 1 + 5 + a + \frac{1}{2} (5a + 3\sqrt{5a+4}\sqrt{a+4})$$

by a and, using the fact that $a \geq 12$, we obtain $k \leq 8$. \square

LEMMA 2.5.

- i) If $\{1, 5, a, 7a\}$ is a $D(4)$ -quadruple, then $a = 96$.
- ii) If $\{1, 5, a, 8a\}$ is a $D(4)$ -quadruple, then $a = 12$.

PROOF. From (2.15), when $k = 7$ we get $a = 96$ and when $k = 8$ we get $a = 12$. \square

Using the theory of Pellian equations, it is easy to see that the only fundamental solutions for the corresponding Pellian equation when extending $\{96, 672\}$ are $(\pm 2, 2)$ and $(\pm 26, 10)$, and when extending $\{12, 96\}$ are $(\pm 2, 2)$ and $(\pm 10, 4)$. So we have two pairs of sequences for each of these pairs that extend them to triples. Since the second element in both of these pairs is less than 10^5 , by Lemma 2.1 we know that all triples from those sequences extend only to regular quadruples.

Lemma 2.3 allows us to further investigate the regularity of $D(4)$ -quadruples $\{a, ka, c, d\}$, i.e., we enhance [1, Theorem 1.4]. In the following Sections, we show that $D(4)$ -triples $\{a, ka, c\}$, $k \in \{7, 8, 10, 11, 12, 13\}$, extend only to regular $D(4)$ -quadruples. Since we separately examined the exceptions, it remains to consider extensions of triples of the form $\{a, ka, c\}$, where $c = c_\nu^\pm$. It is easy to see that $c_4^- > a^3b^3$ and since $ka > 10^5$, $k \in \{7, 8, 10, 11, 12, 13\}$, we get from [4, Theorem 1.6] that $c \in \{c_1^\pm, c_2^\pm, c_3^\pm\}$. Also, it is easy to see that inequalities $a < c_1^- < b$ and $c_1^+, c_2^\pm, c_3^\pm > b$ hold in all our cases.

3. EXTENSIONS OF TRIPLES AND LINEAR FORMS IN THREE LOGARITHMS

In this section, we consider a system of Pellian equations which corresponds to the extension of a $D(4)$ -triple to a $D(4)$ -quadruple. Then, we search for the intersection of linear recurrent sequences that describe solutions to these equations. To help us find these intersections, at the end of this section we use the theory of linear forms in logarithms to obtain some useful lemmas and results. Proofs that differ only in calculations from [1] and [2] will be omitted.

3.1. *System of simultaneous Pellian equations.* Let us consider an extension of a $D(4)$ -triple $\{a, b, c\}$ to a $D(4)$ -quadruple $\{a, b, c, d\}$. We need to find $x, y, z \in \mathbb{N}$ such that

$$ad + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2.$$

By eliminating d from these equations, we obtain a system of generalized Pellian equations

$$(3.16) \quad az^2 - cx^2 = 4(a - c),$$

$$(3.17) \quad bz^2 - cy^2 = 4(b - c),$$

$$(3.18) \quad ay^2 - bx^2 = 4(a - b).$$

Its solutions (z, x) , (z, y) , and (y, x) satisfy

$$(3.19) \quad z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c}) \left(\frac{s + \sqrt{ac}}{2} \right)^m,$$

$$(3.20) \quad z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{a} + y_1\sqrt{c}) \left(\frac{t + \sqrt{bc}}{2} \right)^n,$$

$$(3.21) \quad y\sqrt{a} + x\sqrt{b} = (y_2\sqrt{a} + x_2\sqrt{b}) \left(\frac{r + \sqrt{ab}}{2} \right)^l,$$

where m, n, l are nonnegative integers and (z_0, x_0) , (z_1, y_1) , and (y_2, x_2) are fundamental solutions of (3.16)–(3.18).

Firstly, we consider the solutions of the system of equations (3.19) and (3.20) and determine the intersections $z = v_m = w_n$ of sequences $(v_m)_m$ and $(w_n)_n$ defined by

$$\begin{aligned} v_0 &= z_0, \quad v_1 = \frac{1}{2}(sz_0 + cx_0), \quad v_{m+2} = sv_{m+1} - v_m, \\ w_0 &= z_1, \quad w_1 = \frac{1}{2}(tz_1 + cy_1), \quad w_{n+2} = tw_{n+1} - w_n. \end{aligned}$$

The initial terms of these sequences are described in the following theorem.

THEOREM 3.1. [4, Theorem 1.3] *Suppose that $\{a, b, c, d\}$ is a $D(4)$ -quadruple with $a < b < c < d$ and that w_m and v_n are defined as before.*

- i) *If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$ and $|z_0| = 2$ or $|z_0| = \frac{1}{2}(cr - st)$.*
- ii) *If the equation $v_{2m+1} = w_{2n}$ has a solution, then $|z_0| = t$, $|z_1| = \frac{1}{2}(cr - st)$ and $z_0z_1 < 0$.*
- iii) *If the equation $v_{2m} = w_{2n+1}$ has a solution, then $|z_1| = s$, $|z_0| = \frac{1}{2}(cr - st)$ and $z_0z_1 < 0$.*
- iv) *If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $|z_0| = t$, $|z_1| = s$ and $z_0z_1 > 0$.*

Moreover, if $d > d_+$, case ii) cannot occur.

Under the assumption that some special $D(4)$ -quadruples do not exist, we have the following lemma, which further reduces the number of cases for fundamental solutions we need to examine.

LEMMA 3.2. [1, Lemma 2.2] *Assume that $\{a, b, c, c'\}$ is not a $D(4)$ -quadruple for any c' with $0 < c' < c_{v-1}^\pm$. We have*

- i) *If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1 = \pm 2$ and $x_0 = y_1 = 2$.*
- ii) *If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $z_0 = \pm t$, $z_1 = \pm s$, $x_0 = y_1 = r$ and $z_0z_1 > 0$.*

REMARK 3.3. If $c = c_1^\pm = a + b \pm 2r$, then it is enough to consider the case $v_{2m} = w_{2n}$.

Secondly, we consider the solutions of the system of equations (3.20) and (3.21) and determine the intersections $y = A_n = B_l$ of sequences $(A_n)_n$ and $(B_l)_l$ defined by

$$(3.22) \quad A_0 = y_1, \quad A_1 = \frac{1}{2}(ty_1 + bz_1), \quad A_{n+2} = tA_{n+1} - A_n,$$

$$(3.23) \quad B_0 = y_2, \quad B_1 = \frac{1}{2}(ry_2 + bx_2), \quad B_{l+2} = rB_{l+1} - B_l, \quad n, l \geq 0.$$

The initial terms of these sequences are described in the next lemma, whose proof follows [1, Lemma 2.5].

LEMMA 3.4. *Assume that $\{a, b, c', c\}$ is not a $D(4)$ -quadruple for any c' with $0 < c' < c_{\nu-1}^\pm$ and $b \geq 556881$. Then, $A_{2n} = B_{2l+1}$ has no solution. Moreover, if $A_{2n} = B_{2l}$, then $y_2 = 2$. In other cases, we have $y_2 = \pm 2$.*

Finally, we consider the solutions of the system of equations (3.19) and (3.21) and determine the intersections $x = Q_m = P_l$ of sequences $(Q_m)_m$ and $(P_l)_l$ defined by

$$(3.24) \quad P_0 = x_2, \quad P_1 = \frac{1}{2}(rx_2 + ay_2), \quad P_{l+2} = rP_{l+1} - P_l,$$

$$(3.25) \quad Q_0 = x_0, \quad Q_1 = \frac{1}{2}(sx_0 + az_0), \quad Q_{m+2} = sQ_{m+1} - Q_m.$$

From the above, for the equation $x = P_l = Q_m$, we conclude that only the following two possibilities exist:

Type 1: If $l \equiv m \equiv 0 \pmod{2}$, then $z_0 = \pm 2$, $x_0 = 2$, $y_2 = \pm 2$ and $x_2 = 2$.

Type 2: If $m \equiv 1 \pmod{2}$, then $z_0 = \pm t$, $x_0 = r$, $y_2 = \pm 2$ and $x_2 = 2$.

For the rest of this paper, we will carefully examine the following equation:

$$(3.26) \quad x = Q_m = P_l,$$

while using the fundamental solutions of Types 1 and 2. As we mentioned in Remark 3.3, we only need to consider solutions in Type 1 if $c = c_1^\pm$ since

$$\frac{1}{2}(cr - st) = \frac{1}{2}((a + b \pm 2r)r - (r \pm a)(b \pm r)) = \pm 2.$$

Let's emphasise which solutions of this equation correspond to the regular extension of our triples to quadruples. For the case $c = c_1^-$ we get $(l, m) = (2, 2)$ from fundamental solutions $x_0 = 2, z_0 = 2, x_2 = 2, y_2 = -2$ and for the case $c = c_1^+$ we get $(l, m) = (2, 2)$ from fundamental solutions $x_0 = 2, z_0 = -2, x_2 = 2, y_2 = 2$. Next, we consider the cases $c = c_2^\pm$. Since $d_-(a, ka, c_2^\pm) = c_1^\pm$, that extension comes from the solution $(l, m) = (1, 1)$ from fundamental solutions $x_0 = r, z_0 = t, x_2 = 2, y_2 = -2$ (for c_2^-) and from fundamental solutions $x_0 = r, z_0 = -t, x_2 = 2, y_2 = 2$ (for c_2^+). Next, since

$d_+(a, ka, c_2^\pm) = c_3^\pm$, that extension comes from the solution $(l, m) = (3, 1)$ from fundamental solutions $x_0 = r, z_0 = -t, x_2 = 2, y_2 = -2$ (for c_2^-) and from fundamental solutions $x_0 = r, z_0 = t, x_2 = 2, y_2 = 2$ (for c_2^+). For the cases $c = c_3^\pm$, we will consider a different equation in the final section.

3.2. *A linear form in three logarithms.* Solving recurrences (3.24) and (3.25), we obtain

$$P_l = \frac{1}{2\sqrt{b}} \left((y_2\sqrt{a} + x_2\sqrt{b})\alpha^l - (y_2\sqrt{a} - x_2\sqrt{b})\alpha^{-l} \right),$$

$$Q_m = \frac{1}{2\sqrt{c}} \left((z_0\sqrt{a} + x_0\sqrt{c})\beta^m - (z_0\sqrt{a} - x_0\sqrt{c})\beta^{-m} \right),$$

where

$$(3.27) \quad \alpha = \frac{r + \sqrt{ab}}{2} \quad \text{and} \quad \beta = \frac{s + \sqrt{ac}}{2}.$$

Let us define

$$(3.28) \quad \gamma = \frac{\sqrt{c}(y_2\sqrt{a} + x_2\sqrt{b})}{\sqrt{b}(z_0\sqrt{a} + x_0\sqrt{c})} \quad \text{and} \quad \gamma' = \frac{\sqrt{b}(z_0\sqrt{a} + x_0\sqrt{c})}{\sqrt{c}(y_2\sqrt{a} + x_2\sqrt{b})}.$$

We define the following linear forms in three logarithms.

$$(3.29) \quad \Lambda = l \log \alpha - m \log \beta + \log \gamma \quad \text{for} \quad c > b,$$

and

$$(3.30) \quad \Lambda' = m \log \beta - l \log \alpha + \log \gamma' \quad \text{for} \quad c < b.$$

Notice that Λ' is used only for the case c_1^- .

LEMMA 3.5.

1) Assume that $b = ka$, $k \in \{7, 8, 10, 11, 12, 13\}$ and $c = c_1^-$. If the equation $P_l = Q_m$ has a solution (l, m) of Type 1 with $l \geq 1$, then

$$0 < \Lambda' < 2.6\alpha^{-2l}.$$

2) Assume that $b = ka$, $k \in \{7, 8, 10, 11, 12, 13\}$ and $c \in \{c_1^+, c_2^\pm\}$. If the equation $P_l = Q_m$ has a solution (l, m) of Type 1 with $m \geq 3$, then

$$0 < \Lambda < 2.6\beta^{-2m}.$$

If $P_l = Q_m$ has a solution (l, m) of Type 2 with $m \geq 3$, then

$$0 < \Lambda < 1.5a^2\beta^{-2m}.$$

PROOF. Proof of 2) follows [1, Lemma 2.6] and here we will demonstrate the proof of 1). Let's define

$$E = \frac{z_0\sqrt{a} + x_0\sqrt{c}}{\sqrt{c}}\beta^m, \quad F = \frac{y_2\sqrt{a} + x_2\sqrt{b}}{\sqrt{b}}\alpha^l$$

and then our form is $\Lambda' = \log \frac{E}{F}$. It is easy to see that $E, F > 1$ for $l, m \geq 1$. We can rewrite the equation $P_l = Q_m$ to get

$$E + 4 \left(\frac{c-a}{c} \right) E^{-1} = F + 4 \left(\frac{b-a}{b} \right) F^{-1} \stackrel{b > c}{>} F + 4 \left(\frac{c-a}{c} \right) F^{-1}$$

Then $(E-F)(EF - 4\frac{c-a}{c}) > 0$. Since $EF > 4(\frac{c-a}{c})$ we get $E > F$. It follows that $\Lambda' > 0$ and

$$\Lambda' = \log \left(1 + \frac{E-F}{F} \right) < \frac{E-F}{F} < 4 \left(\frac{c-a}{c} \right) F^{-2} < 4F^{-2}.$$

Now, since we consider only the solutions in Type 1 for $c = c_1^-$, we have

$$\Lambda' < 4 \frac{b}{(\pm 2\sqrt{a} + 2\sqrt{b})^2} \alpha^{-2l} < \frac{k}{(\sqrt{k}-1)^2} \alpha^{-2l} < 2.6\alpha^{-2l}.$$

□

LEMMA 3.6.

- 1) Assume that $b = ka$, $k \in \{7, 8, 10, 11, 12, 13\}$ and $c = c_1^-$. If $P_l = Q_m$ has a solution (l, m) with $l \geq 1$, then $l \leq m$.
- 2) Assume that $b = ka$, $k \in \{7, 8, 10, 11, 12, 13\}$ and $c \in \{c_1^+, c_2^\pm\}$. If $P_l = Q_m$ has a solution (l, m) with $m \geq 3$, then $m \leq l$.

PROOF. Proof of 2) follows [2, Lemma 8] and here we will demonstrate the proof of 1). Since we proved in the previous Lemma that $\Lambda' > 0$, we have

$$\frac{m}{l} > \frac{\log \alpha}{\log \beta} - \frac{\log \gamma'}{l \log \beta}.$$

To prove our statement, we need to show that the right-hand side is greater than $1 - \frac{1}{l}$, which is equivalent to proving that

$$\left(\frac{\alpha}{\beta} \right)^l > \frac{\gamma'}{\beta}.$$

Since $ac > a^2 > 10864^2$, we have

$$\frac{\alpha}{\beta} = \frac{r + \sqrt{ab}}{s + \sqrt{ac}} = \frac{\sqrt{1 + \frac{4}{ab}} + 1}{\sqrt{1 + \frac{4}{ac}} + 1} \sqrt{\frac{b}{c}} > \frac{2}{2.1} \sqrt{\frac{b}{c}}.$$

Also,

$$\begin{aligned} \frac{b}{c} &= \frac{ka}{c_1^-} = \frac{ka}{(k+1)a - 2\sqrt{ka^2} + 4} = \frac{ka^2 \left(k + 1 + 2\sqrt{k + \frac{4}{a^2}} \right)}{(k-1)^2 - \frac{16}{a^2}} \\ &> \frac{k}{(k-1)^2} \left(k + 1 + 2\sqrt{k} \right) = \frac{k}{(\sqrt{k}-1)^2}. \end{aligned}$$

Since $l \geq 1$ we get

$$\left(\frac{\alpha}{\beta}\right)^l > 1.31.$$

On the other hand, since for $c = c_1^-$ we only observe fundamental solutions of Type 1 we get

$$\begin{aligned} \frac{\gamma'}{\beta} &= \frac{2\sqrt{b}(\pm 2\sqrt{a} + 2\sqrt{c})}{(s + \sqrt{ac})\sqrt{c}(2\sqrt{a} + 2\sqrt{b})} < \frac{\sqrt{b}(\sqrt{a} + \sqrt{c})}{c\sqrt{a}(-\sqrt{a} + \sqrt{b})} \\ &= \frac{\sqrt{k}(\sqrt{a} + \sqrt{c})}{c(\sqrt{k} - 1)\sqrt{a}} \leq \frac{2\sqrt{k}}{(\sqrt{k} - 1)a} < 1. \end{aligned}$$

Finally, we have

$$\left(\frac{\alpha}{\beta}\right)^l > 1.31 > 1 > \frac{\gamma'}{\beta}.$$

□

For any nonzero algebraic number α of degree d over \mathbb{Q} whose minimal polynomial over \mathbb{Z} is $a_0 \prod_{j=1}^d (X - \alpha^{(j)})$, we denote by

$$h(\alpha) = \frac{1}{d} \left(\log |a_0| + \sum_{j=1}^d \log \max(1, |\alpha^{(j)}|) \right)$$

its absolute logarithmic height. We recall the following result due to Matveev [11, Theorem 2.2 and Corollary 2.3].

LEMMA 3.7. *Denote by $\alpha_1, \dots, \alpha_j$ algebraic numbers, not 0 or 1, by $\log \alpha_1, \dots, \log \alpha_j$ determinations of their logarithms, by D the degree over \mathbb{Q} of the number field $\mathbb{K} = \mathbb{Q}(\alpha_1, \dots, \alpha_j)$, and by b_1, \dots, b_j integers. Define $B = \max\{|b_1|, \dots, |b_j|\}$ and*

$$A_i = \max\{Dh(\alpha_i), |\log \alpha_i|, 0.16\} \quad (1 \leq i \leq j),$$

where $h(\alpha)$ denotes the absolute logarithmic Weil height of α . Assume that the number

$$\Lambda = b_1 \log \alpha_1 + \dots + b_n \log \alpha_j$$

does not vanish. Then

$$|\Lambda| \geq \exp\{-C(j, \chi) D^2 A_1 \cdots A_j \log(eD) \log(eB)\},$$

where $\chi = 1$ if $\mathbb{K} \subset \mathbb{R}$ and $\chi = 2$ otherwise and

$$C(j, \chi) = \min \left\{ \frac{1}{\chi} \left(\frac{1}{2} e^j \right)^\chi, 30^{j+3} j^{3.5}, 2^{6j+20} \right\}.$$

By applying this result we obtain upper bound for l and m in terms of a .

PROPOSITION 3.8. Assume that $c \in \{c_1^+, c_2^\pm\}$. If $P_l = Q_m$, with $m > 1$ then

$$\frac{l}{\log(el)} < 3.34 \cdot 10^{13} \cdot \log^2(8.09c^2), \quad \text{with solutions of Type 1,}$$

$$\frac{l}{\log(el)} < 6.63 \cdot 10^{13} \cdot \log^2(8.09c^2), \quad \text{with solutions of Type 2.}$$

If $Q_m = P_l$, $l \geq 1$ with $c = c_1^-$, then we get

$$\frac{m}{\log(em)} < 13.36 \cdot 10^{13} \cdot \log^2(21.3a), \quad \text{for } k = 7, 8, 10, 11, 12, 13.$$

PROOF. We will demonstrate the proof for the case $c = c_1^-$ since the proof for the other cases closely follows [1, Proposition 2.9]. We apply Lemma 3.8 with $j = 3$ and $\chi = 1$ to the linear form 3.30 and take

$$D = 4, \quad b_1 = m, \quad b_2 = -l, \quad b_3 = 1, \quad \alpha_1 = \beta, \quad \alpha_2 = \alpha, \quad \alpha_3 = \gamma'.$$

Since $l \leq m$, we can take $B = m$. Also, we have

$$h(\alpha_1) = \frac{1}{2} \log \beta, \quad h(\alpha_2) = \frac{1}{2} \log \alpha.$$

Since $\gamma' = \gamma^{-1}$, then $h(\gamma) = h(\gamma')$ and from [1, Proposition 2.9] we have

$$h(\gamma) < \frac{1}{4} \log \left[\frac{2^4 r^4 c^4 (1 + \sqrt{k})^4}{(c - a)^2} \right].$$

We have

$$(c_1^- - a)^2 = a^2 \left(2\sqrt{k + \frac{4}{a^2}} - k \right)^2 > a^2 (2\sqrt{k} - k)^2,$$

$$r < a\sqrt{1.1k},$$

$$c = c_1^- = a \left(k + 1 - 2\sqrt{k + \frac{4}{a^2}} \right) < a(\sqrt{k} - 1)^2,$$

and now it follows that

$$h(\gamma') = h(\gamma) < \frac{1}{4} \log(93309063a^6) < \frac{3}{2} \log(21.3a).$$

By applying Lemma 3.8 with

$$A_1 = 2 \log \beta, \quad A_2 = 2 \log \alpha, \quad A_3 = 6 \log(21.3a)$$

we get

$$(3.31) \quad \log |\Lambda'| > -1.3901 \cdot 10^{11} \cdot 16 \cdot 2 \cdot \log \beta \\ \cdot 2 \log \alpha \cdot 6 \log(21.3a) \cdot \log(4e) \cdot \log(em).$$

From 1) of Lemma 3.5 and using $l \geq 1$, it is easy to conclude that

$$m \log \beta < l \log \alpha + 2.6\alpha^{-2l} - \log \gamma' < 2l \log \alpha.$$

Also,

$$\log |\Lambda'| < -1.9069l \log \alpha$$

and

$$\log \alpha = \log \left[a \left(\sqrt{k + \frac{4}{a^2}} + \sqrt{k} \right) \right] < \log(7.212a) < \log(21.3a).$$

Now, by combining everything, we get

$$\frac{m}{\log(em)} < 1.336 \cdot 10^{14} \cdot \log^2(21.3a), \text{ for } k = 7, 8, 10, 11, 12, 13.$$

□

4. LOWER BOUNDS FOR m AND l

In this section, we examine the equation

$$x = Q_m = P_l.$$

Firstly, we state a useful result from [1] and then apply it to obtain lower bounds for the indices m and l in terms of a .

LEMMA 4.1. [1, Lemma 3.1] *If a is odd, then*

$$(4.32) \quad Q_{2m} \equiv x_0 + \frac{1}{2}a(cx_0m^2 + sz_0m) \pmod{a^2},$$

$$(4.33) \quad P_{2l} \equiv x_2 + \frac{1}{2}a(bx_2l^2 + ry_2l) \pmod{a^2}.$$

If a is even, then

$$(4.34) \quad Q_{2m} \equiv x_0 + \frac{1}{2}a(cx_0m^2 + sz_0m) \pmod{\frac{1}{2}a^2},$$

$$(4.35) \quad P_{2l} \equiv x_2 + \frac{1}{2}a(bx_2l^2 + ry_2l) \pmod{\frac{1}{2}a^2}.$$

LEMMA 4.2.

1) *Assume that $b = ka$, $k \in \{7, 8, 10, 11, 12, 13\}$ and $c = c_1^-$. If the equation $P_l = Q_m$ has a solution (l, m) (of Type 1), then we have*

$$m \geq \frac{1}{4} \left(-2 + \sqrt{4 + \sqrt{a}} \right).$$

2) *Assume that $b = ka$, $k \in \{7, 8, 10, 11, 12, 13\}$ and $c \in \{c_1^+, c_2^\pm\}$. If the equation $P_l = Q_m$ has a solution (l, m) (of Type 1), then we have*

$$l \geq \frac{1}{12} \left(-2 + \sqrt{4 + 3\sqrt{a}} \right).$$

PROOF. Proof follows [1, Lemma 3.2] and here we will demonstrate the proof of 1). From the recurrent sequence (2.13), we have $s \equiv 2, r \pmod{a}$. Also, $b = ka \equiv 0 \pmod{a}$ and $c = c_1^- \equiv -2r \pmod{a}$. Using the previous Lemma, with solutions in Type 1, we get

$$-4rm^2 \pm 4m \equiv \pm 2rl \pmod{a}, \quad \text{if } s \equiv 2 \pmod{a}$$

and

$$-4m^2 \pm 2m \mp 2l \equiv 0 \pmod{\frac{a}{\gcd(a,r)}}, \quad \text{if } s \equiv r \pmod{a}.$$

In the first case, we multiply the congruence by r and since $r^2 \equiv 4 \pmod{a}$ we have $r \equiv \pm 2 \pmod{a'}$ for some a' that divides a and $a' \geq \sqrt{a}$. So we get

$$-16m^2 \pm 8m \mp 8l \equiv 0 \pmod{a'}.$$

Using $l \leq m$ it follows that

$$16m^2 + 16m \geq |-16m^2 \pm 8m \mp 8l| \geq a' \geq \sqrt{a},$$

which implies

$$(4.36) \quad m \geq \frac{1}{4} \left(-2 + \sqrt{4 + \sqrt{a}} \right).$$

In the second case, since $\gcd(a, r) \leq 2$ and $l \leq m$, we have

$$4m^2 + 4m \geq |-4m^2 \pm 2m \mp 2l| \geq \frac{a}{\gcd(a,r)} \geq \frac{a}{2},$$

which implies

$$(4.37) \quad m \geq \frac{1}{4} \left(-2 + \sqrt{4 + 2a} \right).$$

Combining (4.36) and (4.37) we obtain the desired inequality. \square

In [1] it is demonstrated that $s \equiv \pm 2, \pm a \pmod{r}$ and that the case $s \equiv \pm a \pmod{r}$ leads to a contradiction if $c = c_2^\pm$, which also applies here. Therefore, for solutions of Type 2, we obtain another lower bound on l and m . With only slight changes in calculations, the next result follows similarly to [1, Lemma 3.4].

LEMMA 4.3. *Assume that $c = c_2^\pm$. If the equation $P_l = Q_m$ has a solution (l, m) of Type 2, then we have*

$$m > \begin{cases} (14 - 5\sqrt{7.1})a/4, & k = 7, \\ (6\sqrt{8} - 16)a/4, & k = 8, \\ (20 - 6\sqrt{10.1})a/4, & k = 10, \\ (7\sqrt{11} - 22)a/4, & k = 11, \\ (7\sqrt{12} - 24)a/4, & k = 12, \\ (26 - 7\sqrt{13.1})a/4, & k = 13. \end{cases}$$

5. PROOF OF THEOREM 1.3

In this section, we complete the proof of Theorem 1.3 in two subsections according to the values of c .

5.1. *Proof of Theorem 1.3 for $c = c_1^\pm, c_2^\pm$.*

Combining (2.4)-(2.9) with Proposition 3.8, Lemmas 4.2 and 4.3 we obtain the following result.

LEMMA 5.1.

- 1) For the $D(4)$ -triples $\{a, ka, c_1^-\}$ with $a = a_p^{(k)}$ ($p \geq 1$) defined in (2.4)-(2.9), if the equation $P_l = Q_m$ has a solution (l, m) with $l \geq 1$, then $p \leq E_k$ and $m \leq 2.48 \cdot 10^{20}$ where $E_k \in \{70, 110, 53, 65, 147, 81\}$ for $k \in \{7, 8, 10, 11, 12, 13\}$ respectively.
- 2) For the $D(4)$ -triples $\{a, ka, c_1^+\}$ with $a = a_p^{(k)}$ ($p \geq 1$) defined in (2.4)-(2.9), if the equation $P_l = Q_m$ has a solution (l, m) with $m \geq 3$, then $p \leq E'_k$ and $l \leq 2.56 \cdot 10^{20}$ where $E'_k \in \{71, 111, 54, 65, 149, 82\}$ for $k \in \{7, 8, 10, 11, 12, 13\}$ respectively.
- 3) For the $D(4)$ -triples $\{a, ka, c_2^\pm\}$ with $a = a_p^{(k)}$ ($p \geq 1$) defined in (2.4)-(2.9), if the equation $P_l = Q_m$ has a solution (l, m) in Type 1 with $m \geq 3$, then $p \leq E''_k$ and $l \leq 2.6 \cdot 10^{21}$ where $E''_k \in \{74, 116, 56, 68, 156, 86\}$ for $k \in \{7, 8, 10, 11, 12, 13\}$ respectively. If the equation $P_l = Q_m$ has a solution (l, m) in Type 2 with $m \geq 3$ then $p \leq E'''_k$ and $l \leq 3.13 \cdot 10^{20}$ where $E'''_k \in \{18, 28, 13, 16, 38, 21\}$ for $k \in \{7, 8, 10, 11, 12, 13\}$ respectively.

For the remaining cases, we will use the following lemma which is a slight modification of the original version of Baker-Davenport reduction method (see [7, Lemma 5a]).

LEMMA 5.2. *Assume that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 6M$ and let*

$$\eta = \| \mu q \| - M \cdot \| \kappa q \|,$$

where $\| \cdot \|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution to the inequality

$$0 < l\kappa - m + \mu < AB^{-l}$$

in integers l and m with

$$\frac{\log(Aq/\eta)}{\log(B)} \leq l \leq M.$$

In order to apply Lemma 5.2 we define parameters depending on c .

The case $c = c_1^-$. Dividing $0 < \Lambda' < 2.6\alpha^{-2l}$ by $\log \alpha$ and using the fact that $\alpha^{-2l} < \beta^{-m}$ we get

$$(5.38) \quad 0 < m\kappa - l + \mu < AB^{-m},$$

where

$$\kappa := \frac{\log \beta}{\log \alpha}, \quad \mu := \frac{\log \gamma'}{\log \alpha}, \quad A := \frac{2.6}{\log \alpha}, \quad B := \beta.$$

The case $c \in \{c_1^+, c_2^\pm\}$. Dividing $0 < \Lambda < 2.6\beta^{-2m}$ and $0 < \Lambda < 1.5a^2\beta^{-2m}$ by $\log \beta$ and using the fact that we have $\beta^{-2m} < \alpha^{-l}$ leads to an inequality of the form

$$(5.39) \quad 0 < l\kappa - m + \mu < AB^{-l},$$

where we consider solutions of Type 1

$$\kappa := \frac{\log \alpha}{\log \beta}, \quad \mu := \frac{\log \gamma}{\log \beta}, \quad A := \frac{2.6}{\log \beta}, \quad B := \alpha,$$

and for solutions of Type 2

$$\kappa := \frac{\log \alpha}{\log \beta}, \quad \mu := \frac{\log \gamma}{\log \beta}, \quad A := \frac{1.5a^2}{\log \beta}, \quad B := \alpha.$$

Let's first consider the case $c = c_1^-$. After at most four steps of reduction, we find that $P_l = Q_m$ implies $1 \leq l \leq m \leq 11$ in all cases. Combining this with Lemma 4.2, we get $a \leq 4460544$. Then we explicitly verify these remaining cases and find that the only solution for the equation $P_l = Q_m$ is $(l, m) = (2, 2)$ (from fundamental solutions $x_0 = 2, z_0 = 2, x_2 = 2, y_2 = -2$), which corresponds to the regular extension of a triple to a quadruple. For $l = 0$ we get $x = Q_0 = P_0 = 2$, which gives $d = 0$.

Now, for the cases $c \in \{c_1^+, c_2^\pm\}$, we find that after at most four steps of reduction, $P_l = Q_m$ implies $3 \leq m \leq l \leq 8$ in all cases. Combining this with Lemma 4.2 for solutions in Type 1, we get $a \leq 10240000$. We then explicitly verify these remaining cases and find that the equation $P_l = Q_m$ has no solutions in this range. For solutions in Type 2, we combine $3 \leq m \leq l \leq 8$ with Lemma 4.3, and in all cases, we obtain $a \leq 128$, which contradicts the fact that $b = ka > 10^5$, $k = 7, 8, 10, 11, 12, 13$. Since the relation $m \leq l$ only holds when $m \geq 3$, the final cases $m \in \{0, 1, 2\}$ are solved in the same way as in [1]. We conclude that the only possible intersection $P_l = Q_m$ (besides the trivial $(l, m) = (0, 0)$) is the one that corresponds to the regular extension of a triple to a quadruple.

5.2. *Proof of Theorem 1.3 with $c = c_3^\pm$.* In this case, we examine the equation $z = v_m = w_n$ using Lemma 3.2. By [9, Lemma 5], we know that if this equation has a solution (m, n) , then $n - 1 \leq m \leq 2n + 1$. We now examine the solutions for $2 < n < m < 2n$. The next result follows as in [1, Lemma 4.5].

LEMMA 5.3. *i) If the equation $z = v_{2m} = w_{2n}$ has a solution (m, n) with $n > 2$, then $m > 0.495b^{-0.5}c^{0.5}$.*

ii) If the equation $z = v_{2m+1} = w_{2n+1}$ has a solution (m, n) with $n > 2$, then $m^2 > 0.0625b^{-1}c^{0.5}$.

Filipin proved in [8] that $z = v_m = w_n$, for $n > 2$, implies

$$(5.40) \quad \frac{m}{\log(m+1)} < 6.543 \cdot 10^{15} \log^2 c.$$

Combining this with Lemma 5.3, in the case of even indices, we get

$$(5.41) \quad \frac{2 \cdot 0.495b^{-0.5}c^{0.5}}{\log(2 \cdot 0.495b^{-0.5}c^{0.5} + 1)} < 6.543 \cdot 10^{15} \log^2 c,$$

and in the case of odd indices, we get the inequality

$$(5.42) \quad \frac{2 \cdot 0.0625^{0.5}b^{-0.5}c^{0.25} + 1}{\log(2 \cdot 0.0625^{0.5}b^{-0.5}c^{0.25} + 2)} < 6.543 \cdot 10^{15} \log^2 c.$$

The solutions obtained from these inequalities are summarized in the following lemma.

LEMMA 5.4. *For the $D(4)$ -triples $\{a, ka, c_3^\pm\}$ with $a = a_p^{(k)}$ ($p \geq 1$) defined in (2.4)-(2.9), if $z = v_{2m} = w_{2n}$ has a solution (m, n) , then $p \leq E_k$ and $m \leq 5.2 \cdot 10^{21}$ where $E_k \in \{9, 14, 6, 8, 19, 10\}$ for $k \in \{7, 8, 10, 11, 12, 13\}$ respectively. However, if $z = v_{2m+1} = w_{2n+1}$ has a solution (m, n) , then $p \leq F_k$ and $m \leq 4.3 \cdot 10^{22}$, where $F_k \in \{25, 40, 19, 23, 53, 29\}$ for $k \in \{7, 8, 10, 11, 12, 13\}$ respectively.*

Now, for the remaining small values of p , by using Lemma 5.2 and Lemma 5.4 in the same way as in [1] we get that $z = v_m = w_n$ implies $n \leq m \leq 2$. In these small ranges, we verify that all solutions of $z = v_m = w_n$ give the extension of a $D(4)$ -triple $\{a, b, c\}$ to a regular quadruple. This completes the proof of Theorem 1.3.

REMARK. *The upper bound for b in Lemma 2.3 can be further improved using the same methodology, however, this yields additional finite exceptions that are sometimes non-trivial to eliminate. For example, assuming $a < b \leq 22.956a$ and that $\{1, 5, a, b\}$, $\{3, 4, a, b\}$, $\{2, 6, a, b\}$, and $\{1, 12, a, b\}$ are not $D(4)$ -quadruples, we still obtain $c = c_\nu^\pm$ for some positive integer ν . While this suggests that $D(4)$ -pairs $\{a, ka\}$ for $k > 13$ could be analyzed similarly, managing the exceptions for larger k becomes exceedingly complex. Proving that these exceptional pairs extend strictly to regular quadruples demands separate, extensive research. Additionally, the analogues of all theorems and lemmas presented herein would require independent verification for each k .*

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REFERENCES

- [1] K. N. Adédji, M. Bliznac Trebješanin, A. Filipin and A. Togbé, *On the $D(4)$ -pairs $\{a, ka\}$ with $k \in \{2, 3, 6\}$* , Glas. Mat. Ser. III **58** (2023), 35–57.
- [2] K. N. Adédji, B. He, A. Pintér and A. Togbé, *On the Diophantine pair $\{a, 3a\}$* , J. Number Theory **227** (2021), 330–351.
- [3] Lj. Bačić and A. Filipin, *The extensibility of $D(4)$ -pairs*, Math. Commun. **18** (2013), no. 2, 447–456.
- [4] M. Bliznac Trebješanin, *Extension of a Diophantine triple with the property $D(4)$* , Acta Math. Hungar. **163** (2021), 213–246.
- [5] M. Bliznac Trebješanin and A. Filipin, *Nonexistence of $D(4)$ -quintuples*, J. Number Theory **194** (2019), 170–217.
- [6] A. Dujella, *Number Theory*, Školska knjiga, Zagreb, (2021).
- [7] A. Dujella and A. Pethö, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) **49** (1998), 291–306.
- [8] A. Filipin, *There does not exist a $D(4)$ -sextuple*, J. Number Theory **128** (2008), 1555–1565.
- [9] A. Filipin, *On the size of sets in which $xy + 4$ is always a square*, Rocky Mountain J. Math. **39**(4) (2009), 1195–1224.
- [10] A. Filipin, *The extendibility of $D(4)$ -pair $\{F_{2k}, 5F_{2k}\}$* , Fibonacci Quart. **53** (2015), 124–129.
- [11] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers II*, Izv. Math **64** (2000), 1217–1269.

Proširenja $D(4)$ -parova $\{a, ka\}$, $k \in \{7, 8, 10, 11, 12, 13\}$

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SAŽETAK. U ovom radu se proučavaju proširenja $D(4)$ -parova $\{a, b\}$, gdje je $b = ka$ i $k \in \{7, 8, 10, 11, 12, 13\}$. Prvo se pokaže da se ti parovi mogu proširiti do $D(4)$ -trojke elementom c iz familije prirodnih brojeva koji ovise o elementu a . Zatim se pokaže da se takve $D(4)$ -trojke na jedinstven način proširuju do $D(4)$ -četvorke.

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