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NOVEL VARIANTS OF HERMITE-HADAMARD INEQUALITIES FOR $(\alpha, \eta)-$CONVEX FUNCTIONS OF 1^{st} AND 2^{nd} KINDS

MUHAMMAD BILAL AND ASIF R. KHAN

Abstract. In this article, we aim to present generalized results related to the well-known Hermite-Hadamard dual inequality for two distinct types of (α, η) convex functions, employing various techniques such as Hölder's and Power mean inequalities. Consequently, both established and new results will be encompassed as special cases. Additionally, we intend to explore some relationships between our findings with well-known special means and trapezoidal formula.

1. **Introduction**

In mathematics, the study of inequalities has gained increasing interest among researchers due to its wide range of applications. This field encompasses many important areas, particularly the theory of convex functions, which has received significant attention in the literature over the past few decades. Convexity has numerous practical applications in fields such as arts, architecture, industry, management science, economics, and more. Among the many applications of convexity, Hermite-Hadamard dual inequalities are particularly important due to their extensive use, especially involving various types of convex functions. For further study on this topic, see [1], [2], [6], [8], [9] and [12] – [18] and the references cited therein.

Before proceeding, it is important to introduce some notation that will be used in this article: I represents a real interval, I° denotes the interior of interval I, and $\beta(o_1, o_2) = \int_0^1 \xi^{o_1-1}(1-\xi)^{o_2-1} d\xi$, for $o_1, o_2 > 0$, is the Euler Beta function. Throughout this article, we adopt the convention that $0^0 = 1$.

We shall begin with some important definitions and results:

Theorem 1.1. [10] *Let $\varpi : I \to \mathbb{R}$ be a convex function. Then*

$$\varpi\left(\frac{o_1+o_2}{2}\right) \leq \frac{1}{o_2-o_1}\int_{o_1}^{o_2} \varpi(\zeta)d\zeta \leq \frac{\varpi(o_1)+\varpi(o_2)}{2}. \tag{1.1}$$

This result is known as the Hermite-Hadamard dual inequality for convex functions. For concave functions ϖ, both inequalities are reversed. It should be noted that Hadamard's inequality can be viewed as a refinement and follows naturally from Jensen's inequality.

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Recently, in 2022, Hassan and Khan introduced the concept of $(\alpha, \eta)-$convex functions of the first and second kinds. They further generalized the definitions of $s-$convex functions of the first and second kinds as follows:

Definition 1.2. [11] A function $\varpi : I \subset [0, \infty) \to \mathbb{R}$ is defined as $(\alpha, \eta)-$convex of the 1^{st} kind if

$$\varpi(\xi\zeta_1 + (1-\xi)\zeta_2) \leq \xi^{\alpha}\varpi(\zeta_1) + (1-\xi^{\eta})\varpi(\zeta_2), \tag{1.2}$$

$\forall\ \zeta_1, \zeta_2 \in I, \xi \in [0, 1]$ and $(\alpha, \eta) \in [0, 1]^2$.

Remark 1.3. The following well-known results can be derived by using different combinations of values for α and η.

(1) By selecting $\alpha = \eta = s$ in (1.2), we obtain the definition of an $s-$convex function of the 1^{st} kind [5].
(2) By selecting $\alpha = \eta = 0$ with $\xi \neq 0$ in (1.2), we obtain the refinement of quasi convex function [6].
(3) By selecting $\alpha = \eta = 1$ in (1.2), we obtain the definition of convex function [6].

Definition 1.4. According to [11], a function $\varpi : I \subset [0, \infty) \to \mathbb{R}$ is defined as $(\alpha, \eta)-$convex of the 2^{nd} kind if

$$\varpi(\xi\zeta_1 + (1-\xi)\zeta_2) \leq \xi^{\alpha}\varpi(\zeta_1) + (1-\xi)^{\eta}\varpi(\zeta_2), \tag{1.3}$$

$\forall\ \zeta_1, \zeta_2 \in I, \xi \in [0, 1]$ and $(\alpha, \eta) \in [-1, 1]^2$.

Remark 1.5. Please note that we enhance the definition given in [11] by incorporating specific values for α and η. The well-known results can be derived by using various combinations of these values.

(1) By selecting $\alpha = \eta = s$ in (1.3), we obtain the definition of an extended $s-$convex function of 2^{nd} kind [19].
(2) By selecting $\alpha = \eta = -s$ in (1.3), we obtain the definition of an extended $s-$Godunova Levin function of 2^{nd} kind [19].
(3) By selecting $\alpha = \eta = 0$ with $\xi \in (0, 1)$ in (1.3), we obtain the definition of $P-$convex function [6].
(4) By selecting $\alpha = \eta = 1$ in (1.3), we obtain the definition of convex function [6].
(5) By selecting $\alpha = \eta = -1$ in (1.3), we obtain the definition of Godunova Levin function [19].
(6) When $(\alpha, \eta) \in [-1, 0)$, then equation 1.3 simplifies to:

$$\varpi(\xi\zeta_1 + (1-\xi)\zeta_2) \leq \frac{\varpi(\zeta_1)}{\xi^{\alpha}} + \frac{\varpi(\zeta_2)}{(1-\xi)^{\eta}}, \tag{1.4}$$

where ϖ is referred to as an $(\alpha, \eta)-$Godunova Levin function of the 2^{nd} kind.

The general form of the well-known Hölder's integral inequality is presented as follows [13]:

Theorem 1.6. *Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $\varpi \in L_p$ and $\mu \in L_q$, then $\varpi\mu \in L_1$ and*

$$\int |\varpi(\zeta)\mu(\zeta)|d\zeta \leq \|\varpi\|_p \|\mu\|_q, \tag{1.5}$$

where $\varpi \in L_p$ *if* $\|\varpi\|_p = \left(\int |\varpi(\zeta)|^p d\zeta\right)^{\frac{1}{p}} < \infty$.

Observe that if we set $p = q = 2$, the above inequality simplifies to the Cauchy-Schwarz inequality. Additionally, by setting $q = 1$ and allowing $p \to \infty$, we obtain:

$$\int |\varpi(\zeta)\mu(\zeta)|d\zeta \leq ||\varpi||_\infty ||\mu||_1,$$

where $||\varpi||_\infty$ denotes the essential supremum of $|\varpi|$, which is defined as:

$$||\varpi||_\infty = ess \sup_{\forall \zeta} |\varpi(\zeta)|.$$

Definition 1.7. Let ϖ and μ be real-valued functions defined on $[o_1, o_2]$. If $|\varpi|$ and $|\varpi||\mu|^q$ are integrable on $[o_1, o_2]$, then for $q \geq 1$, we have:

$$\int_{o_1}^{o_2} |\varpi(\zeta)||\mu(\zeta)|d\zeta \leq \left(\int_{o_1}^{o_2} |\varpi(\zeta)|d\zeta\right)^{1-\frac{1}{q}} \left(\int_{o_1}^{o_2} |\varpi(\zeta)||\mu(\zeta)|^q d\zeta\right)^{\frac{1}{q}}.$$

The inequality above is referred to as the power mean inequality (see [16]).

We now present the following identity, taken from [3], which will be utilized to derive the main results of this article.

Lemma 1.8. *According to* [3]*, let* $\varpi : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function on* I° *with* $\varpi'' \in L[o_1, o_2]$*. Then the following identity holds:*

$$\frac{\varpi(o_1) + \varpi(o_2)}{2} - \frac{1}{o_2 - o_1}\int_{o_1}^{o_2} \varpi(\zeta)d\zeta = \frac{(o_2 - o_1)^2}{2}\int_0^1 \xi(1-\xi)\varpi''(\xi o_1 + (1-\xi)o_2)d\xi.$$

This article is organized as follows: In the next two sections, we will estimate the bounds of one of the Hermite-Hadamard inequalities (by evaluating the absolute difference between the middle and last terms of (1.1)) using twice differentiable (α, η)-convex functions of the first and second kinds. These results will include various cases found in [3], [4], [14], and [15]. The fourth and fifth sections explore applications to special means and trapezoidal formulas, respectively, while the sixth section provides concluding remarks. The final section offers additional comments and future directions.

2. Estimations of Right bound of Hermite-Hadamard Inequality for twice differentiable $(\alpha, \eta)-$Convex Function of 1^{st} Kind

We will now state and prove three generalized results concerning Hermite - Hadamard type inequalities for twice differentiable (α, η)-convex functions of the 1^{st} kind. These results will be derived using Definition 1.2, Theorem 1.6, Definition 1.7, and Lemma 1.8.

Theorem 2.1. *Let* $\varpi : I \subset [0, \infty) \to \mathbb{R}$ *be a twice differentiable function on* I° *with* $\varpi'' \in L[o_1, o_2]$*, where* $o_1, o_2 \in I$ *and* $o_1 < o_2$*. If* $|\varpi''|$ *is* (α, η)*-convex of the* 1^{st}

kind on $[o_1, o_2]$ *for some fixed* $(\alpha, \eta) \in [0,1]^2$*, then the following inequality holds:*

$$\left| \frac{\varpi(o_1) + \varpi(o_2)}{2} - \frac{1}{o_2 - o_1} \int_{o_1}^{o_2} \varpi(\zeta) d\zeta \right| \leq \frac{(o_2 - o_1)^2}{2} \left[\frac{|\varpi''(o_1)|}{(\alpha+2)(\alpha+3)} + |\varpi''(o_2)| \left\{ \frac{1}{6} - \frac{1}{(\eta+2)(\eta+3)} \right\} \right]. \quad (2.1)$$

Proof. Using Lemma 1.8 and applying the definition of (α, η)-convexity of the 1^{st} kind to $|\varpi''|$ on $[o_1, o_2]$, we obtain:

$$\begin{aligned} & \left| \frac{\varpi(o_1) + \varpi(o_2)}{2} - \frac{1}{o_2 - o_1} \int_{o_1}^{o_2} \varpi(\zeta) d\zeta \right| \\ \leq & \frac{(o_2 - o_1)^2}{2} \int_0^1 |\xi(1-\xi)| |\varpi''(\xi o_1 + (1-\xi) o_2)| d\xi \\ \leq & \frac{(o_2 - o_1)^2}{2} \int_0^1 |\xi(1-\xi)| \left\{ \xi^\alpha |\varpi''(o_1)| + (1-\xi^\eta) |\varpi''(o_2)| \right\} d\xi \\ = & \frac{(o_2 - o_1)^2}{2} \left[|\varpi''(o_1)| \int_0^1 \xi^{\alpha+1}(1-\xi) d\xi + |\varpi''(o_2)| \int_0^1 \xi(1-\xi)(1-\xi^\eta) d\xi \right]. \end{aligned}$$

By putting

$$\int_0^1 \xi^{\alpha+1}(1-\xi) d\xi = \frac{1}{(\alpha+2)(\alpha+3)} \quad (2.2)$$

and

$$\int_0^1 \xi(1-\xi)(1-\xi^\eta) d\xi = \frac{1}{6} - \frac{1}{(\eta+2)(\eta+3)}, \quad (2.3)$$

we get (2.1). □

Remark 2.2. From Theorem 2.1, one can deduce the following results:

(1) Choosing $\alpha = \eta = 0$ yields the second result of Theorem 3 from [4].
(2) Choosing $\alpha = \eta = 1$, yields the Remark 1 from [15].
(3) Choosing $\alpha = \eta = s$, yields the Corollary 1 from [4].

Theorem 2.3. *Let* $\varpi : I \subset [0, \infty) \to \mathbb{R}$ *be a twice differentiable function on* I° *with* $\varpi'' \in L[o_1, o_2]$*, where* $o_1, o_2 \in I$ *and* $o_1 < o_2$*. If* $|\varpi''|^q$ *is* (α, η)*-convex of the* 1^{st} *kind on* $[o_1, o_2]$ *for some fixed* $(\alpha, \eta) \in [0,1]^2$ *and* $q \geq 1$*, then the following* *inequality holds:*

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2}-\frac{1}{o_2-o_1}\int_{o_1}^{o_2}\varpi(\zeta)d\zeta\right| \leq \frac{(o_2-o_1)^2}{2}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[\frac{|\varpi''(o_1)|^q}{\alpha+1}+\frac{\eta|\varpi''(o_2)|^q}{\eta+1}\right]^{\frac{1}{q}}, \quad (2.4)$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Lemma 1.8, Hölder's integral inequality, and the definition of (α,η)-convexity of $|\varpi''|^q$ of the 1^{st} kind on $[o_1,o_2]$, we obtain:

$$\begin{aligned}
&\left|\frac{\varpi(o_1)+\varpi(o_2)}{2}-\frac{1}{o_2-o_1}\int_{o_1}^{o_2}\varpi(\zeta)d\zeta\right| \\
\leq\ & \frac{(o_2-o_1)^2}{2}\int_0^1|\xi(1-\xi)||\varpi''(\xi o_1+(1-\xi)o_2)|d\xi \\
\leq\ & \frac{(o_2-o_1)^2}{2}\left(\int_0^1|\xi(1-\xi)|^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1|\varpi''(\xi o_1+(1-\xi)o_2)|^q d\xi\right)^{\frac{1}{q}} \\
\leq\ & \frac{(o_2-o_1)^2}{2}\left(\int_0^1\xi^p(1-\xi)^p d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left\{\xi^\alpha|\varpi''(o_1)|^q+(1-\xi^\eta)|\varpi''(o_2)|^q\right\}d\xi\right)^{\frac{1}{q}} \\
=\ & \frac{(o_2-o_1)^2}{2}\left(\int_0^1\xi^p(1-\xi)^p d\xi\right)^{\frac{1}{p}}\left(|\varpi''(o_1)|^q\int_0^1\xi^\alpha d\xi+|\varpi''(o_2)|^q\int_0^1(1-\xi^\eta)d\xi\right)^{\frac{1}{q}}.
\end{aligned}$$

By putting

$$\int_0^1\xi^p(1-\xi)^p dt=\beta(p+1,p+1).$$

$$\int_0^1\xi^\alpha d\xi=\frac{1}{(\alpha+1)}$$

and

$$\int_0^1(1-\xi^\eta)d\xi=\frac{\eta}{\eta+1},$$

we get (2.4). □

Remark 2.4. From Theorem 2.3, one can deduce the following results:

(1) Choosing $\alpha=\eta=0$, yields the 2^{nd} result of Theorem 4 from [4].
(2) Choosing $\alpha=\eta=1$, yields the Corollary 2 from [15].
(3) Choosing $\alpha=\eta=s$, yields the 1^{st} result of Corollary 2 from [4].

Theorem 2.5. *Let $\varpi : I \subset (0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $\varpi'' \in L[o_1, o_2]$, where $o_1, o_2 \in I^\circ$ and $o_1 < o_2$. If $|\varpi''|^q$, $q \geq 1$ is $(\alpha,\eta)-$convex of the 1^{st} kind on $[o_1, o_2]$ for some fixed $(\alpha,\eta) \in [0,1]^2$, then following inequality holds:*

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2} - \frac{1}{o_2-o_1}\int_{o_1}^{o_2}\varpi(\zeta)d\zeta\right|$$
$$\leq \frac{(o_2-o_1)^2}{2(6)^{1-\frac{1}{q}}}\left[\frac{|\varpi''(o_1)|^q}{(\alpha+2)(\alpha+3)} + |\varpi''(o_2)|^q\left\{\frac{1}{6} - \frac{1}{(\eta+2)(\eta+3)}\right\}\right]^{\frac{1}{q}}. \quad (2.5)$$

Proof. By applying Lemma 1.8, the Power Mean Integral Inequality, and the definition of (α,η)-convexity of $|\varpi''|^q$ of the 1^{st} kind on $[o_1, o_2]$, we obtain:

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2} - \frac{1}{o_2-o_1}\int_{o_1}^{o_2}\varpi(\zeta)d\zeta\right|$$
$$\leq \frac{(o_2-o_1)^2}{2}\int_0^1 |\xi(1-\xi)||\varpi''(\xi o_1+(1-\xi)o_2)|d\xi$$
$$\leq \frac{(o_2-o_1)^2}{2}\left(\int_0^1|\xi(1-\xi)|d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1|\xi(1-\xi)||\varpi''(\xi o_1+(1-\xi)o_2)|^q d\xi\right)^{\frac{1}{q}}$$
$$\leq \frac{(o_2-o_1)^2}{2}\left(\int_0^1|\xi(1-\xi)|d\xi\right)^{1-\frac{1}{q}}\left[\int_0^1|\xi(1-\xi)|\left\{\xi^\alpha|\varpi''(o_1)|^q+(1-\xi^\eta)|\varpi''(o_2)|^q\right\}d\xi\right]^{\frac{1}{q}}$$
$$= \frac{(o_2-o_1)^2}{2}\left(\int_0^1|\xi(1-\xi)|d\xi\right)^{1-\frac{1}{q}}\left[|\varpi''(o_1)|^q\int_0^1\xi^{\alpha+1}(1-\xi)d\xi+|\varpi''(o_2)|^q\int_0^1\xi(1-\xi)(1-\xi^\eta)d\xi\right]^{\frac{1}{q}}.$$

By putting

$$\int_0^1 \xi(1-\xi)d\xi = \frac{1}{6}$$

and using (2.2), (2.3) we get (2.5). □

Remark 2.6. From Theorem 2.5, one can deduce the following results:

(1) Choosing $\alpha = \eta = 0$, yields the 2^{nd} result of Theorem 5 from [4].
(2) Choosing $\alpha = \eta = 1$, yields the 2^{nd} result of Corollary 3 from [4].
(3) Choosing $\alpha = \eta = s$, yields the 1^{st} result of Corollary 3 from [4].

3. Estimations of Right bound of Hermite-Hadamard Inequality for twice differentiable $(\alpha,\eta)-$Convex Function of 2^{nd} Kind

We will now present and prove three generalized results concerning Hermite-Hadamard type inequalities for twice differentiable (α,η)-convex functions of the 2^{nd} kind, utilizing Definition 1.4, Theorem 1.6, Definition 1.7, and Lemma 1.8.

Theorem 3.1. *Let $\varpi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° with $\varpi'' \in L[o_1, o_2]$, where $o_1, o_2 \in I$ and $o_1 < o_2$. If $|\varpi''|$ is (α,η)-convex of the 2^{nd} kind on $[o_1, o_2]$ for some fixed $(\alpha,\eta) \in [-1,1]^2$, then the following inequality holds:*

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2} - \frac{1}{o_2-o_1}\int_{o_1}^{o_2}\varpi(\zeta)d\zeta\right| \leq \frac{(o_2-o_1)^2}{2}\left[\frac{|\varpi''(o_1)|}{(\alpha+2)(\alpha+3)} + \frac{|\varpi''(o_2)|}{(\eta+2)(\eta+3)}\right]. \quad (3.1)$$

Proof. The above outcome can be demonstrated through a method analogous to that used in the proof of Theorem 2.1, relying on the definition of $(\alpha,\eta)-$convex functions of the 2^{nd} kind. □

Remark 3.2. From Theorem 3.1, one can deduce the following results:

(1) Choosing $\alpha = \eta = 0$ yields the 3^{rd} result of Corollary 1 from [4].
(2) Choosing $\alpha = \eta = 1$ yields the Remark 1 from [15].
(3) Choosing $\alpha = \eta = s$ yields the 2^{nd} result of Corollary 1 from [4].
(4) Choosing $\alpha = \eta = -s$ yields a specific result of Theorem 4 from [14], applicable only when $\alpha = 1$.
(5) Choosing $\alpha = \eta = -1$ yields a specific result of Theorem 4 from [14], applicable only when $\alpha = s = 1$.

Corollary 3.3. *Theorem 3.1 presents a result concerning the Hermite-Hadamard inequality for the class of (α,η)-GL functions when (α,η) is strictly within the interval $[-1,0)$:*

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2} - \frac{1}{o_2-o_1}\int_{o_1}^{o_2}\varpi(\zeta)d\zeta\right| \leq \frac{(o_2-o_1)^2}{2}\left[\frac{|\varpi''(o_1)|}{(2-\alpha)(3-\alpha)} + \frac{|\varpi''(o_2)|}{(2-\eta)(3-\eta)}\right].$$

Theorem 3.4. *Let $\varpi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° with $\varpi'' \in L[o_1, o_2]$, where $o_1, o_2 \in I$ and $o_1 < o_2$. If $|\varpi''|^q$ is (α,η)-convex of the 2^{nd} kind on $[o_1, o_2]$ for some fixed $(\alpha,\eta) \in (-1,1]^2$ and $q \geq 1$, then the following inequality holds:*

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2} - \frac{1}{o_2-o_1}\int_{o_1}^{o_2}\varpi(\zeta)d\zeta\right| \leq \frac{(o_2-o_1)^2}{2}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[\frac{|\varpi''(o_1)|^q}{\alpha+1} + \frac{|\varpi''(o_2)|^q}{\eta+1}\right]^{\frac{1}{q}}, \quad (3.2)$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The above outcome can be demonstrated through a method analogous to that used in the proof of Theorem 2.3, relying on the definition of (α,η)-convex functions of the 2^{nd} kind. □

Remark 3.5. From Theorem 3.4, one can deduce the following results:

(1) Choosing $\alpha = \eta = 0$ yields the 2^{nd} result of Corollary 2 from [4].
(2) Choosing $\alpha = \eta = 1$ yields Corollary 2 from [15].
(3) Choosing $\alpha = \eta = s$ yields the Theorem 10 from [3].

Corollary 3.6. *Theorem* 3.4 *presents the following result concerning the Hermite-Hadamard inequality:*

(1) *For the class of $(\alpha, \eta)-GL$ function if (α, η) strictly belongs to $(-1, 0)$:*

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2} - \frac{1}{o_2 - o_1}\int_{o_1}^{o_2} \varpi(\zeta)d\zeta\right| \le \frac{(o_2-o_1)^2}{2}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[\frac{|\varpi''(o_1)|^q}{1-\alpha}+\frac{|\varpi''(o_2)|^q}{1-\eta}\right]^{\frac{1}{q}}.$$

(2) *For the class of $s-GL$ function by taking $\alpha = \eta = -s$:*

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2} - \frac{1}{o_2 - o_1}\int_{o_1}^{o_2} \varpi(\zeta)d\zeta\right| \le \frac{(o_2-o_1)^2}{2}\left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}}\left[\frac{|\varpi''(o_1)|^q+|\varpi''(o_2)|^q}{1-s}\right]^{\frac{1}{q}}.$$

Theorem 3.7. *Let $\varpi : I \subset (0, \infty) \to \mathbb{R}$ be a twice differentiable function on I° such that $\varpi'' \in L[o_1, o_2]$, where $o_1, o_2 \in I^\circ$ and $o_1 < o_2$. If $|\varpi''|^q$, with $q \ge 1$, is (α, η)-convex of the 2^{nd} kind on $[o_1, o_2]$ for some fixed $(\alpha, \eta) \in [-1, 1]^2$, then the following inequality holds:*

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2} - \frac{1}{o_2 - o_1}\int_{o_1}^{o_2} \varpi(\zeta)d\zeta\right| \le \frac{(o_2-o_1)^2}{2(6)^{1-\frac{1}{q}}}\left[\frac{|\varpi''(o_1)|^q}{(\alpha+2)(\alpha+3)}+\frac{|\varpi''(o_2)|^q}{(\eta+2)(\eta+3)}\right]^{\frac{1}{q}}. \quad (3.3)$$

Proof. The above outcome can be demonstrated through a method analogous to that used in the proof of Theorem 2.5, relying on the definition of $(\alpha, \eta)-$convex functions of the 2^{nd} kind. □

Remark 3.8. From Theorem 3.7, one can deduce the following results:

(1) Choosing $\alpha = \eta = 0$ yields the 3^{rd} result of Corollary 3 from [4].
(2) Choosing $\alpha = \eta = 1$ yields the 2^{nd} result of Corollary 3 from [4].
(3) Choosing $\alpha = \eta = s$ yields Theorem 8 from [3].
(4) Choosing $\alpha = \eta = -s$ yields a specific result of Theorem 5 from [14], applicable only when $\alpha = 1$.
(5) Choosing $\alpha = \eta = -1$ yields a specific result of Theorem 5 from [14], applicable only when $\alpha = s = 1$.

Corollary 3.9. *Theorem 3.7 presents the following result concerning the Hermite-Hadamard inequality for the class of (α, η)-GL functions, provided (α, η) strictly* *belongs to* $[-1,0)$:

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2}-\frac{1}{o_2-o_1}\int_{o_1}^{o_2}\varpi(\zeta)d\zeta\right|$$
$$\leq \frac{(o_2-o_1)^2}{2\times(6)^{1-\frac{1}{q}}}\left[\frac{|\varpi''(o_1)|^q}{(2-\alpha)(3-\alpha)}+\frac{|\varpi''(o_2)|^q}{(2-\eta)(3-\eta)}\right]^{\frac{1}{q}}.$$

In the subsequent two sections, we will explore the applications of the results we have derived, specifically in the context of the special means and trapezoidal rule.

4. **Application to special means**

We will now introduce the definitions of the following special means, as detailed in [7]:
(1) The Arithmetic mean:

$$A=A(o_1,o_2)=\frac{o_1+o_2}{2}.$$

(2) The Geometric mean:

$$G=G(o_1,o_2)=\sqrt{o_1o_2};\ o_1,o_2\geq 0.$$

(3) The Harmonic mean:

$$H=H(o_1,o_2)=\frac{2o_1o_2}{o_1+o_2};\quad o_1,o_2\in(0,\infty).$$

(4) The Logarithmic mean:

$$L=L(o_1,o_2)=\frac{o_2-o_1}{\ln o_2-\ln o_1};\ o_1\neq o_2\quad \&\quad o_1,o_2\in(0,\infty).$$

We will now establish connections between different means using the results from the previous sections.

Example 4.1. Consider the function ϖ defined by $\varpi(\zeta)=\frac{1}{\zeta}$, where $0<o_1<o_2$. Then, we have:

$$\frac{1}{o_2-o_1}\int_{o_1}^{o_2}\varpi(\zeta)d\zeta=L^{-1}(o_1,o_2)=L^{-1},$$

$$\frac{\varpi(o_1)+\varpi(o_2)}{2}=H^{-1}(o_1,o_2)=H^{-1},$$

$$\frac{|\varpi''(o_1)|+|\varpi''(o_2)|}{2}=2H^{-1}\left({o_1}^3,{o_2}^3\right)$$

and

$$\left(\frac{|\varpi''(o_1)|^q+|\varpi''(o_2)|^q}{2}\right)^{\frac{1}{q}}=2\left[H^{-1}\left({o_1}^{3q},{o_2}^{3q}\right)\right]^{\frac{1}{q}}.$$

(1) Then (2.1) becomes,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(o_2-o_1)^2}{G^2\left({o_1}^3,{o_2}^3\right)}\left[\frac{{o_2}^3}{(\alpha+2)(\alpha+3)}+{o_1}^3\left\{\frac{1}{6}-\frac{1}{(\eta+2)(\eta+3)}\right\}\right]. \tag{4.1}$$

(a) If we put $\alpha=\eta=0$ in (4.1), we get,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{{o_2}^3(o_2-o_1)^2}{6G^2\left({o_1}^3,{o_2}^3\right)}.$$

(b) If we put $\alpha=\eta=s$, in (4.1), we get,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(o_2-o_1)^2}{6G^2\left({o_1}^3,{o_2}^3\right)}\left[\frac{6{o_2}^3+s(s+5){o_1}^3}{(s+2)(s+3)}\right].$$

(c) If we put $\alpha=\eta=1$, in (4.1), we get,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(o_2-o_1)^2}{6}H^{-1}\left({o_1}^3,{o_2}^3\right).$$

(2) Then (3.1) becomes,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(o_2-o_1)^2}{G^2\left({o_1}^3,{o_2}^3\right)}\left[\frac{{o_2}^3}{(\alpha+2)(\alpha+3)}+\frac{{o_1}^3}{(\eta+2)(\eta+3)}\right]. \tag{4.2}$$

(a) If we put $\alpha=\eta=0$ in (4.2), we get,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(o_2-o_1)^2A({o_1}^3,{o_2}^3)}{3G^2\left({o_1}^3,{o_2}^3\right)}=\frac{(o_2-o_1)^2}{3H({o_1}^3,{o_2}^3)}.$$

(b) If we put $\alpha=\eta=s$, in (4.2), we get,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{2(o_2-o_1)^2A\left({o_2}^3,{o_1}^3\right)}{G^2\left({o_1}^3,{o_2}^3\right)(s+2)(s+3)}=\frac{2(o_2-o_1)^2}{H({o_1}^3,{o_2}^3)(s+2)(s+3)}.$$

(c) If we put $\alpha=\eta=1$, in (4.2), we get,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(o_2-o_1)^2A\left({o_2}^3,{o_1}^3\right)}{3G^2\left({o_1}^3,{o_2}^3\right)}=\frac{(o_2-o_1)^2}{3}H^{-1}\left({o_1}^3,{o_2}^3\right).$$

(d) If we put (α,η) strictly belongs to $(-1,0)$, in (4.2), we get,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(o_2-o_1)^2}{G^2\left({o_1}^3,{o_2}^3\right)}\left[\frac{{o_2}^3}{(2-\alpha)(3-\alpha)}+\frac{{o_1}^3}{(2-\eta)(3-\eta)}\right].$$

(e) If we put $\alpha=\eta=-s$, in (4.2), we get,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{2(o_2-o_1)^2A({o_1}^3,{o_2}^3)}{G^2\left({o_1}^3,{o_2}^3\right)(2-s)(3-s)}=\frac{2(o_2-o_1)^2}{H({o_1}^3,{o_2}^3)(2-s)(3-s)}.$$

(f) If we put $\alpha=\eta=-1$, in (4.2), we get,

$$\left|H^{-1}-L^{-1}\right| \leq \frac{(o_2-o_1)^2A({o_1}^3,{o_2}^3)}{G^2\left({o_1}^3,{o_2}^3\right)}=\frac{(o_2-o_1)^2}{H({o_1}^3,{o_2}^3)}.$$

Remark 4.2. Similarly, by considering different convex functions, we can derive additional relationships among various special means.

5. Application to Trapezoidal Rule

Let J be a division of the interval $[o_1, o_2]$, i.e., $J : o_1 = \nu_0 < \nu_1 < ... < \nu_{n-1} < \nu_n = o_2$ and consider the quadrature formula

$$X = \int_{o_1}^{o_2} \varpi(\zeta) d\zeta = T(\varpi, J) + R(\varpi, J)$$

where

$$T(\varpi, J) = \sum_{k=0}^{n-1} \frac{\varpi(\nu_k) + \varpi(\nu_{k+1})}{2} (\nu_{k+1} - \nu_k)$$

is the trapezoidal formula and $R(\varpi, J)$ denotes the associated approximation error of the integral I.

Theorem 5.1. *Let $\varpi : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable function on I° with $\varpi'' \in L[o_1, o_2]$, where $o_1, o_2 \in I$ and $o_1 < o_2$. If $|\varpi''|$ is (α, η)-convex of the 1^{st} kind on $[o_1, o_2]$ for some fixed $(\alpha, \eta) \in [0, 1]^2$ and for every division J of $[o_1, o_2]$ then following inequality holds:*

$$\begin{aligned} &|R(\varpi, J)| \\ &\leq \sum_{k=0}^{n-1} \frac{(\nu_{k+1} - \nu_k)^3}{2} \left[\frac{|\varpi''(\nu_k)|}{(\alpha+2)(\alpha+3)} + |\varpi''(\nu_{k+1})| \left\{ \frac{1}{6} - \frac{1}{(\eta+2)(\eta+3)} \right\} \right]. \end{aligned}$$

Proof. Applying inequality (2.1) on $[\nu_k, \nu_{k+1}]$ and summing over k from 0 to $n-1$ and then by using triangular inequality we get the result of Theorem (5.1). □

Theorem 5.2. *Let $\varpi : I \subset [0, \infty) \to \mathbb{R}$ be a twice differentiable function on I° with $\varpi'' \in L[o_1, o_2]$, where $o_1, o_2 \in I$ and $o_1 < o_2$. If $|\varpi''|^q$ is (α, η)-convex of the 1^{st} kind on $[o_1, o_2]$ for some fixed $(\alpha, \eta) \in [0, 1]^2$ and $q \geq 1$ and for every division J of $[o_1, o_2]$ then following inequality holds:*

$$|R(\varpi, J)| \leq \sum_{k=0}^{n-1} \frac{(\nu_{k+1} - \nu_k)^3}{2} \{\beta(p+1, p+1)\}^{\frac{1}{p}} \left[\frac{|\varpi''(\nu_k)|^q}{\alpha+1} + \frac{\eta|\varpi''(\nu_{k+1})|^q}{\eta+1} \right]^{\frac{1}{q}},$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof of the aforementioned outcome employs a similar technique, applied to the inequality (2.4) instead of (2.1), akin to the approach employed in Theorem 5.1. □

Theorem 5.3. *Let $\varpi : I \subset (0, \infty) \to \mathbb{R}$ be a twice differentiable mapping on I° such that $\varpi'' \in L[o_1, o_2]$, where $o_1, o_2 \in I^\circ$ and $o_1 < o_2$. If $|\varpi''|^q$, $q \geq 1$ is $(\alpha, \eta)-$convex of the 1^{st} kind on $[o_1, o_2]$ for some fixed $(\alpha, \eta) \in [0, 1]^2$ and for every division J of $[o_1, o_2]$ then following inequality holds:*

$$\begin{aligned} &|R(\varpi, J)| \\ &\leq \sum_{k=0}^{n-1} \frac{(\nu_{k+1} - \nu_k)^3}{2(6)^{1-\frac{1}{q}}} \left[\frac{|\varpi''(\nu_k)|^q}{(\alpha+2)(\alpha+3)} + |\varpi''(\nu_{k+1})|^q \left\{ \frac{1}{6} - \frac{1}{(\eta+2)(\eta+3)} \right\} \right]^{\frac{1}{q}}. \end{aligned}$$

Proof. The proof of the aforementioned outcome employs a similar technique, applied to the inequality (2.5) instead of (2.1), akin to the approach employed in Theorem 5.1. □

Theorem 5.4. *Let $\varpi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° with $\varpi'' \in L[o_1, o_2]$, where $o_1, o_2 \in I$ and $o_1 < o_2$. If $|\varpi''|$ is (α,η)-convex of the 2^{nd} kind on $[o_1, o_2]$ for some fixed $(\alpha,\eta) \in [0,1]^2$ and for every division J of $[o_1, o_2]$ then the following inequality holds:*

$$|R(\varpi, J)| \leq \sum_{k=0}^{n-1} \frac{(\nu_{k+1}-\nu_k)^3}{2} \left[\frac{|\varpi''(\nu_k)|}{(\alpha+2)(\alpha+3)} + \frac{|\varpi''(\nu_{k+1})|}{(\eta+2)(\eta+3)} \right].$$

Proof. The proof of the aforementioned outcome employs a similar technique, applied to the inequality (3.1) instead of (2.1), akin to the approach employed in Theorem 5.1. □

Theorem 5.5. *Let $\varpi : I \subset [0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° with $\varpi'' \in L[o_1, o_2]$, where $o_1, o_2 \in I$ and $o_1 < o_2$. If $|\varpi''|^q$ is (α,η)-convex of the 2^{nd} kind on $[o_1, o_2]$ for some fixed $(\alpha,\eta) \in (-1,1]^2$ and $q \geq 1$ and for every division J of $[o_1, o_2]$ then the following inequality holds:*

$$|R(\varpi, J)| \leq \sum_{k=0}^{n-1} \frac{(\nu_{k+1}-\nu_k)^3}{2} \left\{\beta(p+1,p+1)\right\}^{\frac{1}{p}} \left[\frac{|\varpi''(\nu_k)|^q}{\alpha+1} + \frac{|\varpi''(\nu_{k+1})|^q}{\eta+1} \right]^{\frac{1}{q}},$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The proof of the aforementioned outcome employs a similar technique, applied to the inequality (3.2) instead of (2.1), akin to the approach employed in Theorem 5.1. □

Theorem 5.6. *Let $\varpi : I \subset (0,\infty) \to \mathbb{R}$ be a twice differentiable function on I° such that $\varpi'' \in L[o_1, o_2]$, where $o_1, o_2 \in I^\circ$ and $o_1 < o_2$. If $|\varpi''|^q$, with $q \geq 1$, is (α,η)-convex of the 2^{nd} kind on $[o_1, o_2]$ for some fixed $(\alpha,\eta) \in [-1,1]^2$ and for every division J of $[o_1, o_2]$ then the following inequality holds:*

$$|R(\varpi, J)| \leq \sum_{k=0}^{n-1} \frac{(\nu_{k+1}-\nu_k)^3}{2(6)^{1-\frac{1}{q}}} \left[\frac{|\varpi''(\nu_k)|^q}{(\alpha+2)(\alpha+3)} + \frac{|\varpi''(\nu_{k+1})|^q}{(\eta+2)(\eta+3)} \right]^{\frac{1}{q}}.$$

Proof. The proof of the aforementioned outcome employs a similar technique, applied to the inequality (3.3) instead of (2.1), akin to the approach employed in Theorem 5.1. □

6. **Conclusion**

The Hermite-Hadamard dual inequality is among the most well-known inequalities, with numerous generalizations and variants documented in the literature. We have extended this inequality by employing the newly generalized concept of (α,η)-convex functions of the first and second kinds. In Sections 2 and 3, we present three distinct results concerning the estimation of the bound on the difference between the middle and left terms of the Hermite-Hadamard dual inequality in absolute terms, specifically for twice differentiable (α,η)-convex functions of both kinds. We employed various techniques, including Hölder's and power mean inequalities. These results encompass several findings from the articles [3], [4], [14], and [15]. Section 4 and 5 explore some relationships between our derived results with well-known special means and trapezoidal formulas, respectively. Finally, the concluding section provides remarks and outlines future research directions for readers.

We will now provide some observations and suggest potential directions for future research related to our results.

7. Remarks and Future Ideas

(1) All the inequalities presented in this article can also be expressed in reverse for concave functions by using the simple relationship that a function f is concave if and only if $-f$ is convex.

(2) According to [17, p. 140], for a convex function f, we have that

$$\frac{1}{o_2 - o_1}\int_{o_1}^{o_2} \varpi(\zeta)d\zeta - \varpi\left(\frac{o_1+o_2}{2}\right) \leq \frac{\varpi(o_1)+\varpi(o_2)}{2} - \frac{1}{o_2 - o_1}\int_{o_1}^{o_2} \varpi(\zeta)d\zeta \quad (7.1)$$

In all the results presented in Section 2, we derived bounds specifically for

$$\left|\frac{\varpi(o_1)+\varpi(o_2)}{2} - \frac{1}{o_2 - o_1}\int_{o_1}^{o_2} \varpi(\zeta)d\zeta\right|$$

By applying the relation provided in (7.1), we directly obtain bounds for

$$\left|\varpi\left(\frac{o_1+o_2}{2}\right) - \frac{1}{o_2 - o_1}\int_{o_1}^{o_2} \varpi(\zeta)d\zeta\right|$$

(3) One could also explore Fejer's inequality by incorporating weights into the Hermite-Hadamard dual inequality.

(4) Similar work could be undertaken using various other classes of functions.

(5) One could also explore expressing all the results presented in this article in the discrete case.

(6) One could also extend all the results discussed in this article to higher dimensions.

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References

[1] M. A. Ali, Z. Zhang and M. Fečan. Generalization of Hermite-Hadamard–Mercer and Trapezoid Formula Type Inequalities Involving the Beta Function. *Rocky Mountain J. Math.* **54** (2), (2024), 331 – 341.

[2] R. P. Agarwal, A. R. Khan and S. Saadi. Integral Results Related to Similarity Separable Vectors in Separable Hilbert Spaces. *Foundations* **2**, (2022), 813 – 826.

[3] M. I. Bhatti, M. Iqbal and M. Hussain, Hadamard – type inequalities for $s-$convex functions I, *Punjab. Univ. J. Math.*, **41**, (2009), 51 – 60.

[4] M. Bilal and A. R. Khan. Generalized Hermite-Hadamard Type Inequalities for Twice Differentiable $(r,s)-$Convex Functions. *Appl. Math. E-Notes*, **24**(2024), 350 – 361.
[5] M. Bilal, S. S. Dragomir and A. R. Khan. Generalized Hermite-Hadamard Type Inequalities for $(\alpha,\eta,\gamma,\delta)-p-$convex functions. *RAD HAZU. MATEMATIČKE ZNANOSTI*, **29=564**(2025), 145 – 186.
[6] M. Bilal and A. R. Khan. New Generalized Hermite–Hadamard Inequalities for $p-$convex functions in the mixed kind. *EJPAM* **14**(3), (2021), 863 – 880.
[7] M. Bilal, N. Irshad and A. R. Khan. New Version of Generalized Ostrowski – Grüss Type Inequality. *Stud. Univ. Babeş-Bolyai Math.* **66**(3), (2021), 441 – 455.
[8] M. Bohner, H. Budak and H. Kara. Post-Quantum Hermite-Jensen-Mercer Inequalities, *Rocky Mountain J. Math.*, **53** (1), (2023), 17 – 26.
[9] S. S. Dragomir, Hermite-Hadamard Type Inequalities for $MN-$Convex Functions, *Aust. J. Math. Anal. Appl.*, **18** (1), (2021), Art. 1, 127pp.
[10] Ch. Hermite, Sur deux limites d'une int'egrale d'efinie, Mathesis 3 (1883), 82.
[11] A. Hassan and A. R. Khan, Generalized Fractional Ostrowski Type Inequalities via $(\alpha,\beta,\gamma,\delta)$ Convex Functions. *Frac. Diff. Calculus*, **12**(1), (2022), 13 – 36.
[12] Z. X. Mao, W. B. Zhang and J. F. Tian. On Hardy and Hermite-Hadamard Inequalities for N-Tuple Diamond Alpha Integral. *Hacet.. J. Math. Stat.*, **53** (3), (2024), 667 – 689.
[13] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, Classical and new inequalities in analysis, Kluwer Academic Publishers, Dordrecht (1993).
[14] M. A. Noor, K. I. Noor, M. M. U. Awan and S. Khan, Fractional Hermite-Hadamard inequalities for some new classes of Godunova Levin Functions, *Appl. Math. Inf. Sci.*, **8**(6), (2014), 2865 – 2872.
[15] M. E. Özdemír, M. Avci and H. Kavurmaci, Hermite – Hadamard – type inequalities via (α, m) convexity, *Comp. Math. App.*, **61**, (2011), 2614 – 2620.
[16] S. Özcan, and I. Işcan. Some new Hermite – Hadamard type inequalities for $s-$convex functions and their applications, *J. Ineq. App.*, (2019), Article 201.
[17] J. Pečarić, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.
[18] J. Pečarić, J. Barić, Lj. Kvesić and M. Ribičić Penava, Estimates on Some Quadrature Rules Via Weighted Hermite-Hadamard Inequality, *Appl. Anal. Discrete Math.*, **16**, (2022), 232 – 245.
[19] B. Y. Xi and F. Qi, Inequalities of Hermite-Hadamard type for extended s convex functions and applications to means, *J. Nonlin. & Con. Anal.*, **16**(5), (2015), 873 – 890.

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