RAD HRVATSKE AKADEMIJE ZNANOSTI I UMJETNOSTI
MATEMATIČKE ZNANOSTI

M. J. Bertin and O. Lecacheux

Isogenies between some $K3$ surfaces

Manuscript accepted for publication

This is a preliminary PDF of the author-produced manuscript that has been peer-reviewed and accepted for publication. It has not been copy-edited, proofread, or finalized by Rad HAZU Production staff.

ISOGENIES BETWEEN SOME $K3$ SURFACES

Marie José Bertin and Odile Lecacheux

Abstract. Elliptic fibrations of $K3$ surfaces belonging to the Apéry-Fermi pencil (Y_k) may have 3-torsion sections defining on (Y_k) automorphisms τ of order 3. We prove that the generic $K3$ surface Y_k with transcendental lattice $U\oplus <12>$ is transformed in a $K3$ surface Y_k/τ with transcendental lattice $U(3)\oplus <4>$. We also prove that for all the fibrations of Y_2 with 3-torsion sections $Y_2/\tau = Y_{10}$. Results are different for Y_{10} where we can obtain for Y_{10}/τ one of the two surfaces with transcendental lattice $\begin{pmatrix} 4 & 0 \\ 0 & 18 \end{pmatrix}$ or $\begin{pmatrix} 2 & 0 \\ 0 & 36 \end{pmatrix}$. We explicit also the link between 3-isogeny on a fibration and base change on other fibrations.

1. Introduction

The Apéry-Fermi pencil (Y_k) defined by the equation

$$(Y_k) \qquad X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = k,$$

was first studied by Peters and Stienstra [17] who computed its transcendental lattice to be $U\oplus <12>$ (U being the hyperbolic lattice).

Then Bertin was interested in the logarithmic Mahler measure $m(Y_k)$ of the polynomials Y_k which is a height function for polynomials [1] [2]. She proved that, for special values of k, namely k such that the desingularization of the set of zeroes of Y_k defines a singular $K3$ surface (i.e. a $K3$ surface of Picard number 20), $m(Y_k)$ is expressed in terms of the L-series at $s = 3$ of a weight 3 modular form. Even more, she proved that this L-series is the L-series of the transcendental lattice of the singular $K3$ surface Y_k.

2020 *Mathematics Subject Classification.* Primary 11F23, 11G05, 14J28.

Key words and phrases. Elliptic fibrations of $K3$ surfaces, Isogenies between Elliptic Fibrations, Transcendental lattices.

Recall that the transcendental lattice $T(Y)$ of a $K3$ surface Y, orthogonal complement of its Néron-Severi lattice $NS(Y)$ in the unimodular lattice $H^2(Y,\mathbb{Z})$, is a birational invariant of the algebraic surface Y. When Y is singular, the Gram matrix of its transcendental lattice $T(Y)$ has rank 2 and by Liné's modularity theorem, the L-series of $T(Y)$ is modular meaning there exists a weight 3 modular form f, with complex multiplication by $\mathbb{Q}(\sqrt{(-|\det((T(Y))|})$, satisfying $L(T(Y),s) \doteq L(f,s)$ where $\doteq$ means up to a finite number of primes.

In particular, Bertin proved that

$$L(T(Y_2),s) = L(T(Y_{10},s) = L(f_8,s),$$

where f_8 denotes the weight 3 modular form of level 8 and that the corresponding transcendental lattices verify [1],

$$T(Y_2) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \qquad T(Y_{10}) = \begin{pmatrix} 6 & 0 \\ 0 & 12 \end{pmatrix} = T(Y_2)[3].$$

Even more, in their paper ([5], section 5.2.1), Bertin and Lecacheux obtained Weierstrass equations of two rank 0 elliptic fibrations of Y_{10} by 3-isogenies from two rank 0 elliptic fibrations of Y_2.

Isogenies between $K3$ surfaces have been yet considered by many authors. Garbagnati and Prieto-Montañez [9] [10] studied 3-isogenies related to Kummer3 structures. Camere and Garbagnati [7] considered either n isogenies or involutions but are mainly interested in $K3$ surfaces X_n with a symplectic automorphism of order n, which are also isomorphic to another $K3$ surface by an automorphism of order n. Our point of view is quite different and for understanding the geometric link between Y_2 and Y_{10} we were tempted to study two problems.

The first problem is the following. Given a $K3$ surface S with transcendental lattice $T(S)$ and an elliptic fibration with a 3-torsion section, leading by 3-isogeny to a fibration of a K3 surface S', we want to understand when the transcendental lattice $T(S')$ is equal to $T(S)[3]$ since, in case of a pencil of $K3$ surfaces, the transcendental lattice of the specialized surfaces satisfies the same property.

The second problem is to know, in case there are more than one 3-torsion elliptic fibration on the surface S, if by 3-isogeny we get different surfaces.

In this paper, these two problems are completely solved for $S = Y_2$ and for the generic surface Y_k since all their 3-torsion elliptic fibrations are known (see [3] for Y_2 and [4] for Y_k). But it is not the case for Y_{10} for which only partial results can be deduced [5].

We rely the first problem to the following base change. Denote s_3 a 3-torsion section of Y_2 inducing, by 3-isogeny, an elliptic fibration of Y_{10}. Functions fields of Y_2 and Y_{10} are linked by a degree 3 cyclic extension field expressed from Kummer theory by the relation $u^3 = a$, where a is the function on Y_2 associated to the divisor $3((0)-s_3)$ and u the function on Y_{10}. Using a Weierstrass equation of Y_2 of the shape

$$Y^2+a_1(t)XY+a_3(t)Y=X^3,$$

if there exists on Y_2 an elliptic fibration of parameter $Y=a$, the base change $u^3=a$ leads to a fibration of Y_{10}. This construction can be applied to Y_2, thus gives rank 4 and 7 elliptic fibrations of Y_{10}.

Base change techniques have been used in algebraic geometry by many authors. Let us cite, for example, Hindry and Salgado's article [12] concerning Abelian varieties.

Another interesting use of base change is given in Garbagnati and Salgado's paper [11] where applications are given for $K3$ surfaces with low Picard number, namely Picard number 3, and Néron-Severi lattice isomorphic to $U\oplus<-2n>$. Hence it is a totally opposite case to ours, since we consider $K3$ surfaces with transcendental lattice $U\oplus<2n>$. The present paper is concerned with the generic $K3$ surface $U\oplus<12>$ and its specializations.

For the second problem, we use explicit computations of the Néron-Severi lattice. Concerning Y_2 (resp. Y_k) it leads by the various 3-isogenies to the same $K3$ surface, Y_{10} (resp. N_k with transcendental lattice $T(N_k)=U(3)\oplus<4>$). Concerning Y_{10}, different surfaces are obtained namely Y_2, and $K3$ surfaces with transcendental lattice $\begin{pmatrix}2&0\\0&36\end{pmatrix}$ or $\begin{pmatrix}4&0\\0&18\end{pmatrix}$.

The paper is divided into 5 sections. In section 2 we recall definitions and theorems used in the proofs.

Section 3 is devoted to 3-isogenies from generic elliptic fibrations of Y_k and its specialisations to $k=2$ and $k=10$. A degree 3 base change is also studied on a generic elliptic fibration of Y_k with two singular fibers of type II^*.

In section 4 we deal with all the 3-isogenies of Y_2 while we obtain in section 5 high rank elliptic fibrations of Y_{10} from degree 3 base change on elliptic fibrations of Y_2.

Finally section 6 presents some 3-isogenies from elliptic fibrations of Y_{10}.

2. Background

2.1. *Notations.* The singular fibers of type $I_n, D_m, IV^*, \ldots$ at $t=t_1,.,t_m$ or at roots of a polynomial $p(t)$ of degree m are denoted $mI_n(t_1,..,t_m)$ or $mI_n(p(t))$. Concerning the I_n singular fibers, the zero component of a reducible fiber is the component intersecting the zero section and is denoted θ_0 or $\theta_{t_0,0}$. The other components denoted $\theta_{t_0,i}$ satisfy the property $\theta_{t_0,i}\cdot\theta_{t_0,i+1}=1$.

2.2. *Discriminant forms.* Let L be a non-degenerate lattice. The **dual lattice** L^* of L is defined by

$$L^* := \mathrm{Hom}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q}/ \ b(x, y) \in \mathbb{Z} \text{ for all } y \in L\}.$$

and the **discriminant group** G_L by

$$G_L := L^*/L.$$

This group is finite if and only if L is non-degenerate. In the latter case, its order is equal to the absolute value of the lattice determinant $\mid \det(G(e)) \mid$ for any basis e of L. A lattice L is **unimodular** if G_L is trivial.

Let G_L be the discriminant group of a non-degenerate lattice L. The bilinear form on L extends naturally to a $\mathbb{Q}$-valued symmetric bilinear form on L^* and induces a symmetric bilinear form

$$b_L : G_L \times G_L \to \mathbb{Q}/\mathbb{Z}.$$

If L is even, then b_L is the symmetric bilinear form associated to the quadratic form defined by

$$\begin{array}{rcl} q_L : G_L & \to & \mathbb{Q}/2\mathbb{Z} \\ q_L(x + L) & \mapsto & x^2 + 2\mathbb{Z}. \end{array}$$

The latter means that $q_L(na) = n^2 q_L(a)$ for all $n \in \mathbb{Z}$, $a \in G_L$ and $b_L(a, a') = \frac{1}{2}(q_L(a + a') - q_L(a) - q_L(a'))$, for all $a, a' \in G_L$, where $\frac{1}{2} : \mathbb{Q}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is the natural isomorphism. The pair $\boldsymbol{(G_L, b_L)}$ (resp. $\boldsymbol{(G_L, q_L)}$) is called the **discriminant bilinear** (resp. **quadratic**) **form** of L.

When the even lattice L is given by its Gram matrix, we can compute its discriminant form using the following lemma as explained in Shimada [18].

Lemma 2.1. *Let A be the Gram matrix of L and U, $V \in Gl_n(\mathbb{Z})$ such that*

$$UAV = D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

with $1 = d_1 = \ldots = d_k < d_{k+1} \leq \ldots \leq d_n$. Then

$$G_L \simeq \oplus_{i>k} \mathbb{Z}/(d_i).$$

Moreover the ith row vector of V^{-1}, regarded as an element of L^* with respect to the dual basis e_1^*, ..., e_n^* generate the cyclic group $\mathbb{Z}/(d_i)$.

2.3. *Nikulin's results.*

Lemma 2.2 (Nikulin [16], Proposition 1.6.1). *Let L be an even unimodular lattice and T a primitive sublattice. Then we have*

$$G_T \simeq G_{T^\perp} \simeq L/(T \oplus T^\perp), \quad q_{T^\perp} = -q_T.$$

In particular, $|\det T| = |\det T^\perp| = [L : T \oplus T^\perp]$.

2.4. *Shioda's results.*

1. We refer the reader to Shioda and Inose's paper [23].

   Let $\Phi : X \to \mathbb{P}^1$ be an elliptic surface with a section and consider the sections of this elliptic fibration i.e. determined by the rational points of the corresponding elliptic curve defined over the field of rational functions in s.

   Denote by $r(\Phi)$ the rank of the group of sections. Then the Picard number $\rho(X)$ satisfies the equation

$$\rho(X) = r(\Phi) + 2 + \sum_{\nu=1}^{h}(m_\nu - 1)$$

   where h is the number of singular fibers and m_ν the number of irreducible components of the corresponding singular fiber.

   Besides, if $r(\Phi) = 0$ and if $n(\Phi)$ is the order of the group of sections of Φ, one gets

$$\mid \det T_X \mid = \mid \det S_X \mid = (\prod_{\nu=1}^{h} m_\nu^{(1)})/n(\Phi)^2$$

   where $m_\nu^{(1)}$ denotes the number of simple components of the divisor $D_\nu = \Phi^{-1}(t_\nu)$ of a singular fiber, S_X the sublattice of algebraic cycles in $H_2(X, \mathbb{Z})$ and $T_X = S_X^{\perp}$ the orthogonal complement of S_X in $H_2(X, \mathbb{Z})$ called the transcendental lattice.

2. Let $(S, \Phi, \mathbb{P}^1)$ be an elliptic surface with a section Φ, without exceptional curves of first kind (i.e. of self-intersection -1).

   Denote by $NS(S)$ the group of algebraic equivalence classes of divisors of S .

   Let u be the generic point of $\mathbb{P}^1$ and $\Phi^{-1}(u) = E$ the elliptic curve defined over $K = \mathbb{C}(u)$ with a K-rational point $o = o(u)$. Then, $E(K)$ is an abelian group of finite type provided that $j(E)$ is transcendental over $\mathbb{C}$. Let r be the rank of $E(K)$ and $s_1, ..., s_r$ be generators of $E(K)$ modulo torsion. Besides, the torsion group $E(K)_{tors}$ is generated by at most two elements t_1 of order e_1 and t_2 of order e_2 such that $1 \leq e_2$, $e_2 | e_1$ and $\mid E(K)_{tors} \mid = e_1 e_2$.

   The group $E(K)$ of K-rational points of E is canonically identified with the Mordell-Weil group $W(S)$ of sections of S over $\mathbb{P}^1(\mathbb{C})$.

   For $s \in E(K)$, we denote by (s) the curve image in S of the section corresponding to s.

   Let us define

$$D_\alpha := (s_\alpha) - (o) \quad 1 \leq \alpha \leq r$$

$$D'_\beta := (t_\beta) - (o) \quad \beta = 1, 2.$$

Consider now the singular fibers of S over $\mathbb{P}^1$. We set

$$\Sigma := \{v \in \mathbb{P}^1 / C_v = \Phi^{-1}(v) \ \text{ be a singular } \ \text{fiber}\}$$

and for each $v \in \Sigma$, $\Theta_{v,i}$, $0 \leq i \leq m_v - 1$, the m_v irreducible components of C_v.

Let $\Theta_{v,0}$ be the unique component of C_v passing through $o(v)$.

One gets

$$C_v = \Theta_{v,0} + \sum_{i \geq 1} \mu_{v,i} \Theta_{v,i}, \quad \mu_{v,i} \ \geq 1.$$

Let A_v be the matrix of order $m_v - 1$ whose entry of index (i,j) is $(\Theta_{v,i}\Theta_{v,j})$, $i,j \geq 1$, where (DD') is the intersection number of the divisors D and D' along S. Finally f will denote a non singular fiber, i.e. $f = C_{u_0}$ for $u_0 \notin \Sigma$.

Theorem 2.3. *The Néron-Severi group $NS(S)$ of the elliptic surface S is generated by the following divisors*

$$f, \Theta_{v,i} \quad (1 \leq i \leq m_v - 1, \quad v \in \Sigma)$$

$$(o), D_\alpha \quad 1 \leq \alpha \leq r, \quad D'_\beta \quad \beta = 1, 2.$$

The only relations between these divisors are at most two relations

$$e_\beta D'_\beta \approx e_\beta (D'_\beta(o)) f + \sum_{v \in \Sigma} (\Theta_{v,1}, ..., \Theta_{v,m_v-1}) e_\beta A_v^{-1} \begin{pmatrix} (D'_\beta \Theta_{v,1}) \\ . \\ . \\ . \\ (D'_\beta \Theta_{v,m_v-1}) \end{pmatrix}$$

where $\approx$ stands for the algebraic equivalence.

2.5. *Shioda and Kuwata results.*

Theorem 2.4. [21]
Denote $F^{(n)}_{\alpha,\beta}$ the elliptic $K3$ surface defined by the Weierstrass equation

$$F^{(n)}_{\alpha,\beta} \qquad y^2 = x^3 - 3\alpha x + (t^n + \frac{1}{t^n} - 2\beta) \qquad n = 1, ..., 6.$$

$F^{(2)}_{\alpha,\beta} \simeq Km(E_1 \times E_2)$ is the Shioda-Inose structure.

The rank $r^{(n)}$ of $MW(F^{(n)}_{\alpha,\beta})$ satisfies

$$r^{(n)} = h + Min(4(n-1), 16) - \begin{cases} 0 & if \quad j_1 \neq j_2 \\ n & if \quad j_1 = j_2 \neq 0, 1 \\ 2n & if \quad j_1 = j_2 = 0 \ \ or \quad 1 \end{cases}$$

where $h = rk(Hom(E_1, E_2))$, j_1 (resp. j_2) being the j invariant of the elliptic curve E_1 (resp. E_2). For all α, β, for all $n \leq 6$,

$$T(F^{(n)}_{\alpha,\beta}) \simeq T(F^{(1)}_{\alpha,\beta})[n]$$

$$\det(T(F^{(n)}_{\alpha,\beta})) = \det(T(F^{(1)}_{\alpha,\beta})) \cdot n^{\lambda}, \qquad \lambda = 4 - h.$$

Lemma 2.5. [14] *The rank of the Néron-Severi lattice of $Km(E_1 \times E_2)$ is 20 if E_1 and E_2 are isogenous, and have complex multiplication. Moreover $h = 2$ in that case.*

2.6. *Transcendental lattice.* Let X be an algebraic $K3$ surface; the group $H^2(X, \mathbb{Z})$, with the intersection pairing, has a structure of a lattice and by Poincaré duality is unimodular. The Néron-Severi lattice $NS(X) := H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$ and the transcendental lattice $T(X)$, its orthogonal complement in $H^2(X, \mathbb{Z})$ are primitive sublattices of $H^2(X, \mathbb{Z})$ with respective signatures $(1, \rho - 1)$ and $(2, 20 - \rho)$ where ρ is the rank of the Néron-Severi lattice.

By Nikulin's lemma 2.2, their discriminant forms differ just by the sign, that is

$$(G_{T(X)}, q_{T(X)}) \equiv (G_{NS(X)}, -q_{NS(X)}).$$

2.7. 2-*isogenous curves.* [24] Let E be an elliptic curve with a 2-torsion point $\omega = (0, 0)$

$$E : y^2 = x^3 + Ax^2 + Bx, \quad B \neq 0$$

and ϕ the isogeny of kernel $< \omega >$. The 2-isogenous elliptic curve $\phi(E)$ has a Weierstrass equation

$$Y^2 = X^3 - 2AX^2 + (A^2 - 4B)X$$

such that

$$(X = \frac{y^2}{x^2}, \quad Y = y\frac{B - x^2}{x^2}).$$

2.8. 3-*isogenous curves.* We essentially follow Velu's method [26].

2.8.1. *Method.* Let E be an elliptic curve with a 3-torsion point $\omega = (0, 0)$

$$E : Y^2 + AYX + BY = X^3$$

and ϕ the isogeny of kernel $< \omega >$.

To determine a Weierstrass equation for the elliptic curve $E/ < \omega >$ we need two functions x_1 of degree 2 and y_1 of degree 3 invariant by $M \to M + \omega$ where $M = (X_M, Y_M)$ is a general point on E. We compute $M + \omega$ and $M + 2\omega\,(= M - \omega)$ and can choose

$$\begin{aligned}
x_1 &= X_M + X_{M+\omega} + X_{M+2\omega} = \frac{X^3 + ABX + B^2}{X^2} \\
y_1 &= Y_M + Y_{M+\omega} + Y_{M+2\omega} \\
&= \frac{Y\left(X^3 - AXB - 2B^2\right) - B\left(X^3 + A^2X^2 + 2AXB + B^2\right)}{X^3}.
\end{aligned}$$

The relation between x_1 and y_1 gives a Weierstrass equation for $E/ < \omega >$

$$y_1^2 + (Ax_1 + 3B)\, y_1 = x_1^3 - 6ABx_1 - B\left(A^3 + 9B\right).$$

2.8.2. *Formulae.* Denote ζ_3 the 3-th root of unity, $\zeta_3^3 = 1$. Notice that the points with $x_1 = -\frac{A^2}{3}, y_1 = \frac{-3}{2}B + \frac{1}{6}A^3 \frac{\pm i\sqrt{3}}{18}\left(27B - A^3\right)$ are 3-torsion points. Taking one of these points to origin and after some transformation we can obtain a Weierstrass equation $y^2 + ayx + by = x^3$ with the following transformations.

Define

$$\begin{aligned}
S_1 &= 2\left(\zeta_3^2 - 1\right) y + 6Ax - 2\left(\zeta_3 - 1\right)\left(A^3 - 27B\right) \\
S_2 &= 2\left(\zeta_3 - 1\right) y + 6Ax - 2\left(\zeta_3^2 - 1\right)\left(A^3 - 27B\right)
\end{aligned}$$

and

$$X = \frac{-1}{324}\frac{S_1S_2}{x^2}, \qquad Y = \frac{1}{5832}\frac{S_1^3}{x^3}$$

then we get

$$E/ < \omega >: y^2 + (-3A)\, yx + \left(27B - A^3\right) y = x^3.$$

If $A_1 = -3A$, $B_1 = 27B - A^3$, then we define

$$\begin{aligned}
\sigma_1 &= 2\left(\zeta_3^2 - 1\right) 3^6Y + 6A_1 3^4X - 2\left(\zeta_3 - 1\right)\left(A_1^3 - 27B_1\right) \\
\sigma_2 &= 2\left(\zeta_3 - 1\right) 3^6Y + 6A_1 3^4X - 2\left(\zeta_3^2 - 1\right)\left(A_1^3 - 27B_1\right)
\end{aligned}$$

and then

$$x = \frac{-1}{324}\frac{\sigma_1\sigma_2}{3^8X^2} = -\frac{3\,X^3 + A^2X^2 + 3\,BAX + 3\,B^2}{X^2}, \qquad y = \frac{1}{5832}\frac{\sigma_1^3}{3^{12}X^3}.$$

2.8.3. *Other properties of isogenies.* The divisor of the function Y is equal to $-3\,(0) + 3\omega$ so $Y = W^3$ where W is a function on the curve $E/ < \omega >$. If $X = WZ$, the function field of $E/ < \omega >$ is generated by W and Z. So replacing in the equation of E we obtain the relation between Z and W

$$W^3 + AZW + B - Z^3 = 0.$$

From the previous equation, with the transformation

$$W = -\frac{\sqrt{-3}}{9}\frac{y}{x} + \frac{A\zeta_3}{3} + \frac{\sqrt{-3}\zeta_3^2}{9}\frac{\left(27B - A^3\right)}{x}$$
$$Z = -\frac{\sqrt{-3}}{9}\frac{y}{x} - \frac{A\zeta_3^2}{3} + \frac{\sqrt{-3}\zeta_3}{9}\frac{\left(27B - A^3\right)}{x}$$

of inverse

$$x = \frac{27B - A^3}{3W - 3Z + A}$$
$$y = \frac{\zeta_3^2\left(27B - A^3\right)\left(6Z - 6\zeta_3^2 W - 2\zeta_3 A\right)}{2\left(3W - 3Z + A\right)}$$

we recover the Weierstrass equation

$$y^2 - 3Ayx + \left(27B - A^3\right)y = x^3.$$

3. Generic 3-isogenies from elliptic fibrations of Y_k

There are only two generic elliptic fibrations with 3-torsion section on the Apéry-Fermi pencil (see Table 2 of [4]). Their respective Weierstrass equations are given in Table 4 of [4] and we denote them $E_{\#19}(k)$ (resp. $E_{\#20}(k)$).

Theorem 3.1. 1. *The 3-isogenous elliptic fibrations $H_{\#19}(k)$ (resp. $H_{\#20}(k)$) of the two generic fibrations $E_{\#19}(k)$ (resp. $E_{\#20}(k)$) are elliptic fibrations of the same $K3$ surface N_k with transcendental lattice $U(3) \oplus < 4 >$, U being the hyperbolic lattice.*
2. *The specialized $K3$ surface N_2 is the $K3$ surface Y_{10}.*
3. *The specialized $K3$ surface N_{10} is the $K3$ surface with transcendental lattice $\begin{pmatrix} 4 & 0 \\ 0 & 18 \end{pmatrix}$.*

Proof. 1. The 6-torsion elliptic fibration #20 has a Weierstrass equation

$$E_{\#20}(k) \qquad y^2 - (t^2 - tk + 3)xy - (t^2 - tk + 1)y = x^3$$

with singular fibers $I_{12}(\infty)$, $2I_3(t^2{-}kt{+}1)$, $2I_2(0,k)$, $2I_1(t^2{-}kt{+}9)$ and 3-torsion point $(0,0)$. Using 2.8.2, it follows the Weierstrass equation of its 3-isogenous fibration is

$$H_{\#20}(k) = E_{\#20}(k)/\langle(0,0)\rangle$$

with singular fibers $2I_6(0,k)$, $I_4(\infty)$, $2I_3(t^2 - kt + 9)$, $2I_1(t^2 - kt + 1)$. Thus it is a rank 0 and 6-torsion elliptic fibration of a $K3$ surface with Picard number 19 and discriminant $\frac{6\times6\times3\times3\times4}{6\times6} = 12 \times 3$.

Now we shall compute the Gram matrix $NS(20)$ of the Néron-Severi lattice of the $K3$ surface with elliptic fibration $H_{\#20}(k)$ in order to deduce its discriminant form.

Applying Shioda's result 2.4, we order the following elements as, s_0, F, $\theta_{0,i}$, $1 \leq i \leq 4$, s_3, $\theta_{k,i}$, $1 \leq i \leq 5$, $\theta_{\infty,i}$, $1 \leq i \leq 3$, $\theta_{t_0,i}$, $1 \leq i \leq 2$, $\theta_{t_1,i}$, $1 \leq i \leq 2$, where s_0 and s_3 denotes respectively the zero and 3-torsion section, F the generic section, $\theta_{k,i}$ the components of reducible singular fibers, t_0 and t_1 being roots of $t^2 - kt + 9$. We obtain

$$NS(20) = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2
\end{pmatrix}.$$

We get $\det(NS(20)) = 12 \times 3$ and applying Shimada's lemma 2.1, the discriminant form $G_{NS(20)} \simeq \mathbb{Z}/3 \oplus \mathbb{Z}/12$ is generated by vectors L_1 and L_2 satisfying $q_{L_1} = 0$, $q_{L_2} = -\frac{11}{12}$ and $b(L_1, L_2) = \frac{1}{3}$. Denoting $M(20)$ the following Gram matrix of the lattice $U(3) \oplus \langle 4 \rangle$,

$$M(20) = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 0 \end{pmatrix},$$

we find for generators of its discriminant form the vectors

$$g_1 = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{3} \end{pmatrix} \qquad g_2 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{3} \end{pmatrix}$$

satisfying $q_{g_1} = 0$, $q_{g_2} = \frac{11}{12}$ and $b(g_1, g_2) = \frac{1}{3}$. We deduce that $M(20)$ is the transcendental lattice of the $K3$ surface with elliptic fibration $H_{\#20}(k)$.

A Weierstrass equation of the 3-torsion, rank 1, elliptic fibration #19 can be written as

$$E_{\#19}(k) \qquad y^2 + ktxy + t^2(t^2 + kt + 1)y = x^3$$

with singular fibers $2IV^*(0, \infty)$, $2I_3(t^2 + kt + 1)$, $2I_1(k^3t - 27kt - 27t^2 - 27)$ and infinite point $P = (-t^2, -t^2)$ of height $h(P) = \frac{4}{3}$. Its 3-isogenous elliptic fibration $E_{\#19}(k)/\langle(0, 0)\rangle$ has a Weierstrass equation

$$H_{\#19}(k) \qquad Y^2 - 3ktXY - Yt^2(27t^2 - k(k^2 - 27)t + 27) = X^3.$$

It is a 3-torsion, rank 1, elliptic fibration of a $K3$ surface with Picard number 19 and singular fibers $2IV^*(0,\infty)$, $2I_3(27t^2-k(k^2-27)t+27)$, $2I_1(t^2+kt+1)$ and infinite order point Q with x-coordinate $x_Q = -3-3kt-(k^2+3)t^2-3kt^3-3t^4$ and height $h(Q)=4$. This point Q is the image of the point P in the 3-isogeny and non 3-divisible, hence generator of the non torsion part of the Mordell-Weill lattice. We deduce the discriminant of this $K3$ surface $\frac{3\times 3\times 3\times 3\times 4}{3\times 3} = 12\times 3$.

Applying Shioda's result 2.4, we order the components of the singular fibers as, s_0, F, $\theta_{0,i}$, $1\le i\le 6$, $\theta_{\infty,i}$, $1\le i\le 6$, $\theta_{t_0,i}$, $1\le i\le 2$, s_3, $\theta_{t_1,2}$, s_∞,where s_0, s_3, s_∞ denotes respectively the zero, 3-torsion and infinite section, F the generic section, t_0 and t_1 being roots of $27t^2-k(k^2-27)t+27$. The numbering of components of IV^* is done using Bourbaki's notations [6]. It follows the Gram matrix $NS(19)$ of the corresponding $K3$ surface

$$NS(19) = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2
\end{pmatrix}.$$

Its determinant satisfies $\det(NS(19)) = 12\times 3$ and according to Shimada's lemma 2.1, $G_{NS(19)} \simeq \mathbb{Z}/3\oplus\mathbb{Z}/12$ is generated by vectors M_1 and M_2 satisfying $q_{M_1} = -\frac{2}{3}$, $q_{M_2} = \frac{5}{12}$ and $b(M_1,M_2)=0$. We find also generators for the transcendental discriminant form $M(20)$

$$h_1 = \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix} \qquad h_2 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{4} \\ -\frac{1}{3} \end{pmatrix}$$

satisfying $q_{h_1} = \frac{2}{3}$, $q_{h_2} = -\frac{5}{12}$ and $b(h_1,h_2)=0$. We deduce that $M(20)$ is also the transcendental lattice of the $K3$ surface with elliptic fibration $H_{\#19}(k)$.

It follows the discriminant form of its Néron-Severi lattice,

$G_{NS(19)} = \mathbb{Z}/3\mathbb{Z}(-\frac{2}{3})\oplus\mathbb{Z}/12\mathbb{Z}(\frac{5}{12})$, which is also $G_{NS(20)}$ since the generators $L_1' = 15L_1+4L_2$, $L_2' = L_1-L_2$ satisfy $q_{L_1'} = -\frac{2}{3}$, $q_{L_2'} = \frac{5}{12}$, $b(L_1',L_2')=0$.

2. To prove that $N_2 = Y_{10}$ it is sufficient to prove that $H_{\#20}(2)$ is an elliptic fibration of Y_{10} since, by the previous theorem, $H_{\#19}(2)$ is another fibration of the same $K3$ surface.

   But we see easily that $H_{\#20}(2)$ is the 6-torsion extremal fibration of Y_{10} numbered 8 in Shimada and Zhang [20].

3. Similarly, N_{10} is the $K3$ surface with transcendental lattice $\begin{pmatrix} 4 & 0 \\ 0 & 18 \end{pmatrix}$ since $H_{\#20}(10)$ is a fibration of that surface. A Weierstrass equation for $H_{\#20}\,(10)$ is

$$Y^2 + 3\left(t^2-22\right)YX + \left(t^2-25\right)^2\left(t^2-16\right)Y = X^3$$

with singular fibers $I_4\,(\infty)\,,2I_6\,(\pm 5)\,,2I_3\,(\pm 4)\,,2I_1\left(t^2-24\right)$. We have a 2-torsion section $s_2 = \left(-\left(t^2-25\right)^2, \left(t^2-25\right)^3\right)$ and a 6-torsion section
$s_6 = \left(-\left(t^2-25\right)\left(t^2-16\right), -\left(t^2-25\right)\left(t^2-16\right)^2\right)$.
The section $P_t = \left(4\,(t+5)^2\,(t-4)\,, -(t+5)^4\,(t-4)^2\right)$ is of infinite order and is a generator of the Mordell-Weil lattice. Moreover we have $\theta_{\pm 5,1}.s_6 = 1, \theta_{\infty,2}.s_6 = 1, \theta_{\pm 3,1}.s_6 = 1$. So the following divisor is 6-divisible

$$\sum_{i=1}^{5}(6-i)\,\theta_{\pm 5,i} + 3\theta_{\infty,1} + 3\theta_{\infty,3} + 4\theta_{\pm,4,1} + 2\theta_{\pm 4,2} \approx 6s_6$$

and we can replace $\theta_{5,5}$ by s_6.

Moreover we get $s_6.P_t = 1$ (for $t=-3$), $\theta_{5,0}.P_t = 1, \theta_{-5,4}.P_t = 1$ and $\theta_{\infty,0}.P_t = 1, \theta_{4,1}.P_t = 1, \theta_{-4,0}.P_t = 1$. All these computations give the Gram matrix of the Néron-Severi lattice of discriminant -72

$$\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2
\end{pmatrix}.$$

According to Shimada's lemma 2.1, $G_{NS} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/36$ is generated by the vectors L_1 and L_2 satisfying $q_{L_1} = \frac{-1}{2}, q_{L_2} = \frac{-35}{36}$ and $b(L_1, L_2) = \frac{-1}{2}$.

Moreover the following generators of the discriminant group of the lattice with Gram matrix $M_{18} = \left(\begin{smallmatrix} 4 & 0 \\ 0 & 18 \end{smallmatrix}\right)$, namely $f_1 = (0, \frac{1}{2})$,$f_2 = (\frac{1}{4}, \frac{7}{18})$ verify $q_{f_1} = \frac{1}{2}$, $q_{f_2} = \frac{35}{36}$ and $b(f_1, f_2) = \frac{1}{2}$. So the transcendental lattice is $\begin{pmatrix} 4 & 0 \\ 0 & 18 \end{pmatrix}$.

□

THEOREM 3.2. *Define $Y_k^{(3)}$ the elliptic surface obtained by the base change τ of the elliptic fibration of Y_k with two singular fibers of type II^*, where τ is the morphism given by $u \mapsto h = u^3$. Then the $K3$ surface $Y_k^{(3)}$ with transcendental lattice $U(3) \oplus < 36 >$ has a genus one fibration without section such that its Jacobian variety $J\left(Y_k^{(3)}\right) = N_k$ is the $K3$ surface with transcendental lattice $U(3) \oplus < 4 >$.*

PROOF. Recall a Weierstrass equation for fibration #19 of Y_k

$$Y^2 + tkYX + t^2 (t+s)(t+1/s) Y = X^3$$

where $k = s + \frac{1}{s}$. The fibration of Y_k with two singular fibers II^* can be obtained with the parameter $h_k = \frac{Y}{(t+s)^2}$ ([4] Table 3). The surface $Y_k^{(3)}$ is defined by $h = u^3$ and has then the following equation

$$u^3 s (t+s) + tkuWs + t^2 (ts - 1) - W^3 s = 0,$$

where $X = (t+s) uW$.

We consider the fibration

$$\begin{aligned} Y_k^{(3)} &\to \mathbb{P}^1 \\ (u, t, W) &\mapsto t; \end{aligned}$$

this is a genus one fibration since we have a cubic equation in u, W.

However, this fibration seems to have no section. Nevertheless, taking its Jacobian fibration produces an elliptic fibration with section and the same fiber type.

If we make a base change of this fibration: $(t+s) = m^3$ then we obtain the following elliptic fibration with $U = um$,

$$U^3 sm - \left(s - m^3\right) ksWU - W^3 sm - \left(s - m^3\right)^2 \left(s^2 - sm^3 - 1\right) m = 0.$$

The transformation

$$W = \frac{-24y + 12(s^2+1)(s-m^3)x + \left(s-m^3\right)^2 Q}{18sm(4x - m^2(s^2+1)^2)}$$
$$U = \frac{-24y - 12(s^2+1)(s-m^3)x + \left(s-m^3\right)^2 Q}{18sm(4x - m^2(s^2+1)^2)}$$

where $Q = \left(108m^6s^3 + (s^6+105s^2-111s^4-1)m^3 + s(s^2+1)^3\right)$ gives a Weierstrass equation of the Jacobian variety $J(Y_k^{(3)})$, the point π_3 of x coordinate $\frac{1}{4}m^2(s^2+1)^2$ being a 3-torsion point. Taking again $m^3 = (t+s)$, and $\pi_3 = (X=0, Y=0)$, we recover a Weierstrass equation for the 3-isogenous fibration #19

$$Y^2 - 3tkYX - t^2\left(27t^2 - k\left(k^2-27\right)t + 27\right)Y = X^3$$

hence a fibration of N_k. Recall that the transcendental lattice of $Y_k^{(3)}$ is $T(Y_k)[3]$ (Theorem 2.4 [21]). ∎

4. 3-isogenies of Y_2

Let us prove the following theorem

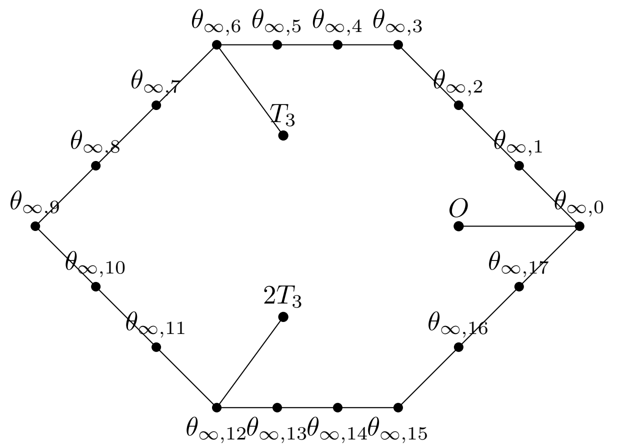
Theorem 4.1. *Given a Weierstrass equation E_t of a $K3$ surface S with a 3-torsion section given by the point* $(0,0)$

$$(E_t) \qquad Y^2 + A(t)XY + B(t)Y = X^3$$

and transcendental lattice $T(S)$, denote $\tilde{S}$ the 3-isogenous $K3$ surface $S/<\omega>$ where $\omega=(0,0)$ and $T(\tilde{S})$ its transcendental lattice.

A sufficient condition to get $T(\tilde{S}) = T(S)[3]$ is that Y will be the elliptic parameter for an elliptic fibration with two II^* fibers at 0 and ∞. In particular this condition can be realized if (E_t) has singular fibers either I_{18} or I_{12} and IV^*.

Proof. Consider first the case of (E_t) with a singular fiber I_{18}. From the components θ_i we can exhibit two fibers of type II^*.



We consider the two divisors D

$$\begin{aligned} D &= 3(O) + 6\theta_{\infty,0} + 5\theta_{\infty,17} + 4\theta_{\infty,16} + 3\theta_{\infty,15} \\ &\quad +2\theta_{\infty,14} + \theta_{\infty,13} + 4\theta_{\infty,1} + 2\theta_{\infty,2} \end{aligned}$$

and D'

$$\begin{aligned} D' &= 3(T_3) + 6\theta_{\infty,6} + 5\theta_{\infty,7} + 4\theta_{\infty,8} + 3\theta_{\infty,9} \\ &\quad +2\theta_{\infty,10} + \theta_{\infty,11} + 4\theta_{\infty,5} + 2\theta_{\infty,4} \end{aligned}$$

We have $D.D' = 0$, $D.\theta_{\infty,12} = 1$, so D and D' are two singular fibers of a new elliptic fibration with two fibers of type II^*. The parameter h of this new fibration can be chosen as $h = Y$ since the horizontal divisor of $D' - D$ is $3(T_3) - 3(0)$ ([15] Propositions 8.1 and 8.2) .

Since E_t has only a singular fiber I_{18} at $t = \infty$, it follows that $B(t) = b$ and $A(t)$ is a degree two polynomial. After some scaling we can take $A(t) = t^2 + a$ and $b = 1$. With $Y = h$ as new parameter and putting $X = -\frac{h^2(h+1)}{x}$, $t = -\frac{y}{hx}$, it follows the Weierstrass equation

$$y^2 = x^3 - ah^2x^2 + h^5(h+1)^2$$

with singular fibers $II^*(0,\infty)$, $I_2(-1)$, $6I_1$. Proceeding as in Shioda and Kuwata's theorem 2.4 with $h = u^3$ we get the fibration E_u of $S^{(3)}$ with transcendental lattice $T(S)[3]$,

$$(E_u) \qquad Y^2 = X^3 - au^2X^2 + u^3(u^3+1)^2,$$

which is a rank 7 elliptic fibration of $S^{(3)}$ with singular fibers $2I_0^*(\infty, 0)$, $3I_2(u^3+1)$, $6I_1$.

The same transformation can be performed directly on the Weierstrass equation (E_t) taking $Y = u^3$ and giving, if $X = ux$, the following cubic equation of $S^{(3)}$

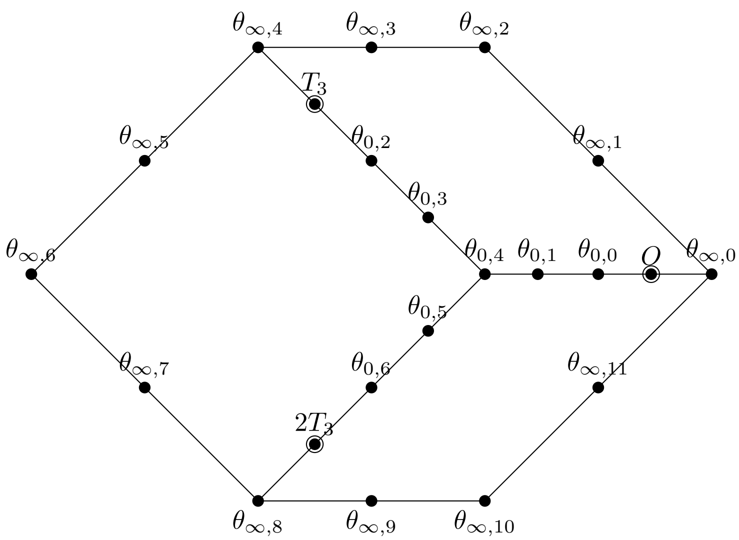
$$u^3 + (t^2 + a)ux + 1 = x^3.$$

Finally applying 2.8.3, it follows a Weierstrass equation (F_t) of this cubic 3-isogenous to (E_t) i.e. $\tilde{S} = S^{(3)}$

$$(F_t) \quad y^2 - 3(t^2 + a)xy + (27 - (t^2 + a)^3)y = x^3.$$

Using Shimada [19], we notice that the elliptic curve (F_t) with singular fibers I_6 and $6I_3$ has $(\mathbb{Z}/3\mathbb{Z})^2)$-torsion.

Consider now the case of (E_t) with singular fibers $I_{12}(\infty)$ and $IV^*(0)$ and draw two fibers of type II^*.



Consider the divisors D

$$\begin{aligned} D &= 3(O) + 6\theta_{\infty,0} + 5\theta_{\infty,11} + 4\theta_{\infty,10} + 3\theta_{\infty,9} \\ &\quad +2\theta_{\infty,8} + \theta_{\infty,7} + 4\theta_{\infty,1} + 2\theta_{\infty,2} \end{aligned}$$

and D'

$$\begin{aligned} D' &= 3\theta_{0,1} + 6\theta_{0,4} + 5\theta_{0,3} + 4\theta_{0,2} + 3(T_3) \\ &\quad +2\theta_{\infty,4}, + \theta_{\infty,5} + 4\theta_{0,5} + 2\theta_{0,6} \end{aligned}$$

They satisfy $D.D' = 0$, $D.\theta_{\infty,6} = 1$ and are two singular fibers of a new elliptic fibration with two fibers of type II^*. The parameter h of this new fibration can be chosen as $h = Y$ since the horizontal divisor of $D' - D$ is $3(T_3) - 3(0)$ ([15] Propositions 8.1 and 8.2).

Moreover a $K3$ surface S having an elliptic fibration with 3-torsion and two fibers $I_{12}(\infty)$, $IV^*(0)$ and $4I_1(t^4+3t^3l+3t^2l^2+tl^3+27)$ has a Weierstrass equation of the form

$$(E_t) \qquad y^2 - t(t+l)xy + t^2y = x^3.$$

Putting $y = h$ we obtain a cubic in t and x with a point $(x = 1, t = \frac{h^2-1}{hl})$; thus it is an elliptic curve with a Weierstrass form

$$Y^2 = X^3 - (1/48l^4 + 3)t^4X + t^7 - 1/864l^2(-648 + l^4)t^6 + t^5.$$

With the base change $h = u^3$ we get a Weierstrass equation of an elliptic fibration of the $K3$ surface $S^{(3)}$ with transcendental lattice $T(S^{(3)}) = T(S)[3]$.

The same transformation can be performed directly on the Weierstrass equation (E_t) taking $y = u^3$ and giving, if $x = uX$, the following equation of a cubic on $S^{(3)}$

$$u^3 - t(t+l)uX + t^2 = X^3.$$

Finally applying 2.8.3 it follows a Weierstrass equation of this cubic 3-isogenous to (E_t)

$$Y^2 + 3t(t+l)YX + t^2(t^4 + 3t^3l + 3t^2l^2 + tl^3 + 27)Y = X^3.$$

$\square$

THEOREM 4.2. *The 4 elliptic fibrations of Y_2 with 3-torsion section give by 3-isogeny elliptic fibrations of Y_{10}.*

PROOF. Recall the results, given in [3], about the 4 elliptic fibrations of Y_2 with 3-torsion.

	Weierstrass Equation Singular Fibers	Rank
#20 $(7-w)$	$E_w : Y^2 - \left(w^2+2\right)YX - w^2Y = X^3$ $I_{12}(\infty), \quad I_6(0), \quad 2I_2(\pm 1), \quad 2I_1$	0
#19 $(8-b)$	$E_b : Y^2 + 2bYX + b^2(b+1)^2 Y = X^3$ $2IV^*(\infty, 0), \quad I_6(-1), \quad 2I_1$	1
$20-j$	$E_j : Y^2 - 4\left(j^2-1\right)YX + 4(j+1)^2 Y = X^3$ $I_{12}(\infty), \quad IV^*(-1), \quad I_2\left(-\frac{1}{2}\right), \quad 2I_1$	0
$21-c$	$E_c : Y^2 + \left(c^2+5\right)YX + Y = X^3$ $I_{18}(\infty), \quad 6I_1$	1

Using 2.8.2 we compute the 3-isogenous elliptic fibrations named H_w, H_b $, H_j$ and H_c and given in the next table. To simplify we denote H_w (resp. H_b) the specialized elliptic fibration $H_{\#20}(2)$ (resp. $H_{\#19}(2)$).

	Weierstrass Equation Singular Fibers
H_w	$Y^2+3\left(w^2+2\right)YX+\left(w^2+8\right)\left(w^2-1\right)^2Y=X^3$ $I_4(\infty),\quad 2I_6(\pm 1),\quad I_2(0),\quad 2I_3$
H_b	$Y^2-6bYX+b^2\left(27b^2+46b+27\right)Y=X^3$ $2IV^*(\infty,0),\quad 2I_3,\quad I_2(-1)$
H_j	$Y^2+12\left(j^2-1\right)YX+4\left(4j^2-12j+11\right)(j+1)^2(2j+1)^2Y=X^3$ $I_4(\infty),\quad IV^*(-1),\quad I_6\left(\frac{-1}{2}\right),\quad 2I_3$
H_c	$Y^2-3\left(c^2+5\right)YX-\left(c^2+2\right)\left(c^2+c+7\right)\left(c^2-c+7\right)Y=X^3$ $I_6(\infty),\quad 6I_3$

We know from Theorem 3.1 that H_w and H_b are elliptic fibrations of Y_{10}; we know also from the classification of elliptic $K3$ surfaces of rank 0 [20] that H_w and H_j, with rank 0, are fibrations of Y_{10}. Fibrations H_j and H_c are fibrations of Y_{10} by applying Theorem 4.1. From Shimada [19] we observe that on H_c with fibers I_6 and $6I_3$ we get $(\mathbb{Z}/3\mathbb{Z})^2$ torsion. We recover this last result using formula of 2.8.3 giving the equation of H_c as $u^3+v^3+w^3=(c^2+5)uvw$ thus showing that H_c is an element of Hesse pencil. The nine inflexion points correspond to 3-torsion sections.

Remark 4.3. On the elliptic fibration $21-c$ there exists an involution $i_c : c \mapsto -c$, $X \mapsto X$, $Y_1 = Y-(c^2+5)/2 \mapsto -Y_1$. We can show that there exists an elliptic fibration of parameter X with a two torsion section, (fibration $24-\psi$ of Y_2 [3] last table), such that i_c corresponds to add the two-torsion section. The same remark is true for the fibration $8-b$.

□

5. 3-isogenies of Y_2 and base change of elliptic fibrations

The 4 fibrations H_w, H_b, H_j, H_c of Y_{10} obtained by 3-isogenies have the same rank as the corresponding fibrations E_w, E_b, E_j, E_c of Y_2. On the contrary, in the previous item 4.1 we study a base change $h=u^3$ of an elliptic fibration with 2 singular fibers of type II^*. This theorem applied with $A(t)=t^2+5$, $B(t)=1$ corresponding to the fibration $21-c$ of Y_2 and then fibration $13-h$ of rank 1 [3] defines another elliptic fibration of Y_{10} increasing the rank to 7. In this paragraph we give also two other examples of base change of elliptic fibrations of Y_2 leading to rank 4 elliptic fibrations $E_{u'}$ and E_n of Y_{10}. We also construct generators of the Mordell-Weil lattice of these fibrations.

Remark 5.1. High rank elliptic fibrations of Y_{10} have been already exhibited in Bertin and Lecacheux's paper [5] (pages 27 and 28), in particular fibration E_u.

Rank 4 elliptic fibration E_n of Y_{10} has been also observed in [5] (page 10). It follows from an embedding in $Ni(A_3^8)$ using a norm 4 glue vector. But no Weierstrass equation was given.

Rank 4 elliptic fibration $E_{u'}$ comes from a primitive embedding into D_6^4.

With some elementary change of variables we obtain Weierstrass equations E_h, E_f, E_g of respective elliptic fibrations $13-h, 11-f, 12-g$ of Y_2 given in [3].

THEOREM 5.2. *Elliptic fibrations of* Y_2

Equations and singular fibers	*rank*
$E_h: y^2 = x^3 - 5h^2x^2 + h^5(h+1)^2$ $2II^*(0,\infty), I_2(-1), 2I_1$	1
$E_f: y^2 = x^3 + f^2x^2 + 2(f-1)f^3x + f^5(f-1)^2$ $II^*(\infty), III^*(0), I_4(1), I_1\left(\frac{32}{27}\right)$	0
$E_g: y^2 = x^3 + 4g^2x^2 + g^3(g+1)^2x$ $2III^*(0,\infty), I_4(-1), I_2(1)$	2

give, by base change, elliptic fibrations of Y_{10} *with the following properties*

Change	*Equations and singular fibers*	*rank*
$h = u^3$	$E_u: Y^2 = X^3 - 5u^2X^2 + u^3\left(u^3+1\right)^2$ $2I_0^*(\infty,0), 3I_2\left(u^3+1\right), 6I_1$	7
$f = u'^3$	$E_{u'}: Y^2 = X^3 + u'^2X^2 + 2u'\left(u'^3-1\right)X^2 + u'^3\left(u'^3-1\right)^2$ $I_0^*(\infty), III(0), 3I_4\left(u'^3-1\right), 3I_1\left(27u'^3-32\right)$	4
$g = n^3$	$E_n: Y^2 = X^3 + 4n^2X^2 + n\left(n^3+1\right)^2X$ $2III(\infty,0), 3I_4\left(n^3+1\right), 3I_2\left(n^3-1\right)$	4

PROOF. For the first fibration E_h with $h = u^3$ we get $E_u: Y^2 = X^3 - 5u^2X^2 + u^3(u^3+1)^2$.

This is a fibration of Y_{10} from Theorem 4.1. Notice that the j-invariant of (E_u) is invariant by the two transformations $u \mapsto \frac{1}{u}$ and $u \mapsto \zeta_3 u$ ($\zeta_3^3 = 1$). These automorphisms of the base $\mathbb{P}^1$ of the fibration E_u can be extended to the sections as explained below.

Let $S_3 =< \gamma, \tau; \gamma^3 = 1, \tau^2 = 1 >$ be the non abelian group of order 6 and define an action of S_3 on the sections of E_u by

$$(X(u), Y(u)) \overset{\tau}{\mapsto} \left(u^4 X\left(\frac{1}{u}\right), u^6 Y\left(\frac{1}{u}\right)\right) \quad (X(u), Y(u)) \overset{\gamma}{\mapsto} (\zeta_3 X(\zeta_3 u), Y(\zeta_3 u)).$$

To obtain generators of (E_u), following Shioda [21], we use the rational elliptic surface $X^{(3)+}$ with $\sigma = u + \frac{1}{u}$ and a Weierstrass equation

$$(E_\sigma): y^2 = x^3 - 5x^2 + (\sigma-1)^2(\sigma+2)$$
$$I_0^*(\infty), I_2(-1), 4I_1$$

of rank 3. The Mordell-Weil lattice of a rational elliptic surface is generated by sections of the form $(a+b\sigma+c\sigma^2, d+e\sigma+f\sigma^2+g\sigma^3)$. Moreover since we have a singular fiber of type I_0^* at ∞ the coefficients c and f, g are 0 [13]. So after an easy computation we find the 3 sections of x coordinate $-(\sigma-1), -\zeta_3(\sigma-1), -\zeta_3^2(\sigma-1)$.

These sections give the sections π_1, π_2, π_3 on (E_u) which are fixed by τ.

$$\begin{aligned}\pi_1 &= \left(-u\left(u^2-u+1\right), i\sqrt{2}u^2\left(u^2-u+1\right)\right)\\ \pi_2 &= \left(-\zeta_3 u\left(u^2-u+1\right), \left(3+\zeta_3\right)u^2\left(u^2-u+1\right)\right)\\ \pi_3 &= \left(-\zeta_3^2 u\left(u^2-u+1\right), \left(3+\zeta_3^2\right)u^2\left(u^2-u+1\right)\right).\end{aligned}$$

We notice $\rho_i=\gamma(\pi_i)$ and $\mu_i=\gamma^2(\pi_i)$ for $1\leq i\leq 3$ which give 9 rational sections with some relations.

Moreover we have another section from the fibration E_h of rank 1, the point of x coordinate $\frac{1}{16}\left(h^2+\frac{1}{h^2}\right)-h-\frac{1}{h}+\frac{29}{24}$. Passing to (E_u) we obtain $\omega=$

$$\left(\frac{(1-16u^3+46u^6-16u^9+u^{12})}{16u^4}, -\frac{(u^6-1)(1-24u^3+126u^6-24u^9+1)}{64u^6}\right).$$

We hope to get a generator system with π_i, ρ_i and ω so we have to compute the height-matrix. The absolute value of its determinant is $\frac{81}{16}$. Since the discriminant of the surface is 72, we obtain a subgroup of index a with $\frac{81}{16}\times\frac{1}{a^2}\times 2^3 4^2=72$ so $a=3$.

After some specialization of $u\in\mathbb{Z}$ (for example if $u=11$, (E_u) has rank 3 on $\mathbb{Q}$) we find other sections with x coordinate of the shape $(au+b)\left(u^2-u+1\right)$

$$\begin{aligned}\mu &= \left(-(u-1)\left(u^2-u+1\right), -\left(u^2-u+1\right)\left(u^2+2u-1\right)\right)\\ \mu_1 &= \left(-(u-9)\left(u^2-u+1\right), \left(u^2-u+1\right)\left(5u^2-18u+27\right)\right)\\ \mu_2 &= \left(-\left(u+\frac{1}{3}\right)\left(u^2-u+1\right), \frac{i\sqrt{3}}{9}\left(u^2-u+1\right)\left(9u^2+4u+1\right)\right).\end{aligned}$$

We deduce the relations

$$3\mu=\omega+\pi_2-\gamma(\pi_3)+\pi_3-\gamma^2(\pi_2)=\omega+2\pi_2+\gamma(\pi_2)+\pi_3-\gamma(\pi_3)$$

so, the Mordell-Weil lattice is generated by $\pi_j, \rho_j=\gamma(\pi_j)$ for $1\leq j\leq 3$ and μ with the following Gram matrix of determinant 9/16 ([22] Corollary 1.7).

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\ -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2}\\ 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & \frac{-1}{2}\\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{-1}{2} & 2 \end{pmatrix}.$$

We consider now the second and third fibration, respectively E_f and E_g.

First we explain why such base changes lead to elliptic fibrations of Y_{10}. Recall (see [3]) that, from the equation

$$Y_2 : X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = 2$$

the fibration with two singular fibers II^* can be obtained with the parameter $h = \frac{(Y+Z)YX^2}{Z^3(X+Z)}$. We can also obtain with the parameter $g = \frac{Y(Y+Z)}{X(X+Z)}$ a fibration with singular fibers $2III^*, I_4(0,\infty)$ and I_2 (fibration $(12-g)$ of Y_2 ([3] last table)). It is also possible to get with the parameter $f = \frac{Y(X+Z)^2(Z+Y)}{Z^3X}$ singular fibers II^*, III^*, I_4 and I_1(fibration $(11-f)$ of Y_2([3] last table)). We observe that $g = \frac{Z^3}{X^3}h$, and $f = \frac{(X+Z)^3}{Z^3}h$. So since the fibration of Y_{10} obtained by a base change $h = u^3$ of the corresponding fibration allow also to construct with base changes $g = r^3$ and $f = u'^3$ two other fibrations of the same surface Y_{10}.

We give some details of how these 2 fibrations are linked to fibrations of Y_2 with 3-torsion sections and also their Mordell-Weil generators.

For the parameter f we start from Weierstrass equation $E_w : Y^2 + \left(w^2+2\right)YX + w^2Y = X^3$ and consider another fibration with the parameter $f = Y$. Then we obtain the fibration E_f. A Weierstrass equation is obtained from E_w with the transformation $X = -\frac{x+f^2(f-1)}{x+f-1}, \quad Y = f, \quad w = -\frac{y}{f(x+f-1)}$.

The base change $f(=Y) = u'^3$ gives an elliptic fibration of Y_{10} with singular fibers $I_0^*(\infty)$, $III\,(0)$, $3I_4\left(1,\zeta,\zeta^2\right)$, $3I_1$, rank 4, a Weierstrass equation and sections

$$\begin{aligned}
&y'^2 = x'^3 + u'^2x'^2 + 2u'\left(u'^3-1\right)x' + u'^3\left(u'^3-1\right)^2 \\
&\quad P = \left(x_P\left(u'\right), y_P\left(u'\right)\right) = \Big(-\left(u'^3-1\right), \left(u'-1\right)^2\left(u'^2+u'+1\right)\Big) \\
&\quad Q = \left(x_Q\left(u'\right), y_Q\left(u'\right)\right) = \Big(-\left(u'+2\right)\left(u'^2+u'+1\right), 2i\sqrt{2}\left(u'^2+u'+1\right)^2\Big).
\end{aligned}$$

Using also the points P' with $x_{P'} = \zeta_3 x_P\left(\zeta_3 u'\right)$ and Q' with $x_{Q'} = \zeta_3 x_Q\left(\zeta_3 u'\right)$ (with $\zeta_3^3 = 1$), we can compute the height matrix and show that the Mordell-Weil lattice is generated by P, P', Q, Q' and is equal to $A_2\left[\frac{1}{4}\right] \oplus A_2\left[\frac{1}{2}\right]$.

The last example is obtained from E_w with the parameter $g = \frac{Y}{w^2}$. We obtain the fibration E_g with singular fibers $2III^*(0,\infty), I_4(-1), I_2(1)$

$$E_g : y^2 = x^3 + 4x^2g^2 + g^3(g+1)^2x$$

with the following transformation

$$X = \frac{x^2\left(x+2g^2\right)(g-1)}{y^2}, Y = \frac{1}{g}\frac{x^2\left(x+2g^2\right)^2}{y^2}, w = \frac{x\left(x+2g^2\right)}{gy}.$$

So in the Weierstrass equation E_g, if we replace the parameter $g(=Y/w^2)$ by n^3 we obtain the following fibration of Y_{10}

$$y^2 = x^3 + 4n^2x^2 + n\left(n+1\right)^2\left(n^2-n+1\right)^2 x \tag{5.1}$$

with singular fibers $2III\left(0,\infty\right), 3I_4\left(-1, n^2-n+1\right), 3I_2\left(1, n^2+n+1\right)$ and rank 4. Notice the two infinite sections with x coordinates $(n+1)^2(n^2-n+1)$ and $-\frac{1}{3}(n-1)^2(n^2-n+1)$. The previous method for the Mordell-Weil generators gives no results but since we have a "self 2-isogeny" that is, by the 2-isogeny of kernel generated by $(0,0)$ we recover the same elliptic fibration up to the automorphism $x \mapsto -x/2, y \mapsto i\sqrt(2)/4y, n \mapsto -n$. Thus we obtain the two other sections of x-coordinates $-1/2(n^2-n+1)^2$ and $1/6(n^2-n+1)^2(n-1)^2/(n+1)^2$ using formulas of 2.7 and the previous automorphism. Computing the height matrix we obtain generators.

□

6. Some 3-isogenies from elliptic fibrations of Y_{10}

We found, along our study of elliptic fibrations of the $K3$ surface Y_{10}, some elliptic fibrations with 3-torsion section. A rank 0 (extremal) elliptic fibration numbered 80 in Shimada-Zhang [20] together with a rank 2 elliptic fibration (11) are given in Bertin and Lecacheux's paper [5]. In the present paper, apart from the two specialized 3-torsion elliptic fibrations of Y_{10}, namely $H_{\#19}(10)$ and $H_{\#20}(10)$, we get the elliptic fibration H_c with $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$-torsion. We are going to compute the transcendental lattices of $K3$ surfaces obtained by 3-isogeny.

Theorem 6.1. 1. *Elliptic fibrations numbered* 80 *(See Shimada's notation* [20]*) and* (11) [5] *induce by* 3*-isogeny an elliptic fibration of a* $K3$ *surface with transcendental lattice* $\begin{pmatrix} 4 & 0 \\ 0 & 18 \end{pmatrix}$.

2. *There is an elliptic fibration of* Y_{10} *with a* 3*-torsion section whose* 3*-isogenous surface is a* $K3$ *surface with transcendental lattice* $\begin{pmatrix} 2 & 0 \\ 0 & 36 \end{pmatrix}$.

Proof. Recall first the corresponding Weierstrass equations.

	Weierstrass Equation Singular Fibers	Rank
80	$Y^2+\left(9t^2+6t-9\right)YX+9t^4\left(3t^2+6t-5\right)Y=X^3$ $I_{12}\left(0\right),3I_3\left(\infty,3t^2+6t-5\right),I_2\left(\frac{3}{5}\right),I_1\left(-\frac{3}{4}\right)$	0
E_{11}	$Y^2+\left(t^2-4\right)YX+t^2\left(2t^2-3\right)Y=X^3$ $2I_6\left(0,\infty\right),2I_3\left(2t^2-3\right),2I_2\left(\pm1\right),2I_1\left(\pm8\right)$	2
#19	$Y^2+10tYX+t^2\left(t^2+10t+1\right)Y=X^3$ $2IV^*\left(0,\infty\right),2I_3\left(t^3+10t+1\right),2I_1\left(\left(t-27\right)\left(27t-1\right)\right)$	2
#20	$Y^2-(t^2-10t+3)YX-\left(t^2-10t+1\right)Y=X^3$ $I_{12}\left(\infty\right),2I_3\left(t^2-10t+1\right),2I_2\left(0,10\right),2I_1\left(\left(t-1\right)\left(t-9\right)\right)$	1
H_c	$Y^2-\left(t^2+11\right)YX-\left(t^2+t+7\right)\left(t^2-t+7\right)Y=X^3$ $I_6\left(\infty\right),6I_3\left(t^2+2,t^2\pm t+7\right)$	1

Notice that from Theorem 3.1 the 3-isogenous $K3$ surfaces of $H_{\#19}$ and $H_{\#20}$ have the lattice $\begin{pmatrix}4 & 0\\ 0 & 18\end{pmatrix}$ as transcendental lattice.

1. **Fibration** 80 **of rank** 0

   A Weierstrass equation for the 3-isogenous fibration is

   $$Y^2+\left(-27t^2-18t+27\right)YX+27\left(4t+3\right)\left(5t-3\right)^2Y=X^3$$

   with singular fibers $I_9\left(\infty\right),I_6\left(\frac{3}{5}\right),I_4,\left(0\right),I_3\left(-\frac{3}{4}\right),2I_1$.

   From singular fibers, torsion and rank we see in Shimada-Zhang table [20] that it is the number 48 case with transcendental lattice $\begin{pmatrix}4 & 0\\ 0 & 18\end{pmatrix}$.

2. **Fibration (11)** In section 6 of Bertin and Lecacheux's paper [5] is exhibited a rank 2 elliptic fibration named (11) with Weierstrass equation

   $$y^2+(t^2-4)xy-t^2(t^2-63)y=(x-9t^2)(x^2+108t^2).$$

   After a translation to put the 3-torsion section in $(0,0)$ we obtain the Weierstrass equation E_{11} given in the table and also p_1 and p_2 generators of the Mordell-Weil lattice

   $$p_1=\left(6t^2,27t^2\right),p_2=\left(6i\sqrt{3}t-3t^2,27t^2\right)$$
   $$2I_6\left(\infty,0\right),\quad 2I_3\left(2t^2-3\right),\quad 2I_2\left(\pm1\right),\quad 2I_1\left(\pm8\right).$$

   We deduce the Weierstrass equation of its 3-isogenous elliptic fibration

   $$H_{11}:Y^2-3\left(t^2-4\right)YX-\left(t^2-1\right)^2\left(t^2-64\right)Y=X^3$$

   with singular fibers

   $$2I_6\left(\pm1\right),\quad 2I_3\left(\pm8\right),\quad 2I_2\left(\infty,0\right),\quad 2I_1\left(2t^2-3\right)\ \text{of rank 2.}$$

Moreover, computing the height matrix of the two sections

$$
\begin{aligned}
\pi_1 = & \left(-\frac{1}{4}\left(t^2-1\right)\left(t^2-64\right), \frac{1}{8}\left(t-8\right)\left(t-1\right)\left(t+1\right)^2\left(t+8\right)^2\right) \\
\omega = & \left(-7\left(t^2-1\right)^2, 49\alpha\left(t^2-1\right)^3\right) \quad \text{where } 49\alpha^2+20\alpha+7=0,
\end{aligned}
$$

we find that the discriminant of the $K3$ surface is 72.

Let us compute the transcendental lattice of H_{11}. For each reducible fiber at $t=i$ we denote (X_i,Y_i) the singular point of H_{11}

$$
\begin{array}{ccc}
t=\pm 1 \quad t=\pm 8 & t=0 & t=\infty \\
(X_{\pm 1}=0, Y_{\pm 1}=0) & (X_0=-16, Y_0=64) & (x_\infty=-1, y_\infty=-1) \\
(X_{\pm 8}=0, Y\pm_8=0) & &
\end{array}
$$

where if $t=\infty$ we substitute $t=\frac{1}{T}$, $x=T^4X, y=T^6Y$. We denote also $\theta_{i,j}$ the j-th component of the reducible fiber at $t=i$. A section $M=(X_M,Y_M)$ intersects the component $\theta_{i,0}$ if and only if $(X_M,Y_M)\not\equiv(X_i,Y_i) \bmod (t-i)$. Using the additivity on the component, we deduce that ω does not intersect $\theta_{i,0}, 2\omega$ intersects $\theta_{i,0}$ and so ω intersects $\theta_{i,3}$ for $i=\pm 1$. Also ω intersects $\theta_{i,0}$ for $i=\pm 8$, $i=0$ and ∞.

For π_1 we compute $k\pi_1$ with $2\leq k\leq 6$. For $i=\pm 1$, only $6\pi_1$ intersects $\theta_{i,0}$ so π_1 intersects $\theta_{i,1}$ (this choice 1, not 5, gives the numbering of components). For $i=\pm 8$, only $3\pi_1$ intersects $\theta_{i,0}$, so π_1 intersects $\theta_{i,1}$. Modulo t, we get $\pi_1=(-16,64)$, so π_1 intersects $\theta_{0,1}$, and π_1 intersects $\theta_{\infty,0}$.

As for the 3-torsion section $s_3=(0,0)$, s_3 intersects $\theta_{i,2}$ or $\theta_{i,4}$ if $i=\pm 1$. Computing $2\pi_1-s_3$, it follows that s_3 intersects $\theta_{1,2}$ and $\theta_{-1,4}$.

For $i=\pm 8$, we compute π_1-s_3, for $t=8$ and show that s_3 intersects $\theta_{8,2}$ and $\theta_{-8,1}$. For $t=0$ and $t=\infty$ s_3 intersects the 0 component.

Since we get the relation $3s_3\approx -2\theta_{1,1}-4\theta_{1,2}-3\theta_{1,3}-2\theta_{1,4}-\theta_{1,5}$, we can choose the following base of the Néron-Severi lattice ordered as $s_0, F, \theta_{1,j}$, with $1\leq j\leq 4$, $s_3, \theta_{-1,k}$ with $1\leq k\leq 5, \theta_{8,k}, k=1,2$, $\theta_{-8,k}, k=1,2$ and $\theta_{0,1}$, $\theta_{\infty,1}, \omega, \pi_1$.

Finally only the two sections ω and π_1 intersect. Hence it follows the Gram matrix Ne of the Néron-Severi lattice,

$$Ne = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & -2
\end{pmatrix}.$$

According to Shimada's lemma 2.1, $G_{Ne} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/36\mathbb{Z}$ is generated by the vectors L_1 and L_2 satisfying $q_{L_1} = -\frac{1}{2}$, $q_{L_2} = \frac{37}{36}$, and $b(L_1, L_2) = \frac{1}{2}$.

Since the following generators of the discriminant group of the lattice with Gram matrix $M_{18} = \left(\begin{smallmatrix} 4 & 0 \\ 0 & 18 \end{smallmatrix}\right)$ namely $f_1 = (0, \frac{1}{2})$, $f_2 = (\frac{1}{4}, \frac{7}{18})$ verify $q_{f_1} = \frac{1}{2}$, $q_{f_2} = -\frac{37}{36}$ and $b(f_1, f_2) = -\frac{1}{2}$, we deduce that the Gram matrix of the transcendental lattice of the surface is $\begin{pmatrix} 4 & 0 \\ 0 & 18 \end{pmatrix}$.

3. **Fibration** H_c We have shown along the proof of Theorem 4.2 that H_c has a $(\mathbb{Z}/3\mathbb{Z})^2$ torsion group.

In the previous Weierstrass equation H_c the X-coordinates of the 3-torsion sections are $0, -\left(c^2 + c + 7\right)\left(c^2 - c + 7\right)$ and roots of

$$X^2 + \left(2c^2 + 13\right)\left(c^2 + 2\right) X + \left(c^2 + c + 7\right)\left(c^2 - c + 7\right)\left(c^2 + 2\right)^2$$

that is $x_3 = -\left(c^2 - \alpha^2\right)\left(c^2 + 2\right)$ and $\overline{x_3}$ where $\alpha = \frac{1+3i\sqrt{3}}{2} = 2 + 3\zeta_3$ and $\bar{\alpha}$ are roots of $c^2 - c + 7$.

By the 3-isogeny of kernel $< (0, 0 >$ we recover the fibration E_c of Y_2.

Let σ_3 be the 3-torsion section of X-coordinate $-\left(c^2 + c + 7\right)\left(c^2 - c + 7\right)$.

Notice that the group $\langle \sigma_3 \rangle$ is stable by the Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$.

After a translation to put this point in $(0, 0)$ and scaling, we obtain a Weierstrass equation

$$Y'^2 - (c^2 + 11)Y'X' - (c^2 + c + 7)(c^2 - c + 7)Y' = X'^3.$$

The 3-isogenous curve of kernel $< \sigma_3 >$ has a Weierstrass equation

$$y^2 + 3\left(c^2 + 11\right) xy + \left(c^2 + 2\right)^3 y = x^3,$$

with singular fibers $I_2(\infty)$, $2I_9\left(c^2 + 2\right)$, $4I_1\left(c^4 + 13c^2 + 49\right)$.

The section $P_c = \left(-\frac{1}{4}\left(c^4+c^2+1\right), -\frac{1}{8}\left(c-\zeta_3\right)^3\left(c+\zeta_3^2\right)^3\right)$ of infinite order, generates the Mordell-Weil lattice.

We consider the components of the reducible fibers in the following order $\theta_{i\sqrt{2},j}$ $j \leq 1 \leq 8$, $\theta_{-i\sqrt{2},k}$ $1 \leq k \leq 8$, $\theta_{\infty,1}$.

The 3-torsion section $t_3 = (0,0)$ and the previous components are linked by the relation

$$3t_3 \approx -\theta_{i\sqrt{2},8} + \sum a_{i,j}\theta_{i,j}.$$

So we can replace, in the previous ordered sequence of components, the element $\theta_{i\sqrt{2},8}$ by t_3. We notice that $(t_3.P_c) = 2$, $\left(P_c.\theta_{\pm i\sqrt{2},0}\right) = 1$ and $(P_c.\theta_{\infty,0}) = 1$, so the Gram matrix of the Néron-Severi lattice is

$$NS = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2
\end{pmatrix}.$$

Its determinant is -72. According to Shimada's lemma 2.1, $G_{NS} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/36$ is generated by the vectors L_1 and L_2 satisfying $q_{L_1} = \frac{-1}{2}$, $q_{L_2} = \frac{5}{36}$ and $b(L_1, L_2) = \frac{1}{2}$.

Moreover the following generators of the discriminant group of the lattice with Gram matrix $M_{36} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 36 \end{smallmatrix}\right)$, namely $f_1 = (\frac{1}{2}, 0)$, $f_2 = (\frac{-1}{2}, \frac{-7}{36})$ verify $q_{f_1} = \frac{1}{2}$, $q_{f_2} = -\frac{5}{36}$ and $b(f_1, f_2) = \frac{-1}{2}$. So the Gram matrix of the transcendental lattice is $\begin{pmatrix} 2 & 0 \\ 0 & 36 \end{pmatrix}$.

Let s_3 be the 3-torsion point of X-coordinate x_3. After a translation to have s_3 in $(0,0)$ and some scaling we obtain the following equation for H_c

$$y^2 + \left(c^2 - 6\zeta_3 + 1\right)yx + \zeta_3^2\left(c^2+2\right)\left(c^2-\alpha^2\right)y = x^3.$$

The 3-isogeny of kernel $< s_3 >$ has a Weierstrass equation

$$y^2 - 3\left(c^2 - 6\zeta_3 + 1\right)yx - \left(c^2 - \bar{\alpha}^2\right)^3 y = x^3$$

with singular fibers $I_2(\infty), 2I_9(\pm\bar{\alpha}), 4I_1$. Notice the section of infinite order

$$\begin{aligned}
Q &= (x_Q, y_Q) \\
x_Q &= -\frac{1}{4}\frac{\left(c^2-\zeta_3^2\right)\left(c^2-\bar{\alpha}^2\right)^3}{\left(c^2-\zeta_3\right)^2} \\
y_Q &= \frac{1}{8}\frac{\left(c-\zeta_3\right)^3\left(c-\bar{\alpha}\right)^6\left(c+\bar{\alpha}\right)^3}{\left(c^2-\zeta_3\right)^3}.
\end{aligned}$$

To construct the Gram matrix of the Néron-Severi lattice we take the following order $\theta_{-\bar{\alpha},i}$,$1 \leq i \leq 8$, $\theta_{\bar{\alpha},i}$,$1 \leq i \leq 8$. We substitute $\theta_{-\bar{\alpha},8}$ by the 3- torsion section $(0,0)$ and the last element is Q. We have $s_\infty.Q = 2 \quad \theta_{-\bar{\alpha},3}.Q = 1$, $\theta_{\bar{\alpha},6}.Q = 1$, $(0,0).Q = 1$. So we have the Gram matrix

$$NS = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2
\end{pmatrix}$$

with discriminant equal to -72. According to Shimada's lemma 2.1 $G_{NS} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/36\mathbb{Z}$ is generated by the two vectors L_1, L_2 with $q_{L_l} = -\frac{1}{2}$, $q_{L_2} = \frac{13}{36}, b(L_1, L_2) = \frac{1}{2}$. So using the results of fibration $H_{\#19}$ we obtain with the vectors $f_1, -f_2$ that Gram matrix of the transcendental lattice is $\begin{pmatrix} 4 & 0 \\ 0 & 18 \end{pmatrix}$.

□

Remark 6.2. Concerning rank 0 elliptic fibrations, using Shimada-Zhang's table [20], we recover easily all our results without using Weierstrass equations. We have only to know the transformation by a 3-isogeny of a type of singular fiber. This can be obtained using Tate's algorithm [25] and an analog of Dokchitser's remark [8].

Acknowledgments We sincerely thank the anonymous referees for their acute reading of our paper. Their remarks and questions have been very useful for improving the presentation of our work.

References

[1] M.-J. Bertin, *Mahler's measure and L-series of $K3$ hypersurfaces* in: Mirror Symmetry V, (Eds. N. Yui, S.-T. Yau, J. D. Lewis) AMS/IP Studies in Advanced Math. **38** (2007).

[2] M.-J. Bertin, *Mesure de Mahler et série L d'une surface $K3$ singulière*, Publ. Math. Besançon, Actes de la Conférence Algèbre et Théorie des Nombres (2010), 5–28.

[3] M.-J. Bertin and O. Lecacheux, *Elliptic fibrations on the modular surface associated to* $\Gamma_1(8)$, in: Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds, Fields Inst.Commun., **67**, Springer, New York, (2013), 153–199.

[4] M.-J. Bertin and O. Lecacheux, *Apéry-Fermi pencil of $K3$ surfaces and 2-isogenies*, J. Math. Soc. Japan (2) **72** (2020), 599–637.

[5] M.-J. Bertin and O. Lecacheux, *Elliptic fibrations of a certain $K3$ surface of the Apéry-Fermi pencil*, Publ. Math. Besançon, Algèbre et Théorie des Nombres, **44**, (2022), 5–36.

[6] N. Bourbaki, *Groupes et algèbres de Lie*, Chap.4, 5, 6, (Masson, 1981).

[7] C. Camere and A. Garbagnati, *On certain isogenies between $K3$ surfaces*, Trans. Amer. Math. Soc. **373** (2020), 2913–2931. doi:https//doi.org/10.1090/tran/8022, 19p. elec. 28 Jan. 2020
also
arXiv:1905.08859v1 [math.AG], 21 May 2019, 28 p.

[8] T. and V. Dokchitser, *Local invariants of isogenous elliptic curves*, Trans. Amer. Math. Soc. **367** (2015), 4339–4358.

[9] A. Garbagnati and J. Prieto-Montañez, *Order 3 symplectic automorphisms on $K3$ surfaces*, Math. Z., **301** (2022), 225–253.

[10] A. Garbagnati and J. Prieto-Montañez, *Generalized Shioda-Inose structures of order* 3, Advances in Geometry, (2) **24** (2024), 183–207. https://doi.org/10.1515/advgeom-2024-0005
also
arXiv:2209.10141v1 [math.AG], 21 Sep. 2022, 32 p.

[11] A. Garbagnati and C. Salgado, *Rank jumps and multisections of elliptic fibrations on $K3$ surfaces*, Forum of Math., Sigma (2026), **14**, 1-27.

[12] M. Hindry and C. Salgado, *Lower bounds for the rank of families of abelian varieties under base change*, Act. Arithm., **189** (2019), 263-282.

[13] A. Kumar, and M. Kuwata, *Elliptic $K3$ surfaces associated with the product of two elliptic curves: Mordell-Weil lattices and their fields of definition*, Nagoya Math. J. **228** (2017), 124–185.

[14] M. Kuwata, *Elliptic $K3$ surfaces with given Mordell-Weil rank*, Comment. Math. Univ. St. Paul. **49** (2000), 91–100.

[15] O. Lecacheux, *Weierstrass equations for all elliptic fibrations on the modular $K3$ surface associated to* $\Gamma_1(7)$, Rocky Mountain J. Math. **45** (5) (2015), 1481–1509.

[16] V. V. Nikulin, *Integral symmetric bilinear forms and some of their applications*, Izv. Math. (1) **14**, (1980), 103–167.

[17] C. Peters and J. Stienstra, *A pencil of $K3$ surfaces related to Apéry's recurrence for zeta(3) and Fermi surfaces for potential zero*, Arithmetic and Complex manifolds (Erlangen, 1988) (W.-P. Barth and H. Lange, eds.), Lecture Notes in Math., vol. 1399, Springer, Berlin 1989, 110-127.

[18] I. Shimada, *$K3$ surfaces and Lattice Theory, Talk Dec. 2, 2014, Hanoï Vietnam Institute for Advavanced Study in Mathematics, (Slides on Shimada's web page)*.

[19] I. Shimada, *On elliptic $K3$ surfaces*, arXiv:math 0505140v3 [math.AG] 11Apr 2006, 1–114.

[20] I. Shimada, and D. Q. Zhang, *Classification of extremal elliptic $K3$ surfaces and fundamental groups of open $K3$ surfaces*, Nagoya Math. J. **161** (2001), 23–54.

[21] T. Shioda, $K3$ *surface and Sphere packing*, J. Math. Soc. Japan **60** (2008), 1083–1105.
[22] T. Shioda, *On elliptic modular surfaces*, J. Math. Soc. Japan, **24** (1972), 20-59.
[23] T. Shioda & H. Inose, *On singular* $K3$ *surfaces* in: Complex analysis and algebraic geometry (Baily, Shioda T. ed.), Cambridge (1977), 119-135.
[24] J. Silverman, The Arithmetic of Elliptic Curves, Grad. Texts in Math., **106**, Springer-Verlag, New York, (1986).
[25] J. Tate, *Algorithm for determining the type of a singular fiber in an elliptic pencil*, in: Modular Functions of One Variable IV, Lect. Notes in Math. 476, B. J. Birch and W. Kuyk, eds, Springer-Verlag, Berlin, 1975, 33-52.
[26] J. Vélu, *Isogénies entre courbes elliptiques*, C. R. Acad. Sc. Paris, Série A, t. **273** (1971), 238-241.