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SOME INEQUALITIES ON TIME SCALES SIMILAR TO REVERSE HARDY'S INEQUALITY

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Abstract. In this paper, we give new inverted Hardy inequalities on a general time scale by introducing two parameters in the Theorem 3.4 and a new inequality with negative parameters in the Theorem 3.5. The main results are demonstrated using two new lemmas and an interesting proposition. We derive some new corollaries of continuous and discrete choice on time scale.

1. Introduction

Hardy inequality in the integral form plays an important role in the modern analysis of partial differential equations and is an indispensable tool in spectral theory of partial differential operators.

In 2012, Sulaiman presented a reverse Hardy's inequality [5, Theorem 3.1]: Let h be positive function defined on $[a, b]$, $H(x) = \int_a^x h(\tau)d\tau$, then

1. for $p \geq 1$,

$$p\int_a^b \left(\frac{H(x)}{x}\right)^p dx \leq (b-a)^p \int_a^b \left(\frac{h(x)}{x}\right)^p dx - \int_a^b (1-\frac{a}{x})h^p(x)dx,$$

2. for $0 < p < 1$,

$$p\int_a^b \left(\frac{H(x)}{x}\right)^p dx \geq (1-\frac{a}{b})^p \int_a^b \left(\frac{h(x)}{x}\right)^p dx - \frac{1}{b^p}\int_a^b (x-a)^p h^p(x).$$

In 2013, B. Sroysang extends the inverse Hardy inequality [6, Theorem 2.1 and Theorem 2.2]:

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Let h be positive function defined on $[a,b]$, $H(x) = \int_a^x f(\tau)d\tau$ and $q > 0$, then
1. for $p \geq 1$,
$$p\int_a^b \frac{H(x)^p}{x^q}dx \leq (b-a)^p \int_a^b \frac{h(x)^p}{x^q}dx - \int_a^b \frac{(x-a)^p}{x^q}h^p(x)dx,$$
2. for $0 < p < 1$,
$$p\int_a^b \frac{H(x)^p}{x^q}dx \geq \frac{(b-a)^p}{b^q}\int_a^b h^p(x)dx - \frac{1}{b^q}\int_a^b (x-a)^p h^p(x).$$

In 2018, B. Benaissa gives a further generalization [1, theorem 2.2]. Let h, ϕ be positive functions defined on $[a,b]$ and $H(x) = \int_a^x h(\tau)d\tau$. If ϕ is non-decreasing then
(i) for $p \geq 1$,
$$p\int_a^b \frac{H(x)^p}{\phi(x)}dx \leq (b-a)^p \int_a^b \frac{h(x)^p}{\phi(x)}dx - \int_a^b \frac{(x-a)^p}{\phi(x)}h^p(x)dx, \tag{1.1}$$
(ii) for $0 < p < 1$,
$$p\int_a^b \frac{H(x)^p}{\phi(x)}dx \geq \frac{(b-a)^p}{\phi(b)}\int_a^b h^p(x)dx - \frac{1}{\phi(b)}\int_a^b (x-a)^p h^p(x)dx. \tag{1.2}$$

The aim of this work is to give a generalization for (1.1), (1.2) and to conclude new results using calculus on time scales.

2. PRELIMINARIES

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with standard topology.

DEFINITION 2.1. [2] *Let $\mathbb{T}$ be a time scales. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by:*
$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$
We put $\inf \emptyset = \sup \mathbb{T}, \quad \sup \emptyset = \inf \mathbb{T}$.

Delta derivative

DEFINITION 2.2. *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. We define $f^\Delta(t)$ to be the number, if it exists, defined as follows: for every $\epsilon > 0$ there is a neighborhood U of t, $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$, such that*
$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \ \textit{for all}\ s \in U,\ s \neq \sigma(t).$$
We call $f^\Delta(t)$ the delta derivative of f at t.

THEOREM 2.3. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following.*

1. *If f is Δ-differentiable at t, then f is continuous at t.*
2. *If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t with*
$$f^{\Delta}(t) = \frac{f^{\sigma}(t) - f(t)}{\mu(t)}.$$
3. *If t is right-dense, then f is Δ-differentiable iff the limit*
$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$
exists as a finite number. In this case,
$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$
4. *If f is Δ-differentiable at t, then*
$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

For $a, b \in \mathbb{T}$ and a delta differentiable function h, the Cauchy integral of h^{Δ} is defined by
$$\int_a^b h^{\Delta}(\tau)\Delta\tau = h(b) - h(a).$$

THEOREM 2.4. [4, Theorem 1.1.2] *Let h, $\phi \in C_{rd}(\mathbb{T}, \mathbb{R})$ be rd-continuous functions, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then, the following are true:*

1. $\int_a^b \left(\alpha h^{\Delta}(\tau) + \beta \phi^{\Delta}(\tau)\right) \Delta\tau = \alpha \int_a^b h^{\Delta}(\tau)\Delta\tau + \beta \int_a^b \phi^{\Delta}(\tau)\Delta\tau,$
2. $\int_a^b h^{\Delta}(\tau)\Delta\tau = -\int_b^a h^{\Delta}(\tau)\Delta\tau,$
3. $\int_a^c h^{\Delta}(\tau)\Delta\tau = \int_a^b h^{\Delta}(\tau)\Delta\tau + \int_b^c h^{\Delta}(\tau)\Delta\tau,$
4. $\left|\int_a^b h^{\Delta}(\tau)\Delta\tau\right| \le \int_a^b \left|h^{\Delta}(\tau)\right| \Delta\tau.$

Chain Rule

THEOREM 2.5. [2, Theorem 1.90] *.Let $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f o g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*
$$(f o g)^{\Delta} = g^{\Delta} \int_0^1 f'\left(s\, g^{\sigma} + (1 - s) g\right) ds,$$
holds.

Lemma 2.6. *(Hölder's inequality)* [4, Theorem 1.1.10; (1.1.8)]. *Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and h, $\phi \in C_{rd}(\mathbb{T}, \mathbb{R})$ be two positive functions. If $\frac{1}{p} + \frac{1}{p'} = 1$ with $p \geq 1$, then*

$$\int_a^b h(\tau)\phi(\tau)\Delta\tau \leq \left(\int_a^b h^p(\tau)\Delta\tau\right)^{\frac{1}{p}} \left(\int_a^b \phi^{p'}(\tau)\Delta\tau\right)^{\frac{1}{p'}}, \tag{2.3}$$

this inequality is reversed if $0 < p < 1$.

Proposition 2.7. [4, Fubini's Theorem]

Let f be $\Delta\Delta$-integrable on $\mathbf{R} = [a, b) \times [c, d)$ and suppose that the single integral

$$I(x) = \int_c^d f(x, y)\Delta_2 y,$$

exists for each $x \in [a, b)$ and the single integral

$$K(y) = \int_a^b f(x, y)\Delta_1 x,$$

exists for each $y \in [c, d)$. Then the iterated integrals

$$\int_a^b I(x)\Delta_1 x = \int_a^b \Delta_1 x \int_c^d f(x, y)\Delta_2 y \text{ and } \int_c^d K(y)\Delta_2 y = \int_c^d \Delta_2 y \int_a^b f(x, y)\Delta_1 x,$$

exists and the equalities

$$\int\int_{\mathbf{R}} f(x, y)\Delta_1 x\Delta_2 y = \int_a^b \Delta_1 x \int_c^d f(x, y)\Delta_2 y = \int_c^d \Delta_2 y \int_a^b f(x, y)\Delta_1 x \tag{2.4}$$

holds.

In this paper, we study some inverted Hardy dynamic integral inequalities on time scales and we give new ones with a negative parameter, also extend some continuous inequalities and their discrete analogues. We assume that all the integrals of the right and left side of the inequalities are convergent.

3. Main Results

We state the following lemmas which are useful in the proof of the main theorem. Firstly, we extended the lemma [3, 3.6 p 15] for $p \neq 0$.

Lemma 3.1. *Let ψ be a nonnegative monotone function on $[a,\ b]_{\mathbb{T}}$. If ψ in non-decreasing on $[a,\ b]_{\mathbb{T}}$, then*

$$for\ 1 \leq p < \infty : \ (\psi^p)^\Delta \geq p\,\psi^\Delta \psi^{p-1}, \tag{3.5}$$

$$\text{(3.6)} \qquad \textit{for } 0 < p < 1 : \ (\psi^p)^\Delta \leq p\,\psi^\Delta \psi^{p-1}.$$

If ψ is non-increasing function on $[a,\, b]_{\mathbb{T}}$, then

$$\text{(3.7)} \qquad \textit{for } p < 0 : \ (\psi^p)^\Delta \geq p\,\psi^\Delta \psi^{p-1},$$

PROOF. Applying the chain rule for $1 \leq p < \infty$, we get

$$\begin{aligned}(\psi^p)^\Delta &= p\,\psi^\Delta \int_0^1 (s\,\psi^\sigma \ + (1-s)\psi)^{p-1}\,ds \\ &\geq p\,\psi^\Delta \int_0^1 (s\,\psi \ + (1-s)\psi)^{p-1}\,ds \\ &= p\,\psi^\Delta \psi^{p-1}.\end{aligned}$$

For $0 < p < 1$, since $p-1 < 0$ we have

$$\begin{aligned}(\psi^p)^\Delta &\leq p\,\psi^\Delta \int_0^1 (s\,\psi \ + (1-s)\psi)^{p-1}\,ds \\ &= p\,\psi^\Delta \psi^{p-1}.\end{aligned}$$

Since $p < 0$ and $\psi^\Delta \leq 0$, then

$$\begin{aligned}(\psi^p)^\Delta &\geq p\,\psi^\Delta \int_0^1 (s\,\psi \ + (1-s)\psi)^{p-1}\,ds \\ &= p\,\psi^\Delta \psi^{p-1}.\end{aligned}$$

□

LEMMA 3.2. *Let $0 < p \leq q < \infty$ and h, ν are nonnegative and rd-continuous functions on $[a,\, b]_{\mathbb{T}}$ and we suppose that $0 < \int_a^b h^q(\tau)\nu(\tau)\Delta\tau < \infty$, then*

$$\text{(3.8)} \qquad \int_a^b h^p(\tau)\nu(\tau)\Delta\tau \leq \left(\int_a^b \nu(\tau)\Delta\tau\right)^{\frac{q-p}{q}} \left(\int_a^b h^q(\tau)\nu(\tau)\Delta\tau\right)^{\frac{p}{q}}.$$

The inequality (3.8) hold for $-\infty < q \leq p < 0$ and inverted for $0 < q \leq p < \infty$.

PROOF. By Hölder integral inequality (2.3) for using the parameter $\dfrac{q}{p} \geq 1$, we have

$$\begin{aligned}\int_a^b h^p(\tau)\nu(\tau)\Delta\tau &= \int_a^b \left(\nu^{\frac{q-p}{q}}(\tau)\right)\left(h^p(\tau)\nu^{\frac{p}{q}}(\tau)\right)\Delta\tau \\ &\leq \left(\int_a^b \nu(\tau)\Delta\tau\right)^{\frac{q-p}{q}} \left(\int_a^b h^q(\tau)\nu(\tau)\Delta\tau\right)^{\frac{p}{q}}.\end{aligned}$$

□

PROPOSITION 3.3. *Let $0 < B < A$ are two positive numbers, then*

$$\text{for} \quad p \geq 1: \qquad (A-B)^p \leq A^p - B^p, \tag{3.9}$$

$$\text{for} \quad 0 < p < 1: \quad (A-B)^p > A^p - B^p. \tag{3.10}$$

PROOF. we can write

$$\text{for } p \geq 1; \qquad 1 \leq \left(\tfrac{A}{A-B}\right)^p - \left(\tfrac{B}{A-B}\right)^p,$$

$$\text{for } 0 < p < 1; \quad 1 \geq \left(\tfrac{A}{A-B}\right)^p - \left(\tfrac{B}{A-B}\right)^p,$$

so for $p = 1$ we get equality.

Now we take $t = \dfrac{A}{A-B}$, then we get $t > 1$ and $B = A(1 - \dfrac{1}{t})$, thus

$$\left(\frac{A}{A-B}\right)^p - \left(\frac{B}{A-B}\right)^p = t^p - (t-1)^p = g(t),$$

for all $t > 1$, we have

$$g'(t) = p\, t^p \left(1 - \left(\frac{t-1}{t}\right)^{p-1}\right).$$

Since $0 < \frac{t-1}{t} < 1$, then

- for $p > 1$ yields g is increasing function, hence $t > 1 \Rightarrow g(t) \geq g(1) = 1$, consequently $1 \leq \left(\frac{A}{A-B}\right)^p - \left(\frac{B}{A-B}\right)^p$.
- For $0 < p < 1$ gives us that g is decreasing function, hence $t > 1 \Rightarrow g(t) \leq g(1) = 1$, which is same as $1 \geq \left(\frac{A}{A-B}\right)^p - \left(\frac{B}{A-B}\right)^p$.

□

We can be rewritten the above inequalities (3.9) and (3.10) in the following form

$$\text{for} \quad p \geq 1: \qquad A - B \leq (A^p - B^p)^{\frac{1}{p}}, \tag{3.11}$$

$$\text{for} \quad 0 < p < 1: \qquad A - B \geq (A^p - B^p)^{\frac{1}{p}}. \tag{3.12}$$

3.1. *Reverse Hardy's Inequality.* Now we give the dynamic reverse Hardy's Inequality version in time scales.

THEOREM 3.4. *Let $\mathbb{T}$ be a time scales, $a, b \in \mathbb{T}$ with $a < b$ and h, ϕ be non-negative continuous functions on $[a, b]_{\mathbb{T}}$, let*

$$H(\tau) = \int_a^\tau h(s)\Delta s.$$

If ϕ is increasing function and $(\sigma(s) - a)^\Delta = 1$, then
• *for $1 \leq p \leq q < \infty$:*

$$\begin{aligned} p\int_a^b \frac{(H^\sigma(\tau))^p}{\phi^\sigma(\tau)}\Delta\tau &\leq \left(\int_a^b \frac{1}{\phi(\sigma(s))}\Delta s\right)^{\frac{q-p}{q}} \\ &\times \left(\int_a^b \frac{h^q(s)\left[(b-a)^q - (\sigma(s)-a)^q\right]}{\phi(\sigma(s))}\Delta s\right)^{\frac{p}{q}} \end{aligned} \tag{3.13}$$

• *for $0 < q \leq p < 1$:*
(3.14)
$$p\int_a^b \frac{(H^\sigma(\tau))^p}{\phi^\sigma(\tau)}\Delta\tau \geq \frac{(b-a)^{\frac{q-p}{q}}}{\phi(b)}\left(\int_a^b h^q(s)\left[(b-a)^q - (\sigma(s)-a)^q\right]\Delta s\right)^{\frac{p}{q}}.$$

PROOF. Let $p \geq 1$,using Hölder inequality for $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, thus

$$\begin{aligned} H(\sigma(\tau)) = \int_a^{\sigma(\tau)} h(s)\Delta s &\leq \left(\int_a^{\sigma(\tau)} h^p(s)\Delta s\right)^{\frac{1}{p}} \left(\int_a^{\sigma(\tau)} \Delta s\right)^{\frac{1}{p'}} \\ &= \left(\int_a^{\sigma(\tau)} h^p(s)\Delta s\right)^{\frac{1}{p}} (\sigma(\tau) - a)^{\frac{p-1}{p}}, \end{aligned}$$

hence by using Fubini Theorem, we have

$$\begin{aligned}
\int_a^b \frac{H(\sigma(\tau))^p}{\phi(\sigma(\tau))}\Delta\tau &= \int_a^b \frac{1}{\phi(\sigma(\tau))}\left(\int_a^{\sigma(\tau)} h(s)\Delta s\right)^p \Delta\tau \\
&\leq \int_a^b \frac{1}{\phi(\sigma(\tau))}\left(\sigma(\tau)-a\right)^{p-1}\left(\int_a^{\sigma(\tau)} h^p(s)\Delta s\right)\Delta\tau \\
&= \int_a^b \Delta\tau \int_a^{\sigma(\tau)} \frac{1}{\phi(\sigma(\tau))}(\sigma(\tau)-a)^{p-1}h^p(s)\Delta s \\
&= \int_a^b h^p(s)\Delta s\left(\int_{\sigma(s)}^b \frac{1}{\phi(\sigma(\tau))}(\sigma(\tau)-a)^{p-1}\Delta\tau\right),
\end{aligned}$$

the hypothesis ϕ is increasing, yield

$$\forall\tau\in[\sigma(s),b]: \frac{1}{\phi(\sigma(\tau))}\leq\frac{1}{\phi(\tau)}\leq\frac{1}{\phi(\sigma(s))},$$

this gives us that

$$\int_a^b \frac{H(\sigma(\tau))^p}{\phi(\sigma(\tau))}\Delta\tau \leq \int_a^b \frac{h^p(s)}{\phi(\sigma(s))}\Delta s\left(\int_{\sigma(s)}^b (\sigma(\tau)-a)^{p-1}\Delta\tau\right). \tag{3.15}$$

Let $\psi(\tau)=\sigma(\tau)-a$, by applied (3.5) with the supposition that $\psi^\Delta=1$, we get

$$(\sigma(\tau)-a)^{p-1}\leq\frac{1}{p}((\sigma(\tau)-a)^p)^\Delta$$

Integrating on $[\sigma(s),b]_{\mathbb{T}}$, then

$$\begin{aligned}
\int_{\sigma(s)}^b (\sigma(\tau)-a)^{p-1}\Delta\tau &\leq \frac{1}{p}\int_{\sigma(s)}^b (\psi^p)^\Delta(\tau)\Delta t \\
&= \frac{1}{p}\left[(b-a)^p-(\sigma(s)-a)^p\right],
\end{aligned}$$

using the inequality (3.11) for $A=(b-a)^p$, $B=(\sigma(s)-a)^p$ and $\frac{q}{p}\geq 1$, we get

$$(b-a)^p-(\sigma(s)-a)^p\leq((b-a)^q-(\sigma(s)-a)^q)^{\frac{p}{q}},$$

therefore

$$\int_{\sigma(s)}^b (\sigma(\tau)-a)^{p-1}\Delta\tau\leq\frac{1}{p}\left[(b-a)^q-(\sigma(s)-a)^q\right]^{\frac{p}{q}} \tag{3.16}$$

put (3.19) in (3.18) and apply (3.8), we get

$$
\begin{aligned}
p\int_a^b \frac{H(\sigma(\tau))^p}{\phi(\sigma(\tau))}\Delta\tau \quad &\leq \int_a^b \frac{h^p(s)}{\phi(\sigma(s))}\left[(b-a)^q-(\sigma(s)-a)^q\right]^{\frac{p}{q}}\Delta s \\
&= \int_a^b \frac{\left(h(s)\left[(b-a)^q-(\sigma(s)-a)^q\right]^{\frac{1}{q}}\right)^p}{\phi(\sigma(s))}\Delta s \\
&\leq \left(\int_a^b \frac{1}{\phi(\sigma(s))}\Delta s\right)^{\frac{q-p}{q}} \\
&\quad\times\left(\int_a^b \frac{h^q(s)\left[(b-a)^q-(\sigma(s)-a)^q\right]}{\phi(\sigma(s))}\Delta s\right)^{\frac{p}{q}},
\end{aligned}
$$

this compleat the proof.

(ii) for $0<q\leq p<1$, by using the reverse Hölder inequality and the assumption ϕ is increasing function on $[s,b]$ we get

$$\int_a^b \frac{H(\sigma(\tau))^p}{\phi(\sigma(\tau))}\Delta\tau \geq \frac{1}{\phi(b)}\int_a^b h^p(s)\Delta s\left(\int_{\sigma(s)}^b (\sigma(\tau)-a)^{p-1}\Delta\tau\right).$$

Let $\psi(\tau)=\sigma(\tau)-a$, apply (3.6) with the supposition that $\psi^{\Delta}=1$, then

$$(\sigma(\tau)-a)^{p-1}\geq\frac{1}{p}((\sigma(\tau)-a)^p)^{\Delta}$$

Integrating on $[\sigma(s),b]_{\mathbb{T}}$, we obtain

$$\int_{\sigma(s)}^b (\sigma(\tau)-a)^{p-1}\Delta\tau \quad \geq \quad \frac{1}{p}\left[(b-a)^p-(\sigma(s)-a)^p\right],$$

using the inequality (3.11) and we take $\nu=1$, so the proof is similar to the proof of inequality (3.13). □

THEOREM 3.5. *Let $\mathbb{T}$ be a time scales, a, $b\in\mathbb{T}$ with $a<b$ and h,ϕ be non-negative continuous functions on $[a,\, b]_{\mathbb{T}}$, let*

$$\widetilde{H}(\tau)=\int_\tau^b h(s)\Delta s.$$

If ϕ is decreasing function and $(b-\sigma(s))^{\Delta}=-1$, then
• *for $-\infty < q \leq p < 0$:*
(3.17)

$$\begin{aligned} -p\int_a^b \frac{(\widetilde{H}^{\sigma}(\tau))^p}{\phi^{\sigma}(\tau)}\Delta\tau \quad &\leq \left(\int_a^b \frac{1}{\phi(\sigma(s))}\Delta s\right)^{\frac{q-p}{q}} \\ &\times\left(\int_a^b \frac{h^q(s)\left[(b-\sigma(s))^q-(b-a)^q\right]}{\phi(\sigma(s))}\Delta s\right)^{\frac{p}{q}}. \end{aligned}$$

Proof. Let $p<0$, by using Hölder inequality , we have

$$\begin{aligned} \widetilde{H}(\sigma(\tau)) = \int_{\sigma(\tau)}^b h(s)\Delta s \quad &\geq \quad \left(\int_{\sigma(\tau)}^b h^p(s)\Delta s\right)^{\frac{1}{p}}\left(\int_{\sigma(\tau)}^b \Delta s\right)^{\frac{1}{p'}} \\ &= \quad \left(\int_{\sigma(\tau)}^b h^p(s)\Delta s\right)^{\frac{1}{p}}(b-\sigma(\tau))^{\frac{p-1}{p}}, \end{aligned}$$

now we apply Fubini theorem, then

$$\begin{aligned} \int_a^b \frac{\widetilde{H}(\sigma(\tau))^p}{\phi(\sigma(\tau))}\Delta\tau \quad &= \quad \int_a^b \frac{1}{\phi(\sigma(\tau))}\left(\int_{\sigma(\tau)}^b h(s)\Delta s\right)^p\Delta\tau \\ &\leq \quad \int_a^b \frac{1}{\phi(\sigma(\tau))}(b-\sigma(\tau))^{p-1}\left(\int_{\sigma(\tau)}^b h^p(s)\Delta s\right)\Delta\tau \\ &= \quad \int_a^b h^p(s)\Delta s\left(\int_a^{\sigma(s)} \frac{1}{\phi(\sigma(\tau))}(b-\sigma(\tau))^{p-1}\Delta\tau\right), \end{aligned}$$

since ϕ is decreasing, yields

$$\forall\tau\in[a,\sigma(s)] : \frac{1}{\phi(\tau)} \leq \frac{1}{\phi(\sigma(\tau))} \leq \frac{1}{\phi(\sigma(s))},$$

therefore

$$\int_a^b \frac{\widetilde{H}(\sigma(\tau))^p}{\phi(\sigma(\tau))}\Delta\tau \leq \int_a^b \frac{h^p(s)}{\phi(\sigma(s))}\Delta s\left(\int_a^{\sigma(s)}(b-\sigma(\tau))^{p-1}\Delta\tau\right). \tag{3.18}$$

Let $\psi(\tau)=b-\sigma(\tau)$, by applied (3.7) with the supposition that $\psi^{\Delta}=-1$, we get

$$(b-\sigma(\tau))^{p-1} \leq -\frac{1}{p}(b-(\sigma(\tau))^p)^{\Delta}$$

Integrating on $[a, \sigma(s)]_{\mathbb{T}}$, then

$$\begin{aligned}\int_a^{\sigma(s)} (b-\sigma(\tau))^{p-1}\Delta\tau &\leq -\frac{1}{p}\int_a^{\sigma(s)} (\psi^p)^{\Delta}(\tau)\Delta\tau \\ &= -\frac{1}{p}\left[(b-\sigma(s))^p-(b-a)^p\right],\end{aligned}$$

using the inequality (3.11) for $A=(b-\sigma(s))^p$, $B=(b-a)^p$ and $\frac{q}{p}\geq 1$, we obtain

$$(b-\sigma(s))^p-(b-a)^p \leq \left[(b-\sigma(s))^q-(b-a)^q\right]^{\frac{p}{q}},$$

hence

$$\int_a^{\sigma(s)} (b-\sigma(\tau))^{p-1}\Delta\tau \leq -\frac{1}{p}\left[(b-\sigma(s))^q-(b-a)^q\right]^{\frac{p}{q}}. \tag{3.19}$$

Put (3.19) in (3.18) and apply (3.8), we get

$$\begin{aligned}-p\int_a^b \frac{\widetilde{H}(\sigma(\tau))^p}{\phi(\sigma(\tau))}\Delta\tau &\leq \int_a^b \frac{h^p(s)}{\phi(\sigma(s))}\left[(b-\sigma(s))^q-(b-a)^q\right]^{\frac{p}{q}}\Delta s \\ &= \int_a^b \frac{\left(h(s)\left[b-\sigma(s))^q-(b-a)^q\right]^{\frac{1}{q}}\right)^p}{\phi(\sigma(s))}\Delta s \\ &\leq \left(\int_a^b \frac{1}{\phi(\sigma(s))}\Delta s\right)^{\frac{q-p}{q}} \\ &\quad \times \left(\int_a^b \frac{h^q(s)\left[b-\sigma(s))^q-(b-a)^q\right]}{\phi(\sigma(s))}\Delta s\right)^{\frac{p}{q}},\end{aligned}$$

this compleat the proof.

$\square$

4. Applications

By taking $q=p$ in the Theorem 3.4 and the Theorem 3.5, we obtain the following Corollaries.

Corollary 4.1. *Let $\mathbb{T}$ be a time scales, a, $b \in \mathbb{T}$ with $a<b$ and suppose that*
$(\sigma(s)-a)^{\Delta}=1$. *Let h, ϕ be non-negative continuous functions on $[a, b]_{\mathbb{T}}$ and* $H(\tau)=\displaystyle\int_a^{\tau} h(s)\Delta s$. *If ϕ is increasing function, then*

• *for* $1 \le p < \infty$ *:*

$$p\int_a^b \frac{(H^\sigma(\tau))^p}{\phi^\sigma(\tau)}\Delta\tau \le \int_a^b \frac{h^p(s)\left[(b-a)^p-(\sigma(s)-a)^p\right]}{\phi(\sigma(s))}\Delta s, \tag{4.20}$$

• *for* $0 < p < 1$ *:*

$$p\int_a^b \frac{(H^\sigma(\tau))^p}{\phi^\sigma(\tau)}\Delta\tau \ge \frac{1}{\phi(b)}\int_a^b h^p(s)\left[(b-a)^p-(\sigma(s)-a)^p\right]\Delta s. \tag{4.21}$$

Corollary 4.2. *Let $\mathbb{T}$ be a time scales, $a, b \in \mathbb{T}$ with $a < b$ and suppose that*
$(b-\sigma(s))^\Delta = -1$. *Let h, ϕ be non-negative continuous functions on $[a, b]_\mathbb{T}$ and*
$\widetilde{H}(\tau) = \displaystyle\int_\tau^b h(s)\Delta s$. *If ϕ is decreasing function, then*

• *for* $-\infty < q \le p < 0$ *:*
(4.22)

$$\begin{aligned} -p\int_a^b \frac{(\widetilde{H}^\sigma(\tau))^p}{\phi^\sigma(\tau)}\Delta\tau \quad &\le \left(\int_a^b \frac{1}{\phi(\sigma(s))}\Delta s\right)^{\frac{q-p}{q}} \\ &\times\left(\int_a^b \frac{h^q(s)\left[(b-\sigma(s))^q-(b-a)^q\right]}{\phi(\sigma(s))}\Delta s\right)^{\frac{p}{q}}. \end{aligned}$$

If we put $\mathbb{T} = \mathbb{R}$ in the Theorem 3.4 and the Theorem 3.5, we get the following Corollaries.

Corollary 4.3. *Let $a, b \in \mathbb{R}$ with $a < b$ and h, ϕ be non-negative continuous functions on $[a, b]$, let $H(\tau) = \displaystyle\int_a^\tau h(s)d\,s$. If ϕ is increasing function, then*

• *for* $1 \le p \le q < \infty$ *:*
(4.23)

$$p\int_a^b \frac{(H(\tau))^p}{\phi(\tau)}d\,\tau \le \left(\int_a^b \frac{1}{\phi(s)}d\,\tau\right)^{\frac{q-p}{q}}\left(\int_a^b \frac{h^q(\tau)\left[(b-a)^q-(\tau-a)^q\right]}{\phi(\tau)}d\,\tau\right)^{\frac{p}{q}},$$

• *for* $0 < q \le p < 1$ *:*
(4.24)

$$p\int_a^b \frac{(H(\tau))^p}{\phi(\tau)}d\,\tau \ge \frac{(b-a)^{\frac{q-p}{q}}}{\phi(b)}\left(\int_a^b h^q(\tau)\left[(b-a)^q-(\tau-a)^q\right]d\,\tau\right)^{\frac{p}{q}}.$$

The inequalities (4.23) and (4.24) are the new generalizations of the reverse Hardy inequalities.

Remark 4.4. By taking $q = p$, then the Corollary 4.3 also coincides with the Theorem 2.2 in [1].

Corollary 4.5. *Let $a,\, b \in \mathbb{R}$ with $a < b$ and h, ϕ be non-negative continuous functions on $[a,\, b]$, let $\widetilde{H}(\tau) = \int_\tau^b h(s)\Delta s$. If ϕ is decreasing function, then*
• *for $-\infty < q \leq p < 0$:*
(4.25)

$$-p\int_a^b \frac{(\widetilde{H}(\tau))^p}{\phi(\tau)}d\,\tau \leq \left(\int_a^b \frac{1}{\phi(s)}d\,s\right)^{\frac{q-p}{q}} \left(\int_a^b \frac{h^q(s)\left[(b-s)^q-(b-a)^q\right]}{\phi(s)}d\,s\right)^{\frac{p}{q}}.$$

Remark 4.6. By putting $q = p$ in the Corollary 4.5, we get for $p < 0$:

$$\int_a^b \frac{(\widetilde{H}(\tau))^p}{\phi(\tau)}d\,\tau \leq -\frac{1}{p}\int_a^b \frac{h^p(s)\left[(b-s)^p-(b-a)^p\right]}{\phi(s)}d\,s. \tag{4.26}$$

The inequalities (4.25) and (4.26) are the new inverse Hardy inequalities for negative parameters.
If we put $\mathbb{T} = \mathbb{Z}$ in the Theorem 3.4 and the Theorem 3.5, we get the following Corollary.

Corollary 4.7. *Let $\{u_j\}$, $\{U_j\}$, $\{V_j\}$ and $\{w_j\}$ for $j = 0, 1, 2, ..., n,$ $n \in \mathbb{N}^\star$ be positive sequences of real numbers where $U_j = \sum\limits_{i=0}^{j-1} u_i$ and $V_j = \sum\limits_{i=j}^{n-1} u_i$. If $\{w_j\}$ is increasing, then*
• *for $1 \leq p \leq q < \infty$;*

$$p\sum_{j=0}^{n-1}\frac{U_j^p}{w_j} \leq \left(\sum_{j=0}^{n-1}\frac{1}{w_j}\right)^{\frac{q-p}{q}} \left(\sum_{j=0}^{n-1}\frac{u_j^q\left(n^q-j^q\right)}{w_j}\right)^{\frac{p}{q}}, \tag{4.27}$$

• *for $0 < q \leq p < 1$:*

$$p\sum_{j=0}^{n-1}\frac{U_j^p}{w_j} \geq \frac{n^{\frac{q-p}{q}}}{w_{n-1}} \left(\sum_{j=0}^{n-1}u_j^q\left(n^q-j^q\right)\right)^{\frac{p}{q}}. \tag{4.28}$$

If $\{w_j\}$ is decreasing, then for $-\infty < q \leq p < 0$;

$$-p\sum_{j=0}^{n-1}\frac{V_j^p}{w_j} \leq \left(\sum_{j=0}^{n-1}\frac{1}{w_j}\right)^{\frac{q-p}{q}} \left(\sum_{j=0}^{n-1}\frac{u_j^q\left((n-j)^q-n^q\right)}{w_j}\right)^{\frac{p}{q}}. \tag{4.29}$$

The inequalities (4.27), (4.28) and (4.29) are the new ones of the reverse Hardy inequalities in the discrete form.

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