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SPECTRAL EXPANSION FOR IMPULSIVE DYNAMIC DIRAC SYSTEM ON THE WHOLE LINE

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ABSTRACT. In this study, we consider a impulsive dynamic Dirac system on the whole line. A spectral function of this system is constructed. We establish a Parseval equality and expansion formula in terms of the spectral function.

1. INTRODUCTION

F. V. Atkinson, in his book written in the 1960s [3], states that neither differential equations nor difference equations alone are sufficient for boundary value problems, and he mentioned that it would be beneficial to have these two types of equations in a single theory. Years after this book, in the 1990s, this wish was fulfilled with the concept of time scale. Differential equations and difference equations began to be studied under a single roof. The need to investigate all the problems discussed in the theory of differential equations on the time scale has arisen. For more detailed information on time scales see the excellent book by Bohner and Peterson [4].

Impulsive differential equations are one of the important equations in the theory of differential equations. It is well-known that these equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. For this reason, it is being studied extensively by researchers today ([5, 6, 7, 8, 9, 10, 12, 13, 14, 16]).

On the other hand, spectral expansion theorems play a very important role in solving problems expressed with partial differential equations in mathematics and physics. Especially when solving partial differential equations

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with the Fourier method, such theorems are needed. There are many studies on this subject in the literature (see [1, 2, 3, 5, 11, 15]).

Recently Allahverdiev and Tuna [2] studied the classical Dirac equation under impulsive conditions on the time scale. They investigated the fundamental properties of this type of problem in the finite interval and regular case.

In this paper, a spectral function for impulsive dynamic Dirac systems on the interval $(-\infty,\infty)$ is constructed. Later, we establish a Parseval equality and expansion formula in terms of the spectral function.

2. Main Results

We assume that the reader is familiar with the basic facts of time scales [4]. Let us consider the following impulsive dynamic Dirac system

$$\left\{ \begin{array}{l} -y_2^{\nabla}+p(\zeta)\, y_1=\lambda y_1, \\ y_1^{\Delta}+r(\zeta)\, y_2=\lambda y_2, \end{array} \right. \quad \zeta\in I, \tag{2.1}$$

$$y_2^{\rho}(a,\lambda)\cos\beta+y_1(a,\lambda)\sin\beta=0, \tag{2.2}$$

$$y_1(d-)-k_1y_1(d+)=0, \tag{2.3}$$

$$y_2^{\rho}(d-)-k_2y_2^{\rho}(d+)=0, \tag{2.4}$$

$$y_2^{\rho}(b,\lambda)\cos\gamma+y_1(b,\lambda)\sin\gamma=0, \tag{2.5}$$

where $\mathbb{T}$ be a Sturmian time scale, $k_1,k_2,\beta,\gamma\in\mathbb{R},\ y_2^{\rho}(.)=y_2(\rho(.)),\ -\infty< a<d<b<\infty,\ I_1:=[a,d),\ I_2:=(d,b],\ I:=I_1\cup I_2,\ I\subset\mathbb{T},$ and λ is a complex eigenvalue parameter.

Our basic assumptions throughout the paper are the following:

(A_1) Let $k_1k_2=\alpha>0$.

(A_2) $p,r:I\to\mathbb{R}$ are continuous functions and have finite limits $p(d\pm),\ r(d\pm)$.

A similar problem has been investigated in [1] without impulsive conditions.

Let $H_1=L_{\Delta}^2(I_1)\dot{+}L_{\Delta}^2(I_2)$ be a Hilbert space endowed with the following inner product

$$\langle h,\omega\rangle_{H_1}:=\int_{I_1}(h,\omega)_{\mathbb{C}^2}\,\Delta\zeta+\alpha\int_{I_2}(h,\omega)_{\mathbb{C}^2}\,\Delta\zeta,$$

where

$$h(\zeta)=\begin{pmatrix} h_1(\zeta,\lambda) \\ h_2(\zeta,\lambda)\end{pmatrix},\ h_1(\zeta,\lambda)=\left\{ \begin{array}{ll} h_{11}(\zeta,\lambda), & \zeta\in I_1 \\ h_{12}(\zeta,\lambda), & \zeta\in I_2, \end{array}\right.$$

$$h_2(\zeta,\lambda)=\left\{ \begin{array}{ll} h_{21}(\zeta,\lambda), & \zeta\in I_1 \\ h_{22}(\zeta,\lambda), & \zeta\in I_2, \end{array}\right.$$

and

$$\omega(\zeta)=\left(\begin{array}{c}\omega_1(\zeta,\lambda)\\ \omega_2(\zeta,\lambda)\end{array}\right),\ \omega_1(\zeta,\lambda)=\begin{cases}\omega_{11}(\zeta,\lambda), & \zeta\in I_1\\ \omega_{12}(\zeta,\lambda), & \zeta\in I_2,\end{cases}$$

$$\omega_2(\zeta,\lambda)=\begin{cases}\omega_{21}(\zeta,\lambda), & \zeta\in I_1\\ \omega_{22}(\zeta,\lambda), & \zeta\in I_2.\end{cases}$$

It follows from [2] that there is an orthonormal system $\{\varphi_n\}$ $(n\in\mathbb{Z}:=\{0,\pm1,\pm2,...\})$ of eigenvectors of (2.1)-(2.5) with corresponding nonzero eigenvalues λ_n such that

$$\int_a^d\|\omega(\zeta)\|^2_{\mathbb{C}^2}\,\Delta\zeta+\alpha\int_d^b\|\omega(\zeta)\|^2_{\mathbb{C}^2}\,\Delta\zeta=\sum_{n=-\infty}^{\infty}|a_n|^2, \tag{2.6}$$

which is called the *Parseval equality*, where $\omega\in H_1$, $a_n=\langle\omega,\varphi_n\rangle_{H_1}$, $n\in\mathbb{Z}$.

Denote by

$$\psi_1(\zeta,\lambda)=\left(\begin{array}{c}\psi_{11}(\zeta,\lambda)\\ \psi_{12}(\zeta,\lambda)\end{array}\right),\ \psi_{11}(\zeta,\lambda)=\begin{cases}\psi_{11}^{(1)}(\zeta), & \zeta\in I_3\\ \psi_{11}^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

$$\psi_{12}(\zeta,\lambda)=\begin{cases}\psi_{12}^{(1)}(\zeta), & \zeta\in I_3\\ \psi_{12}^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

(where $I_3:=(-\infty,d)\subset\mathbb{T}$, $I_4:=(d,\infty)\subset\mathbb{T}$) and

$$\psi_2(\zeta,\lambda)=\left(\begin{array}{c}\psi_{21}(\zeta,\lambda)\\ \psi_{22}(\zeta,\lambda)\end{array}\right),\ \psi_{21}(\zeta,\lambda)=\begin{cases}\psi_{21}^{(1)}(\zeta), & \zeta\in I_3\\ \psi_{21}^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

$$\psi_{22}(\zeta,\lambda)=\begin{cases}\psi_{22}^{(1)}(\zeta), & \zeta\in I_3\\ \psi_{22}^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

the solutions of (2.1) $(\zeta\in I_3\cup I_4)$ which satisfy conditions

$$\begin{array}{l}\psi_{11}^{(1)}(c,\lambda)=1,\ \psi_{12}^{(1)}(c,\lambda)=0,\\ \psi_{21}^{(1)}(c,\lambda)=0,\ \psi_{22}^{(1)}(c,\lambda)=1,\ a<c<d.\end{array} \tag{2.7}$$

and impulsive conditions (2.3)-(2.4).

Let λ_n $(n\in\mathbb{Z})$ be the eigenvalues and y_n $(n\in\mathbb{Z})$ be the corresponding eigenfunctions of the self-adjoint problem (2.1)-(2.5), where

$$y_n(\zeta)=\left(\begin{array}{c}y_{n1}(\zeta)\\ y_{n2}(\zeta)\end{array}\right),\ y_{n1}(\zeta)=\begin{cases}y_{n1}^{(1)}(\zeta), & \zeta\in I_1\\ y_{n2}^{(2)}(\zeta), & \zeta\in I_2,\end{cases}$$

$$y_{n2}(\zeta)=\begin{cases}y_{n2}^{(1)}(\zeta), & \zeta\in I_1\\ y_{n2}^{(2)}(\zeta), & \zeta\in I_2.\end{cases}$$

Since the solutions $\psi_1(\zeta,\lambda)$ and $\psi_2(\zeta,\lambda)$ of the system (2.1) are linearly independent, we find

$$y_n(\zeta)=u_n\psi_1(\zeta,\lambda_n)+v_n\psi_2(\zeta,\lambda_n),\ n\in\mathbb{Z}.$$

Without loss of generality, we can assume that $|u_n| \leq 1$ and $|v_n| \leq 1$ $(n \in \mathbb{Z})$.
Write

$$\alpha_n^2 = \int_a^d \|y_n(\zeta)\|_{\mathbb{C}^2}^2 \Delta\zeta + \alpha \int_d^b \|y_n(\zeta)\|_{\mathbb{C}^2}^2 \Delta\zeta, \ n \in \mathbb{Z}.$$

Let

$$\omega(.) = \begin{pmatrix} \omega_1(.) \\ \omega_2(.) \end{pmatrix} \in H_1,$$

is a real vector-valued function, where

$$\omega_1(\zeta) = \begin{cases} \omega_1^{(1)}(\zeta), & \zeta \in I_1 \\ \omega_1^{(2)}(\zeta), & \zeta \in I_2, \end{cases} \quad \omega_2(\zeta) = \begin{cases} \omega_2^{(1)}(\zeta), & \zeta \in I_1 \\ \omega_2^{(2)}(\zeta), & \zeta \in I_2. \end{cases}$$

By (2.6), we see that

$$\begin{aligned} &\int_a^d \|\omega(\zeta)\|_{\mathbb{C}^2}^2 \Delta\zeta + \alpha \int_d^b \|\omega(\zeta)\|_{\mathbb{C}^2}^2 \Delta\zeta \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n^2} \left\{ \int_a^d (\omega(\zeta), y_n(\zeta))_{\mathbb{C}^2} \Delta\zeta + \alpha \int_d^b (\omega(\zeta), y_n(\zeta))_{\mathbb{C}^2} \Delta\zeta \right\}^2 \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n^2} \left\{ \begin{array}{c} \int_a^d (\omega(\zeta), u_n\psi_1(\zeta,\lambda_n) + v_n\psi_2(\zeta,\lambda_n))_{\mathbb{C}^2} \Delta\zeta \\ +\alpha \int_d^b (\omega(\zeta), u_n\psi_1(\zeta,\lambda_n) + v_n\psi_2(\zeta,\lambda_n))_{\mathbb{C}^2} \Delta\zeta \end{array} \right\}^2 \\ &= \sum_{n=-\infty}^{\infty} \frac{u_n^2}{\alpha_n^2} \left\{ \begin{array}{c} \int_a^d (\omega(\zeta), \psi_1(\zeta,\lambda_n))_{\mathbb{C}^2} \Delta\zeta \\ +\alpha \int_d^b (\omega(\zeta), \psi_1(\zeta,\lambda_n))_{\mathbb{C}^2} \Delta\zeta \end{array} \right\}^2 \\ &+2 \sum_{n=-\infty}^{\infty} \frac{u_n v_n}{\alpha_n^2} \left\{ \begin{array}{c} \int_a^d (\omega(\zeta), \psi_1(\zeta,\lambda_n))_{\mathbb{C}^2} \Delta\zeta \\ +\alpha \int_d^b (\omega(\zeta), \psi_1(\zeta,\lambda_n))_{\mathbb{C}^2} \Delta\zeta \end{array} \right\} \\ &\times \left\{ \int_a^d (\omega(\zeta), \psi_2(\zeta,\lambda_n))_{\mathbb{C}^2} \Delta\zeta + \alpha \int_d^b (\omega(\zeta), \psi_2(\zeta,\lambda_n))_{\mathbb{C}^2} \Delta\zeta \right\} \\ &+ \sum_{n=-\infty}^{\infty} \frac{v_n^2}{\alpha_n^2} \left\{ \begin{array}{c} \int_a^d (\omega(\zeta), \psi_2(\zeta,\lambda_n))_{\mathbb{C}^2} \Delta\zeta \\ +\alpha \int_d^b (\omega(\zeta), \psi_2(\zeta,\lambda_n))_{\mathbb{C}^2} \Delta\zeta \end{array} \right\}^2. \end{aligned} \tag{2.8}$$

The step function $\mu_{ij,[a,b]}$ $(i,j=1,2)$ on $\mathbb{R}$ is defined by

$$\mu_{11,[a,b]}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} \frac{u_n^2}{\alpha_n^2}, & \text{for } \lambda \leq 0 \\ \sum\limits_{0\leq\lambda_n<\lambda} \frac{u_n^2}{\alpha_n^2}, & \text{for } \lambda > 0, \end{cases}$$

$$\mu_{12,[a,b]}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} \frac{u_n v_n}{\alpha_n^2}, & \text{for } \lambda \le 0 \\ \sum\limits_{0\le\lambda_n<\lambda} \frac{u_n v_n}{\alpha_n^2}, & \text{for } \lambda > 0, \end{cases}$$

$$\mu_{12,[a,b]}(\lambda) = \mu_{21,[a,b]}(\lambda),$$

$$\mu_{22,[a,b]}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} \frac{u_n^2}{\alpha_n^2}, & \text{for } \lambda \le 0 \\ \sum\limits_{0\le\lambda_n<\lambda} \frac{u_n^2}{\alpha_n^2}, & \text{for } \lambda > 0. \end{cases}$$

From (2.8), we deduce that

$$\int_a^d \|\omega(\zeta)\|_{\mathbb{C}^2}^2 \Delta\zeta + \alpha \int_d^b \|\omega(\zeta)\|_{\mathbb{C}^2}^2 \Delta\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \Omega_i(\lambda)\,\Omega_j(\lambda)\, d\mu_{ij,[a,b]}(\lambda), \tag{2.9}$$

where

$$\Omega_1(\lambda) = \int_a^d (\omega(\zeta), \psi_1(\zeta,\lambda))_{\mathbb{C}^2} \Delta\zeta + \alpha \int_d^b (\omega(\zeta), \psi_1(\zeta,\lambda))_{\mathbb{C}^2} \Delta\zeta,$$

and

$$\Omega_2(\lambda) = \int_a^d (\omega(\zeta), \psi_2(\zeta,\lambda))_{\mathbb{C}^2} \Delta\zeta + \alpha \int_d^b (\omega(\zeta), \psi_2(\zeta,\lambda))_{\mathbb{C}^2} \Delta\zeta.$$

LEMMA 2.1. *For any positive ξ, there is a positive constant $\Lambda = \Lambda(\xi)$ not depending on b such that*

$$\bigvee_{-\xi}^{\xi} \left\{\mu_{ij,[a,b]}(\lambda)\right\} < \Lambda \;\; (i,j=1,2). \tag{2.10}$$

PROOF. By (2.7), we infer that $\psi_{ij}^{(1)}(d_0,\lambda) = \delta_{ij}$, where δ_{ij} $(i,j=1,2)$ is the Kronecker delta. It is clear that $\psi_{ij}^{(1)}(\zeta,\lambda)$ $(i,j=1,2)$ are continuous both with respect to $\zeta \in [a,d)$ and $\lambda \in \mathbb{R}$. Then for every $\varepsilon > 0$ there is a $d_0 < k < d$ such that

$$\left|\psi_{ij}^{(1)}(\zeta,\lambda) - \delta_{ij}\right| < \varepsilon,\; |\lambda| < \xi,\; \text{where } \zeta \in [d_0,k]. \tag{2.11}$$

Let

$$\omega_k(\zeta) = \begin{pmatrix} \omega_{k1}(\zeta) \\ \omega_{k2}(\zeta) \end{pmatrix}$$

be a nonnegative vector-valued function such that $\omega_{k1}(\zeta)$ vanishes outside the interval $[d_0,k]$ with

$$\int_{d_0}^{k}\omega_{k1}(\zeta)\,\Delta\zeta=1, \tag{2.12}$$

and $\omega_{k2}(\zeta)=0$. Let

$$\omega_{ik}(\lambda)=\int_{d_0}^{k}(\omega_k(\zeta),\psi_i)_{\mathbb{C}^2}\,\Delta\zeta$$

$$=\int_{d_0}^{k}\omega_{k1}(\zeta)\,\psi_{i1}^{(1)}(\zeta,\lambda)\,\Delta\zeta,$$

where $i=1,2$. By virtue of (2.11) and (2.12), we conclude that

$$|\omega_{1k}(\lambda)-1|<\varepsilon,\ |\omega_{2k}(\lambda)|<\varepsilon,\ \text{and}\ |\lambda|<\xi. \tag{2.13}$$

It follows from (2.9) that

$$\int_{d_0}^{k}\omega_{k1}^2(\zeta)\,\Delta\zeta\geq\int_{-\xi}^{\xi}\Omega_{1k}^2(\lambda)\,d\mu_{11,[a,b]}(\lambda)$$
$$+2\int_{-\xi}^{\xi}\Omega_{1k}(\lambda)\,\Omega_{2k}(\lambda)\,d\mu_{12,[a,b]}(\lambda)$$
$$+\int_{-\xi}^{\xi}\Omega_{2k}^2(\lambda)\,d\mu_{22,[a,b]}(\lambda)\geq\int_{-\xi}^{\xi}\Omega_{1k}^2(\lambda)\,d\mu_{11,[a,b]}(\lambda)$$
$$-2\int_{-\xi}^{\xi}|\Omega_{1k}(\lambda)|\,|\Omega_{2k}(\lambda)|\left|d\mu_{12,[a,b]}(\lambda)\right|.$$

By (2.13), we obtain

$$\int_{d_0}^{k}\omega_{k1}^2(\zeta)\,\Delta\zeta>\int_{-\xi}^{\xi}(1-\varepsilon)^2\,d\mu_{11,[a,b]}(\lambda)$$
$$-2\int_{-\xi}^{\xi}\varepsilon(1+\varepsilon)\left|d\mu_{12,[a,b]}(\lambda)\right|$$
$$=(1-\varepsilon)^2\left(\mu_{11,[a,b]}(\xi)-\mu_{11,[a,b]}(-\xi)\right)$$
$$-2\varepsilon(1+\varepsilon)\mathop{V}_{-\xi}^{\xi}\left\{\mu_{12,[a,b]}(\lambda)\right\}.$$

Hence

$$\int_{d_0}^{k}\omega_{k1}^2(\zeta)\,\Delta\zeta>(1-3\varepsilon)\left\{\mu_{11,[a,b]}(\xi)-\mu_{11,[a,b]}(-\xi)\right\}$$
$$-\varepsilon(1+\varepsilon)\left\{\mu_{22,[a,b]}(\xi)-\mu_{22,[a,b]}(-\xi)\right\}. \tag{2.14}$$

due to

$$\bigvee_{-\xi}^{\xi}\left\{\mu_{12,[a,b]}\left(\lambda\right)\right\}\leq\frac{1}{2}\left[\begin{array}{c}\mu_{11,[a,b]}\left(\xi\right)-\mu_{11,[a,b]}\left(-\xi\right)\\+\mu_{22,[a,b]}\left(\xi\right)-\mu_{22,[a,b]}\left(-\xi\right)\end{array}\right]. \tag{2.15}$$

Let

$$\chi_k\left(\zeta\right)=\left(\begin{array}{c}\chi_{k1}\left(\zeta\right)\\\chi_{k2}\left(\zeta\right)\end{array}\right)$$

be a nonnegative vector-valued function such that $\chi_{k2}\left(\zeta\right)$ vanishes outside the interval $[d_0,k]$ with $\int_{d_0}^{k}\chi_{k2}\left(\zeta\right)\Delta\zeta=1$, and $\chi_{k1}\left(\zeta\right)=0$. Similarly, we get

$$\begin{aligned}\int_{d_0}^{k}\chi_{k2}^2\left(\zeta\right)\Delta\zeta &> \left(1-3\varepsilon\right)\left\{\mu_{22,[a,b]}\left(\xi\right)-\mu_{22,[a,b]}\left(-\xi\right)\right\}\\&-\varepsilon\left(1+\varepsilon\right)\left\{\mu_{11,[a,b]}\left(\xi\right)-\mu_{11,[a,b]}\left(-\xi\right)\right\}.\end{aligned} \tag{2.16}$$

From (2.14) and (2.16), we find

$$\begin{gathered}\int_{d_0}^{k}\left\{\omega_{k1}^2\left(\zeta\right)+\chi_{k2}^2\left(\zeta\right)\right\}\Delta\zeta\\>\left(1-4\varepsilon-\varepsilon^2\right)\left\{\begin{array}{c}\mu_{11,[a,b]}\left(\xi\right)-\mu_{11,[a,b]}\left(-\xi\right)\\+\mu_{22,[a,b]}\left(\xi\right)-\mu_{22,[a,b]}\left(-\xi\right)\end{array}\right\}.\end{gathered}$$

If the number $\varepsilon>0$ is selected such that $1-4\varepsilon-\varepsilon^2>0$, then the statement follows for the functions $\mu_{11,[a,b]}\left(-\xi\right)$ and $\mu_{22,[a,b]}\left(-\xi\right)$, relying on their monotonicity. For the function $\mu_{12,[a,b]}\left(-\xi\right)$, it follows from the Cauchy–Schwarz inequality. □

Now let's define the following spaces.

$H:=L^2\left(I_3;\mathbb{C}^2\right)\dot{+}L^2\left(I_4;\mathbb{C}^2\right)$, be a Hilbert space endowed with the following inner product

$$\langle\omega,\chi\rangle_H:=\int_{I_3}\left(\omega\left(\zeta\right),\chi\left(\zeta\right)\right)_{\mathbb{C}^2}\Delta\zeta+\alpha\int_{I_4}\left(\omega\left(\zeta\right),\chi\left(\zeta\right)\right)_{\mathbb{C}^2}\Delta\zeta,$$

where $I_3=(-\infty,d)\subset\mathbb{T},\ I_4=(d,\infty)\subset\mathbb{T}$,

$$\omega\left(\zeta\right)=\left(\begin{array}{c}\omega_1\left(\zeta\right)\\\omega_2\left(\zeta\right)\end{array}\right),\ \chi\left(\zeta\right)=\left(\begin{array}{c}\chi_1\left(\zeta\right)\\\chi_2\left(\zeta\right)\end{array}\right),$$

$$\omega_1(\zeta)=\left\{\begin{array}{ll}\omega_1^{(1)}(\zeta), & \zeta\in I_3\\\omega_1^{(2)}(\zeta), & \zeta\in I_4,\end{array}\right.\quad\omega_2(\zeta)=\left\{\begin{array}{ll}\omega_2^{(1)}(\zeta), & \zeta\in I_3\\\omega_2^{(2)}(\zeta), & \zeta\in I_4,\end{array}\right.$$

$$\chi_1(\zeta)=\left\{\begin{array}{ll}\chi_1^{(1)}(\zeta), & \zeta\in I_3\\\chi_1^{(2)}(\zeta), & \zeta\in I_4,\end{array}\right.\quad\chi_2(\zeta)=\left\{\begin{array}{ll}\chi_2^{(1)}(\zeta), & \zeta\in I_3\\\chi_2^{(2)}(\zeta), & \zeta\in I_4.\end{array}\right.$$

Let ϱ be any non-decreasing function on $-\infty < \lambda < \infty$. $L_{\varrho}^{2}(\mathbb{R})$ be a Hilbert space of all functions $\omega : \mathbb{R} \to \mathbb{R}$ which are measurable with respect to the Lebesque–Stieltjes measure defined by ϱ and such that

$$\int_{-\infty}^{\infty} \omega^{2}(\lambda)\, d\varrho(\lambda) < \infty,$$

with the inner product

$$(\omega, \chi)_{\varrho} := \int_{-\infty}^{\infty} \omega(\lambda)\, \chi(\lambda)\, d\varrho(\lambda).$$

THEOREM 2.2. *Let ω is a real vector-valued function and $\omega \in H$. There are two monotone functions $\mu_{11}(\lambda)$ and $\mu_{22}(\lambda)$, and a function $\mu_{12}(\lambda)$ with variation bounded in each finite interval, none of which depends ω, and such that the following Parseval equality holds*

$$\int_{I_3} \|\omega(\zeta)\|_{\mathbb{C}^2}^{2} \Delta\zeta + \alpha \int_{I_4} \|\omega(\zeta)\|_{\mathbb{C}^2}^{2} \Delta\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \Omega_i(\lambda)\, \Omega_j(\lambda)\, d\mu_{ij}(\lambda), \tag{2.17}$$

where

$$\Omega_i(\lambda) = \lim_{n\to\infty} \left\{ \begin{array}{c} \int_{-n}^{d} (\omega(\zeta), \psi_i(\zeta,\lambda))_{\mathbb{C}^2} \Delta\zeta \\ \\ +\alpha \int_{d}^{n} (\omega(\zeta), \psi_i(\zeta,\lambda))_{\mathbb{C}^2} \Delta\zeta \end{array} \right\} \ (i = 1, 2).$$

We note that the matrix-valued function $\mu = (\mu_{ij})_{i,j=1}^{2}$ $(\mu_{12} = \mu_{21})$ is called a *spectral function* for the system (2.1), (2.3), (2.4).

PROOF. Let

$$\omega_m(\zeta) = \begin{pmatrix} \omega_{1m}(\zeta) \\ \omega_{2m}(\zeta) \end{pmatrix},$$

where

$$\omega_{1m}(\zeta) = \begin{cases} \omega_{1m}^{(1)}(\zeta), & \zeta \in I_3 \\ \omega_{1m}^{(2)}(\zeta), & \zeta \in I_4, \end{cases} \quad \omega_{2m}(\zeta) = \begin{cases} \omega_{2m}^{(1)}(\zeta), & \zeta \in I_3 \\ \omega_{2m}^{(2)}(\zeta), & \zeta \in I_4, \end{cases}$$

satisfies the following conditions:

1) $\omega_m(\zeta)$ vanishes outside the interval $[-m, d) \cup (d, m]$, where $a < -m < d < m < b$.

2) The real vector-valued functions $\omega_m(\zeta)$ and $\omega_m^{\Delta}(\zeta)$ are continuous.

3) $\omega_m(\zeta)$ satisfies conditions (2.2)-(2.5).

By (2.6), we have

$$\int_{m}^{d} \|\omega_m(\zeta)\|_{\mathbb{C}^2}^{2} \Delta\zeta + \alpha \int_{d}^{m} \|\omega_m(\zeta)\|_{\mathbb{C}^2}^{2} \Delta\zeta$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{\alpha_k^2} \left\{ \begin{array}{c} \int_a^d \left(\omega_m\left(\zeta\right), y_k\left(\zeta\right)\right)_{\mathbb{C}^2} \Delta\zeta \\ +\alpha \int_d^b \left(\omega_m\left(\zeta\right), y_k\left(\zeta\right)\right)_{\mathbb{C}^2} \Delta\zeta \end{array} \right\}^2. \tag{2.18}$$

Δ-integrating by parts gives

$$\int_a^d \left(\omega_m\left(\zeta\right), y_k\left(\zeta\right)\right)_{\mathbb{C}^2} \Delta\zeta + \alpha \int_d^b \left(\omega_m\left(\zeta\right), y_k\left(\zeta\right)\right)_{\mathbb{C}^2} \Delta\zeta$$

$$= \frac{1}{\lambda_k} \int_a^d \omega_{1m}^{(1)}\left(\zeta\right) \left[-y_{k2}^{(1)^{\nabla}}\left(\zeta\right) + p\left(\zeta\right) y_{k1}^{(1)}\left(\zeta\right)\right] \Delta\zeta$$

$$+\frac{1}{\lambda_k} \alpha \int_d^b \omega_{1m}^{(2)}\left(\zeta\right) \left[-y_{k2}^{(2)^{\nabla}}\left(\zeta\right) + p\left(\zeta\right) y_{k1}^{(2)}\left(\zeta\right)\right] \Delta\zeta$$

$$+\frac{1}{\lambda_k} \int_a^d \omega_{2m}^{(1)}\left(\zeta\right) \left[y_{k1}^{(1)^{\Delta}}\left(\zeta\right) + r\left(\zeta\right) y_{k2}^{(1)}\left(\zeta\right)\right] \Delta\zeta$$

$$+\frac{1}{\lambda_k} \alpha \int_d^b \omega_{2m}^{(2)}\left(\zeta\right) \left[y_{k1}^{(2)^{\Delta}}\left(\zeta\right) + r\left(\zeta\right) y_{k2}^{(2)}\left(\zeta\right)\right] \Delta\zeta$$

$$= \frac{1}{\lambda_k} \int_a^d \left[-\omega_{2m}^{(1)^{\nabla}}\left(\zeta\right) + p\left(\zeta\right) \omega_{1m}^{(1)}\left(\zeta\right)\right] y_{k1}^{(1)}\left(\zeta\right) \Delta\zeta$$

$$+\frac{1}{\lambda_k} \alpha \int_d^b \left[-\omega_{2m}^{(2)^{\nabla}}\left(\zeta\right) + p\left(\zeta\right) \omega_{1m}^{(2)}\left(\zeta\right)\right] y_{k1}^{(2)}\left(\zeta\right) \Delta\zeta$$

$$+\frac{1}{\lambda_k} \int_a^d \left[\omega_{1m}^{(1)^{\Delta}}\left(\zeta\right) + r\left(\zeta\right) \omega_{2m}^{(1)}\left(\zeta\right)\right] y_{k2}^{(1)}\left(\zeta\right) \Delta\zeta$$

$$+\frac{1}{\lambda_k} \alpha \int_d^b \left[\omega_{1m}^{(2)^{\Delta}}\left(\zeta\right) + r\left(\zeta\right) \omega_{2m}^{(2)}\left(\zeta\right)\right] y_{k2}^{(2)}\left(\zeta\right) \Delta\zeta.$$

It follows from condition 1 that

$$\sum_{|\lambda_k| \geq s} \frac{1}{\alpha_k^2} \left\{ \begin{array}{c} \int_m^d \left(\omega_m\left(\zeta\right), y_k\left(\zeta\right)\right)_{\mathbb{C}^2} \Delta\zeta \\ +\alpha \int_d^m \left(\omega_m\left(\zeta\right), y_k\left(\zeta\right)\right)_{\mathbb{C}^2} \Delta\zeta \end{array} \right\}^2$$

$$\leq \frac{1}{s^2} \sum_{|\lambda_k| \geq s} \frac{1}{\alpha_k^2} \left\{ \begin{array}{c} \int_m^d \left[-\omega_{2m}^{(1)^{\nabla}}\left(\zeta\right) + p\left(\zeta\right) \omega_{1m}^{(1)}\left(\zeta\right)\right] y_{k1}^{(1)}\left(\zeta\right) \Delta\zeta \\ +\alpha \int_d^m \left[-\omega_{2m}^{(2)^{\nabla}}\left(\zeta\right) + p\left(\zeta\right) \omega_{1m}^{(2)}\left(\zeta\right)\right] y_{k1}^{(2)}\left(\zeta\right) \Delta\zeta \\ + \int_m^d \left[\omega_{1m}^{(1)^{\Delta}}\left(\zeta\right) + r\left(\zeta\right) \omega_{2m}^{(1)}\left(\zeta\right)\right] y_{k2}^{(1)}\left(\zeta\right) \Delta\zeta \\ +\alpha \int_d^m \left[\omega_{1m}^{(2)^{\Delta}}\left(\zeta\right) + r\left(\zeta\right) \omega_{2m}^{(2)}\left(\zeta\right)\right] y_{k2}^{(2)}\left(\zeta\right) \Delta\zeta \end{array} \right\}^2$$

$$\leq \frac{1}{s^2}\sum_{k=-\infty}^{\infty}\frac{1}{\alpha_k^2}\left\{\begin{array}{c}\int_{-m}^{d}\left[-\omega_{2m}^{(1)^{\nabla}}(\zeta)+p(\zeta)\,\omega_{1m}^{(1)}(\zeta)\right]y_{k1}^{(1)}(\zeta)\,\Delta\zeta\\ +\alpha\int_{d}^{m}\left[-\omega_{2m}^{(2)^{\nabla}}(\zeta)+p(\zeta)\,\omega_{1m}^{(2)}(\zeta)\right]y_{k1}^{(2)}(\zeta)\,\Delta\zeta\\ +\int_{-m}^{d}\left[\omega_{1m}^{(1)^{\Delta}}(\zeta)+r(\zeta)\,\omega_{2m}^{(1)}(\zeta)\right]y_{k2}^{(1)}(\zeta)\,\Delta\zeta\\ +\alpha\int_{d}^{m}\left[\omega_{1m}^{(2)^{\Delta}}(\zeta)+r(\zeta)\,\omega_{2m}^{(2)}(\zeta)\right]y_{k2}^{(2)}(\zeta)\,\Delta\zeta\end{array}\right\}^2$$

$$=\frac{1}{s^2}\left\{\begin{array}{c}\int_{-m}^{d}\left[-\omega_{2m}^{(1)^{\nabla}}(\zeta)+p(\zeta)\,\omega_{1m}^{(1)}(\zeta)\right]^2\Delta\zeta\\ +\alpha\int_{d}^{m}\left[-\omega_{2m}^{(2)^{\nabla}}(\zeta)+p(\zeta)\,\omega_{1m}^{(2)}(\zeta)\right]^2\Delta\zeta\\ +\int_{-m}^{d}\left[\omega_{1m}^{(1)^{\Delta}}(\zeta)+r(\zeta)\,\omega_{2m}^{(1)}(\zeta)\right]^2\Delta\zeta\\ +\alpha\int_{d}^{m}\left[\omega_{1m}^{(2)^{\Delta}}(\zeta)+r(\zeta)\,\omega_{2m}^{(2)}(\zeta)\right]^2\Delta\zeta\end{array}\right\}.$$

By (2.18), we find

$$\left|\begin{array}{c}\int_{-m}^{d}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^2\,\Delta\zeta+\alpha\int_{d}^{m}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^2\,\Delta\zeta\\ -\sum_{-N\leq\lambda_k\leq N}\frac{1}{\alpha_k^2}\left\{\langle\omega_m(.),y_k(.)\rangle_H\right\}^2\end{array}\right|$$

$$\leq\sum_{-N\leq\lambda_k\leq N}\frac{1}{\alpha_k^2}\left\{\langle\omega_m(.),(u_k\psi_1(.,\lambda_k)+v_k\psi_2(.,\lambda_k))\rangle_H\right\}^2$$

$$=\int_{-N}^{N}\sum_{i,j=1}^{2}\Omega_{im}(\lambda)\,\Omega_{jm}(\lambda)\,d\mu_{ij,[a,b]}(\lambda),$$

where

$$\Omega_{im}(\lambda)=\langle\omega_m(.),\psi_i(.,\lambda)\rangle_H\ \ (i=1,2).$$

Therefore, we obtain

$$\left|\begin{array}{c}\int_{-m}^{d}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^2\,\Delta\zeta+\alpha\int_{d}^{m}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^2\,\Delta\zeta\\ -\int_{-N}^{N}\sum_{i,j=1}^{2}\Omega_{im}(\lambda)\,\Omega_{im}(\lambda)\,d\mu_{ij,[a,b]}(\lambda)\end{array}\right|$$

$$\le \frac{1}{N^2}\int_{-m}^{d}\left[-\omega_{2m}^{(1)^{\nabla}}(\zeta)+p(\zeta)\,\omega_{1m}^{(1)}(\zeta)\right]^2\Delta\zeta$$
$$+\alpha\frac{1}{N^2}\int_{d}^{m}\left[-\omega_{2m}^{(2)^{\nabla}}(\zeta)+p(\zeta)\,\omega_{1m}^{(2)}(\zeta)\right]^2\Delta\zeta$$
$$+\frac{1}{N^2}\int_{-m}^{d}\left[\omega_{1m}^{(1)^{\Delta}}(\zeta)+r(\zeta)\,\omega_{2j}^{(1)}(\zeta)\right]^2\Delta\zeta$$
$$+\frac{1}{N^2}\alpha\int_{d}^{m}\left[\omega_{1m}^{(2)^{\Delta}}(\zeta)+r(\zeta)\,\omega_{2m}^{(2)}(\zeta)\right]^2\Delta\zeta. \tag{2.19}$$

By Helly's theorems and Lemma 2.1, we can find sequences $\{a_k\}$ and $\{b_k\}$ such that the functions $\mu_{ij,[a_k,b_k]}(\lambda)$ converge ($a_k\to-\infty$, $b_k\to\infty$) to a function $\mu_{ij}(\lambda)$ $(i,j=1,2)$. It follows from (2.19) that

$$\left|\begin{array}{c}\int_{-m}^{d}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^2\Delta\zeta+\alpha\int_{d}^{m}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^2\Delta\zeta\\ -\int_{-N}^{N}\sum_{i,j=1}^{2}\Omega_{im}(\lambda)\,\Omega_{jm}(\lambda)\,d\mu_{ij}(\lambda)\end{array}\right|$$
$$\le \frac{1}{N^2}\int_{-m}^{d}\left[-\omega_{2m}^{(1)^{\nabla}}(\zeta)+p(\zeta)\,\omega_{1m}^{(1)}(\zeta)\right]^2\Delta\zeta$$
$$+\alpha\frac{1}{N^2}\int_{d}^{m}\left[-\omega_{2m}^{(2)^{\nabla}}(\zeta)+p(\zeta)\,\omega_{1m}^{(2)}(\zeta)\right]^2\Delta\zeta$$
$$+\frac{1}{N^2}\int_{-m}^{d}\left[\omega_{1m}^{(1)^{\Delta}}(\zeta)+r(\zeta)\,\omega_{2m}^{(1)}(\zeta)\right]^2\Delta\zeta$$
$$+\frac{1}{N^2}\alpha\int_{d}^{m}\left[\omega_{1m}^{(2)^{\Delta}}(\zeta)+r(\zeta)\,\omega_{2m}^{(2)}(\zeta)\right]^2\Delta\zeta.$$

As $N\to\infty$, we see that

$$\int_{-m}^{d}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^2\Delta\zeta+\alpha\int_{d}^{m}\|\omega_m(\zeta)\|_{\mathbb{C}^2}^2\Delta\zeta$$
$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_{im}(\lambda)\,\Omega_{jm}(\lambda)\,d\mu_{ij}(\lambda).$$

Now let

$$\omega(\zeta)=\begin{cases}\omega^{(1)}(\zeta), & \zeta\in I_3\\ \omega^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

is a real vector-value function and $\omega\in H$. Choose vector-valued functions

$$\omega_\eta(\zeta)=\begin{cases}\omega_\eta^{(1)}(\zeta), & \zeta\in I_3\\ \omega_\eta^{(2)}(\zeta), & \zeta\in I_4,\end{cases}$$

satisfying conditions 1-3 and such that

$$\lim_{\eta\to\infty}\int_{-\infty}^{d}\left\|\omega^{(1)}(\zeta)-\omega_\eta^{(1)}(\zeta)\right\|_{\mathbb{C}^2}^2\Delta\zeta$$

$$+\alpha \lim_{\eta\to\infty} \int_{d}^{\infty} \left\| \omega^{(2)}(\zeta) - \omega_{\eta}^{(2)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta = 0.$$

Let

$$\Omega_{i\eta}(\lambda) = \int_{-\infty}^{d} \left(\omega_{\eta}^{(1)}(\zeta), \psi_i(\zeta,\lambda) \right)_{\mathbb{C}^2} \Delta\zeta$$

$$+\alpha \int_{d}^{\infty} \left(\omega_{\eta}^{(2)}(\zeta), \psi_i(\zeta,\lambda) \right)_{\mathbb{C}^2} \Delta\zeta \quad (i=1,2).$$

Then, we find

$$\int_{-\infty}^{d} \left\| \omega_{\eta}^{(1)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta + \alpha \int_{d}^{\infty} \left\| \omega_{\eta}^{(2)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \Omega_{i\eta}(\lambda) \Omega_{j\eta}(\lambda) d\mu_{ij}(\lambda).$$

Since

$$\int_{-\infty}^{d} \left\| \omega_{\eta_1}^{(1)}(\zeta) - \omega_{\eta_2}^{(1)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta + \alpha \int_{d}^{\infty} \left\| \omega_{\eta_1}^{(2)}(\zeta) - \omega_{\eta_2}^{(2)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta \to 0$$

as $\eta_1, \eta_2 \to \infty$, we conclude that

$$\int_{-\infty}^{\infty} \sum_{i=1}^{2} \left(\Omega_{i\eta_1}(\lambda) \Omega_{j\eta_1}(\lambda) - \Omega_{i\eta_2}(\lambda) \Omega_{j\eta_2}(\lambda) \right) d\mu_{ij}(\lambda)$$

$$= \int_{-\infty}^{d} \left\| \omega_{\eta_1}^{(1)}(\zeta) - \omega_{\eta_2}^{(1)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta + \alpha \int_{d}^{\infty} \left\| \omega_{\eta_1}^{(2)}(\zeta) - \omega_{\eta_2}^{(2)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta \to 0$$

as $\eta_1, \eta_2 \to \infty$. Therefore, there is a limit function Ω_i $(i=1,2)$ which satisfies

$$\int_{-\infty}^{d} \left\| \omega^{(1)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta + \alpha \int_{d}^{\infty} \left\| \omega^{(2)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \Omega_i(\lambda) \Omega_j(\lambda) d\mu_{ij}(\lambda),$$

by the completeness of the space $L_{\mu}^{2}(\mathbb{R})$.

Now, we shall prove that the sequence

$$K_{\eta i}(\lambda) = \int_{-\eta}^{d} (\omega^{(1)}(\zeta), \psi_i(\zeta,\lambda))_{\mathbb{C}^2} \Delta\zeta$$

$$+ \int_{d}^{\eta} (\omega^{(2)}(\zeta), \psi_i(\zeta,\lambda))_{\mathbb{C}^2} \Delta\zeta \ (i=1,2)$$

converges as $\eta \to \infty$ to Ω_i $(i=1,2)$ in $L_{\mu}^{2}(\mathbb{R})$. Let χ be another function in H. $\Sigma(\lambda)$ can be defined by χ. It is obvious that

$$\int_{-\infty}^{d} \left\| \omega^{(1)}(\zeta) - \chi^{(1)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta + \alpha \int_{d}^{\infty} \left\| \omega^{(2)}(\zeta) - \chi^{(2)}(\zeta) \right\|_{\mathbb{C}^2}^{2} \Delta\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left\{ \left(\Omega_i(\lambda) - \Sigma_i(\lambda) \right) \left(\Omega_j(\lambda) - \Sigma_j(\lambda) \right) \right\} d\mu_{ij}(\lambda).$$

Let

$$\chi(\zeta) = \begin{cases} \omega(\zeta), & \zeta \in [-\eta, d) \cup (d, \eta] \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we obtain

$$\int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left\{ \left(\Omega_i(\lambda) - K_{\eta i}(\lambda) \right) \left(\Omega_j(\lambda) - K_{\eta j}(\lambda) \right) \right\} d\mu_{ij}(\lambda)$$

$$= \int_{-\infty}^{-\eta} \left\| \omega^{(1)}(\zeta) \right\|_{\mathbb{C}^2}^2 \Delta\zeta + \alpha \int_{\eta}^{\infty} \left\| \omega^{(2)}(\zeta) \right\|_{\mathbb{C}^2}^2 \Delta\zeta \to 0 \ (\eta \to \infty),$$

which proves that $(K_{\eta i})$ converges to Ω_i $(i = 1, 2)$ in $L_\mu^2(\mathbb{R})$ as $\eta \to \infty$. ◻

THEOREM 2.3. *Suppose that the real vector-valued functions*

$$\omega(\zeta) = \begin{cases} \omega^{(1)}(\zeta), & \zeta \in I_3 \\ \omega^{(2)}(\zeta), & \zeta \in I_4, \end{cases} \quad \chi(\zeta) = \begin{cases} \chi^{(1)}(\zeta), & \zeta \in I_3 \\ \chi^{(2)}(\zeta), & \zeta \in I_4, \end{cases}$$

$\omega, \chi \in H$, *and* $\Omega_i(\lambda)$, $\Sigma_i(\lambda)$ $(i = 1, 2)$ *are their Fourier transforms. Then, the following generalized Parseval equality holds*

$$\int_{-\infty}^{d} \left(\omega^{(1)}(\zeta), \chi^{(1)}(\zeta) \right)_{\mathbb{C}^2} \Delta\zeta + \alpha \int_{d}^{\infty} \left(\omega^{(2)}(\zeta), \chi^{(2)}(\zeta) \right)_{\mathbb{C}^2} \Delta\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \Omega_i(\lambda) \Sigma_j(\lambda) d\mu_{ij}(\lambda).$$

PROOF. Since $\Omega \mp \Sigma$ are transforms of $\omega \mp \chi$, we see that

$$\int_{-\infty}^{d} \left\| \omega^{(1)}(\zeta) + \chi^{(1)}(\zeta) \right\|_{\mathbb{C}^2}^2 \Delta\zeta + \alpha \int_{d}^{\infty} \left\| \omega^{(2)}(\zeta) + \chi^{(2)}(\zeta) \right\|_{\mathbb{C}^2}^2 \Delta\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left(\Omega_i(\lambda) + \Sigma_i(\lambda) \right) \left(\Omega_j(\lambda) + \Sigma_j(\lambda) \right) d\mu_{ij}(\lambda) \tag{2.20}$$

and

$$\int_{-\infty}^{d} \left\| \omega^{(1)}(\zeta) - \chi^{(1)}(\zeta) \right\|_{\mathbb{C}^2}^2 \Delta\zeta + \alpha \int_{d}^{\infty} \left\| \omega^{(2)}(\zeta) - \chi^{(2)}(\zeta) \right\|_{\mathbb{C}^2}^2 \Delta\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left(\Omega_i(\lambda) - \Sigma_i(\lambda) \right) \left(\Omega_j(\lambda) - \Sigma_j(\lambda) \right) d\mu_{ij}(\lambda). \tag{2.21}$$

By (2.20) and (2.21), we get the desired result. ◻

THEOREM 2.4. *Let ω is a real vector-valued function*

$$\omega(\zeta)=\begin{cases}\omega^{(1)}(\zeta), & \zeta\in I_3\\ \omega^{(2)}(\zeta), & \zeta\in I_4\end{cases}$$

and $\omega\in H$. Then, the integrals

$$\int_{-\infty}^{\infty}\Omega_i(\lambda)\,\psi_j(\zeta,\lambda)\,d\mu_{ij}(\lambda)\ \ (i,j=1,2)$$

converge in H. Thus, we obtain the spectral expansion formula

$$\omega(\zeta)=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_i(\lambda)\,\psi_j(\zeta,\lambda)\,d\mu_{ij}(\lambda).$$

PROOF. Write

$$\omega_s(\zeta)=\int_{-s}^{s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\,\psi_j(\zeta,\lambda)\,d\mu_{ij}(\lambda),$$

where $s>0,\ \omega_s\in H$ and

$$\omega_s(\zeta)=\begin{cases}\omega_s^{(1)}(\zeta), & \zeta\in I_3\\ \omega_s^{(2)}(\zeta), & \zeta\in I_4.\end{cases}$$

Let

$$\chi(\zeta)=\begin{cases}\chi^{(1)}(\zeta), & \zeta\in I_3\\ \chi^{(2)}(\zeta), & \zeta\in I_4,\end{cases}\quad \chi\in H$$

be a real vector-valued function which is equal to zero outside the finite interval $[-\tau,d)\cup(d,\tau]$, where $\tau\geq m$. Hence we get

$$\begin{aligned}&\int_{-\tau}^{d}\left(\omega_s^{(1)}(\zeta),\chi^{(1)}(\zeta)\right)_{\mathbb{C}^2}\Delta\zeta+\alpha\int_{d}^{\tau}\left(\omega_s^{(2)}(\zeta),\chi^{(2)}(\zeta)\right)_{\mathbb{C}^2}\Delta\zeta\\ &=\int_{-\tau}^{d}\left(\int_{-s}^{s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\,\psi_j(\zeta,\lambda)\,d\mu_{ij}(\lambda),\chi^{(1)}(\zeta)\right)_{\mathbb{C}^2}\Delta\zeta\\ &+\alpha\int_{d}^{\tau}\left(\int_{-s}^{s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\,\psi_j(\zeta,\lambda)\,d\mu_{ij}(\lambda),\chi^{(2)}(\zeta)\right)_{\mathbb{C}^2}\Delta\zeta\\ &=\int_{-s}^{s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\left\{\begin{array}{c}\int_{-\tau}^{d}\left(\chi^{(1)}(\zeta),\psi_j(\zeta,\lambda)\right)_{\mathbb{C}^2}\Delta\zeta\\ +\alpha\int_{d}^{\tau}\left(\chi^{(2)}(\zeta),\psi_j(\zeta,\lambda)\right)_{\mathbb{C}^2}\Delta\zeta\end{array}\right\}\end{aligned}$$

$$=\int_{-s}^{s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\,\Sigma_j(\lambda)\,d\mu_{ij}(\lambda). \tag{2.22}$$

From Theorem 2.3, we find

$$\int_{-\infty}^{d}\left(\omega^{(1)}(\zeta),\chi^{(1)}(\zeta)\right)_{\mathbb{C}^2}\Delta\zeta+\alpha\int_{d}^{\infty}\left(\omega^{(2)}(\zeta),\chi^{(2)}(\zeta)\right)_{\mathbb{C}^2}\Delta\zeta$$

$$=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Omega_i(\lambda)\Sigma_j(\lambda)\,d\mu_{ij}(\lambda). \tag{2.23}$$

By virtue of (2.22) and (2.23), we deduce that

$$\langle\omega-\omega_s,\chi\rangle_H=\int_{|\lambda|>s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\Sigma_j(\lambda)\,d\mu_{ij}(\lambda). \tag{2.24}$$

Let

$$\chi(\zeta)=\begin{cases}\omega(\zeta)-\omega_s(\zeta), & \zeta\in[-s,d)\cup(d,s]\\ 0, & \text{otherwise.}\end{cases}$$

It follows from (2.24) that

$$\|\omega-\omega_s\|_H^2=\int_{|\lambda|>s}\sum_{i,j=1}^{2}\Omega_i(\lambda)\Omega_j(\lambda)\,d\mu_{ij}(\lambda).$$

Letting $s\to\infty$ gives the desired result. □

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