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Manuscript accepted for publication

ON b-REPDIGITS AS PRODUCTS OR SUMS OF FIBONACCI, PELL, BALANCING, AND JACOBSTHAL NUMBERS

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Abstract. Let $b \geq 2$ be an integer. In this paper, we study the repdigits in base b that can be expressed as sums or products of Fibonacci, Pell, Balancing and Jacobsthal numbers. The proofs of our main theorems use lower bounds for linear forms in logarithms of algebraic numbers and a version of the Baker-Davenport reduction method.

1. Introduction

For an integer $b \geq 2$, a positive integer N is called a base b-repdigit if it has only one digit in its base b representation. That is,

$$N = d\left(\frac{b^{\ell}-1}{b-1}\right),$$

for some integers $\ell \geq 1$ and $d \in \{1, \ldots, b-1\}$. When $b = 10$, one usually omits to mention b and simply calls this number a repdigit. The sequence of numbers with repeated digits is the sequence A010785 in Sloane's *On-Line Encyclopedia of Integer Sequences* (OEIS) [19].

Let P, Q be non-zero integers. The polynomial $X^2 - PX + Q$, called the characteristic polynomial, has two roots

$$\frac{P+\sqrt{D}}{2}, \frac{P-\sqrt{D}}{2}.$$

2010 *Mathematics Subject Classification.* 11D09, 11B37, 11J68, 11Y50.

Key words and phrases. Fibonacci, Pell, Balancing and Jacobsthal numbers, b-repdigits, logarithmic height, reduction method.

The number $D = P^2 - 4Q$ is called its discriminant. Now, assume that $D \neq 0$. For each $n \geq 0$, we define the sequence $U_n = U_n(P, Q)$ by

$$U_0 = 0,\ U_1 = 1,\ U_n = PU_{n-1} - QU_{n-2}.$$

This sequence $(U_n)_{n\geq 0}$ is called a Lucas sequence with parameters (P, Q). In this paper, we discuss four particular Lucas sequences: Fibonacci, Pell, Balancing, and Jacobsthal sequences. The Fibonacci sequence denoted by $(F_n)_{n\geq 0}$ is the Lucas sequence corresponding to the parameters $(P, Q) = (1, -1)$ and the Pell sequence $(P_n)_{n\geq 0}$ is the Lucas sequence with the parameters $(P, Q) = (2, -1)$. Also, the Balancing sequence $(B_n)_{n\geq 0}$ and Jacobsthal sequence $(J_n)_{n\geq 0}$ correspond to the parameters $(P, Q) = (6, 1)$ and $(P, Q) = (1, -2)$, respectively. In the literature, we note that many theorists are interested in solving Diophantine equations involving linear recurrent sequences and repdigits, essentially using Baker's method based on linear forms in logarithms (see [1], [2], [4], [5] [6], [9], [11], [12], [13], [15],[16], [18]). Recently, the authors of [6] and [20] found all repdigits that can be written as sums or products of Fibonacci and Tribonacci numbers with the same index. Later, the two previous results were generalized in [3]. This study deals with a similar work with an extension to four Lucas sequences. More precisely, we fully solve the following Diophantine equations

$$F_nP_nB_nJ_n = d\left(\frac{b^\ell - 1}{b-1}\right) \tag{1.1}$$

and

$$F_n + P_n + B_n + J_n = d\left(\frac{b^\ell - 1}{b-1}\right), \tag{1.2}$$

in positive integers b, n, ℓ, d with $b \geq 2$ and $d \in \{1, \ldots, b-1\}$.

We organize this paper as follows. In Section 2, we recall some elementary properties of Fibonacci, Pell, Balancing, and Jacobsthal numbers, a result due to Matveev on lower bounds of linear forms in logarithms of algebraic numbers. The proofs of our main results are given in Section 3.

2. Some useful tools

In this section, we recall some tools on the considered sequences and Matveev's result [17].

2.1. *Linear forms in logarithms.* Let η be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\eta = \eta^{(1)}, \ldots, \eta^{(d)}$ denote its conjugates. The logarithmic height of η is defined by

$$h(\eta) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d} \log\max\left(1, \left|\eta^{(j)}\right|\right)\right).$$

This height has the following basic properties. For η_1, η_2 algebraic numbers and $m \in \mathbb{Z}$, we have

$$\begin{aligned} h(\eta_1 \pm \eta_2) &\leq h(\eta_1) + h(\eta_2) + \log 2, \\ h(\eta_1 \eta_2^{\pm 1}) &\leq h(\eta_1) + h(\eta_2), \\ h(\eta_1^m) &= |m| h(\eta_1). \end{aligned}$$

Now, let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\gamma_1, \ldots, \gamma_s \in \mathbb{L}$ and $b_1, \ldots, b_s \in \mathbb{Z} \setminus \{0\}$. Let $B \geq \max\{|b_1|, \ldots, |b_s|\}$ and

$$\Gamma = \gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1.$$

Let $A_1, \ldots, A_s$ be real numbers with

$$A_i \geq \max\{d_{\mathbb{L}} h(\gamma_i), |\log \gamma_i|, 0.16\}, \quad i = 1, 2, \ldots, s.$$

With the notations above, Bugeaud, Mignotte and Siksek [8, Theorem 9.4] proved the following result which is a variant of Matveev's result [17].

Theorem 2.1. *Assume that* $\Gamma \neq 0$. *Then*

$$\log|\Gamma| > -1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 \cdot (1 + \log d_{\mathbb{L}}) \cdot (1 + \log B) \cdot A_1 \cdots A_s.$$

2.2. *Some properties of Fibonacci, Pell, Balancing, and Jacobsthal sequences.* We recall here some useful properties of Fibonacci, Pell, Balancing and Jacobsthal sequences. The Binet's formula for the Fibonacci sequence is

$$F_n = \frac{\alpha^n - (-\alpha^{-1})^n}{\sqrt{5}}, \quad \text{for} \quad n \geq 0, \tag{2.3}$$

where $\alpha = (1 + \sqrt{5})/2$. Using this formula, we can deduce that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}, \quad \text{for} \quad n > 1. \tag{2.4}$$

It is also possible to see that

$$F_n = \frac{\alpha^n}{\sqrt{5}} + v \quad \text{with} \quad |v| \leq \frac{1}{\sqrt{5}}, \quad \text{for } n \geq 0. \tag{2.5}$$

In the case of Pell sequence, we have

$$P_n = \frac{\beta^n - (-\beta^{-1})^n}{2\sqrt{2}}, \quad \text{for} \quad n \geq 0, \tag{2.6}$$

where $\beta = 1 + \sqrt{2}$. Moreover, we can deduce that

$$\beta^{n-2} \leq P_n \leq \beta^{n-1}, \quad \text{for} \quad n > 1. \tag{2.7}$$

We also have

$$P_n = \frac{\beta^n}{2\sqrt{2}} + w \quad \text{with} \quad |w| \leq \frac{1}{2\sqrt{2}}, \quad \text{for } n \geq 0. \tag{2.8}$$

Next, the Binet's formula for the general terms of Balancing sequence is given by

$$B_n = \frac{\gamma^n - (-\gamma^{-1})^n}{4\sqrt{2}}, \quad \text{for} \quad n \geq 0, \tag{2.9}$$

where $\gamma = 3 + 2\sqrt{2}$. Using the above formula, we deduce the following inequalities

$$\gamma^{n-2} \leq B_n \leq \gamma^{n-1}, \quad \text{for} \quad n > 1. \tag{2.10}$$

It follows from (2.9) that

$$B_n = \frac{\gamma^n}{4\sqrt{2}} + r \quad \text{with} \quad |r| \leq \frac{1}{4\sqrt{2}}, \quad \text{for } n \geq 0. \tag{2.11}$$

Finally, the Binet's formula for the general terms of the Jacobsthal sequence is given by

$$J_n = \frac{2^n - (-1)^n}{3}, \quad \text{for} \quad n \geq 0 \tag{2.12}$$

and by induction on n, we see that

$$2^{n-2} \leq J_n \leq 2^{n-1}, \quad \text{for} \quad n > 1. \tag{2.13}$$

3. Main results

3.1. *On b-repdigits as products of Fibonacci Pell, Balancing and Jacobsthal numbers.* In this subsection, we will prove our first main result.

THEOREM 3.1. *Let $b \geq 2$ be an integer. Then, the Diophantine equation* (1.1) *has only finitely many solutions in integers (n, b, d, ℓ) such that $n, \ell \geq 1$ and $1 \leq d \leq b - 1$. Moreover, we have*

$$n < 7.5 \times 10^{18} \log^3 b \quad \textit{and} \quad \ell < 4.2 \times 10^{19} \log^3 b.$$

Note that if $n = 1$, then all solutions of equation (1.1) are of the form $(n, b, d, \ell) = (1, b, 1, 1)$ with $b \geq 2$. For the remaining of the proof, we consider $n \geq 2$. The following result will be useful in proving Theorem 3.1, which gives a relation between ℓ, n and b of equation (1.1).

LEMMA 3.2. *All solutions of Diophantine equation* (1.1) *satisfy*

$$(\ell - 1)\frac{\log b}{\log(2\alpha\beta\gamma)} + 1 < n < \ell\frac{\log b}{\log(2\alpha\beta\gamma)} + 2.$$

PROOF. From inequalities (2.4), (2.7), (2.10), and (2.13), we get

$$\alpha^{n-2}\beta^{n-2}\gamma^{n-2}2^{n-2} \leq F_n P_n B_n J_n = d\left(\frac{b^\ell - 1}{b - 1}\right) < b^\ell. \tag{3.14}$$

Taking the logarithm of both sides of (3.14), we get

$$n \log(2\alpha\beta\gamma) < \ell \log b + 2\log(2\alpha\beta\gamma). \tag{3.15}$$

For the lower bound, from (2.4), (2.7), (2.10), and (2.13), we have

$$b^{\ell-1} < d\left(\frac{b^\ell - 1}{b-1}\right) = F_nP_nB_nJ_n \le \alpha^{n-1}\beta^{n-1}\gamma^{n-1}2^{n-1}.$$

Taking the logarithm of both sides, we get

$$(\ell-1)\log b < (n-1)\log(2\alpha\beta\gamma),$$

which leads to

$$(\ell-1)\log b + \log(2\alpha\beta\gamma) < n\log(2\alpha\beta\gamma). \tag{3.16}$$

Combining (3.15) and (3.16), we obtain the desired inequalities. $\square$

Now, we are ready to prove Theorem 3.1. Substituting (2.5), (2.8), (2.11), and (2.12) in (1.1), we have

$$\begin{aligned}
\frac{db^\ell}{b-1} - \frac{d}{b-1} &= \left(\frac{\alpha^n}{\sqrt{5}} + v\right)\left(\frac{\beta^n}{2\sqrt{2}} + w\right)\left(\frac{\gamma^n}{4\sqrt{2}} + r\right)\left(\frac{2^n}{3} - \frac{(-1)^n}{3}\right)\\
&= \left(\frac{(\alpha\beta)^n}{2\sqrt{10}} + \frac{w\alpha^n}{\sqrt{5}} + \frac{v\beta^n}{2\sqrt{2}} + vw\right)\left(\frac{(2\gamma)^n}{12\sqrt{2}} - \frac{(-\gamma)^n}{12\sqrt{2}} + \frac{r2^n}{3} - \frac{r(-1)^n}{3}\right)\\
&= \frac{(2\alpha\beta\gamma)^n}{24\sqrt{20}} - \frac{(-\alpha\beta\gamma)^n}{24\sqrt{20}} + r\frac{(2\alpha\beta)^n}{6\sqrt{10}} - r\frac{(-\alpha\beta)^n}{6\sqrt{10}} + w\frac{(2\alpha\gamma)^n}{12\sqrt{10}}\\
&\quad - w\frac{(-\alpha\gamma)^n}{12\sqrt{10}} + rw\frac{(2\alpha)^n}{3\sqrt{5}} - rw\frac{(-\alpha)^n}{3\sqrt{5}} + v\frac{(2\beta\gamma)^n}{48} - v\frac{(-\beta\gamma)^n}{48}\\
&\quad + rv\frac{(2\beta)^n}{6\sqrt{2}} - rv\frac{(-\beta)^n}{6\sqrt{2}} + vw\frac{(2\gamma)^n}{12\sqrt{2}} - vw\frac{(-\gamma)^n}{12\sqrt{2}} + vwr\frac{2^n}{3} - vwr\frac{(-1)^n}{3},
\end{aligned}$$

which leads to

$$\begin{aligned}
\frac{(2\alpha\beta\gamma)^n}{24\sqrt{20}} - \frac{db^\ell}{b-1} &= -\frac{d}{b-1} + \frac{(-\alpha\beta\gamma)^n}{24\sqrt{20}} - r\frac{(2\alpha\beta)^n}{6\sqrt{10}} + r\frac{(-\alpha\beta)^n}{6\sqrt{10}} - w\frac{(2\alpha\gamma)^n}{12\sqrt{10}}\\
&\quad + w\frac{(-\alpha\gamma)^n}{12\sqrt{10}} - rw\frac{(2\alpha)^n}{3\sqrt{5}} + rw\frac{(-\alpha)^n}{3\sqrt{5}} - v\frac{(2\beta\gamma)^n}{48} + v\frac{(-\beta\gamma)^n}{48}\\
&\quad - rv\frac{(2\beta)^n}{6\sqrt{2}} + rv\frac{(-\beta)^n}{6\sqrt{2}} - vw\frac{(2\gamma)^n}{12\sqrt{2}} + vw\frac{(-\gamma)^n}{12\sqrt{2}} - vwr\frac{2^n}{3}\\
&\quad + vwr\frac{(-1)^n}{3}.
\end{aligned}$$

Taking the absolute value and dividing both sides of the above equality by $(2\alpha\beta\gamma)^n/24\sqrt{20}$, we get

$$\begin{aligned}\left|1-\frac{db^\ell}{b-1}\cdot\frac{48\sqrt{5}}{(2\alpha\beta\gamma)^n}\right| &< \frac{48\sqrt{5}}{(2\alpha\beta\gamma)^n}+\frac{1}{2^n}+\frac{1}{\gamma^n}+\frac{1}{(2\gamma)^n}+\frac{1}{\beta^n}+\frac{1}{(2\beta)^n}+\frac{1}{(\beta\gamma)^n}\\ &\quad+\frac{1}{(2\beta\gamma)^n}+\frac{1}{\alpha^n}+\frac{1}{(2\alpha)^n}+\frac{1}{(\alpha\gamma)^n}+\frac{1}{(2\alpha\gamma)^n}+\frac{1}{(\alpha\beta)^n}\\ &\quad+\frac{1}{(2\alpha\beta)^n}+\frac{1}{(\alpha\beta\gamma)^n}+\frac{1}{(2\alpha\beta\gamma)^n}<\frac{16}{\alpha^n},\end{aligned}$$

which holds, for $n \geq 2$. Thus, we obtain

$$\left|1-\frac{48d\sqrt{5}}{b-1}\cdot b^\ell\cdot(2\alpha\beta\gamma)^{-n}\right|<\frac{16}{\alpha^n}. \tag{3.17}$$

Put

$$\Gamma_1 := 1-\gamma_1^{b_1}\cdot\gamma_2^{b_2}\cdot\gamma_3^{b_3},$$

with

$$(\gamma_1,b_1):=\left(\frac{48d\sqrt{5}}{b-1},1\right),\ (\gamma_2,b_2):=(b,\ell)\text{ and }(\gamma_3,b_3):=(2\alpha\beta\gamma,-n).$$

Next, we will apply Theorem 2.1 on Γ_1. First, we need to check that $\Gamma_1 \neq 0$. Indeed, we have

$$(2\alpha\beta\gamma)^{2n}=\frac{5(48d)^2}{(b-1)^2}b^{2\ell}$$

and so $(2\alpha)^{2n}\in\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{5})$. Since $\mathbb{Q}(\sqrt{2})\cap\mathbb{Q}(\sqrt{5})=\mathbb{Q}$, then we conclude that $(2\alpha)^{2n}\in\mathbb{Q}$, which is not possible. Therefore, $\Gamma_1\neq 0$. Note that $\mathbb{L}:=\mathbb{Q}(\gamma_1,\gamma_2,\gamma_3)=\mathbb{Q}(\sqrt{2},\sqrt{5})$, so $d_{\mathbb{L}}:=4$. Moreover, we have $h(\gamma_2)=\log b$ and

$$h(\gamma_3)=h(2\alpha\beta\gamma)\leq h(2)+h(\alpha)+h(\beta)+h(\gamma)=\log 2+\frac{1}{2}\log(\alpha\beta\gamma).$$

Furthermore, we get

$$\begin{aligned}h(\gamma_1) &= h\left(\frac{48d\sqrt{5}}{b-1}\right)\leq h\left(\frac{d}{b-1}\right)+h(48)+h(\sqrt{5})\\ &= \log(\max\{b-1,d\})+\log 48+\frac{1}{2}\log 5\\ &\leq \log(b-1)+\log(48\sqrt{5})\\ &< \log b+\log(48\sqrt{5}).\end{aligned}$$

Thus, we can take

$$A_1:=4\log b+4\log(48\sqrt{5}),\ A_2:=4\log b\text{ and }A_3:=4\log 2+2\log(\alpha\beta\gamma).$$

As $n\geq 2$ and

$$B\geq\max\{|b_1|,|b_2|,|b_3|\}=\max\{\ell,n,1\},$$

we can take $B := \max\{\ell, n\}$. With the previous data and $s := 3$, Theorem 2.1 tells us that

$$\log|\Gamma_1| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 4^2(1+\log 4)(1+\log B) \cdot A_1 \cdot A_2 \cdot A_3, \tag{3.18}$$

where

$$\begin{aligned} A_1 \cdot A_2 \cdot A_3 &= \left(4\log b + 4\log(48\sqrt{5})\right) \cdot 4\log b \cdot (4\log 2 + 2\log(\alpha\beta\gamma)) \\ &< 1.16 \cdot 10^3 \log^2 b. \end{aligned} \tag{3.19}$$

In the above inequality, we have used the fact that $4\log b + 4\log(48\sqrt{5}) < 32\log b$, which holds for all $b \geq 2$. Combining (3.17), (3.18), and (3.19), we get

$$n\log\alpha - \log 16 < 6.35 \times 10^{15} \cdot \log^2 b \cdot (1+\log B). \tag{3.20}$$

Case 1: $B = n$. First, for $n \geq 2$ we have $1 + \log n < 2.5\log n$. Therefore, from (3.20) it follows that

$$n\log\alpha - \log 16 < 6.35 \times 10^{15} \cdot \log^2 b \cdot (1+\log n)$$

and then

$$n < 3.3 \times 10^{16} \cdot \log^2 b \cdot \log n, \tag{3.21}$$

which is valid for $n \geq 2$.

Case 2: $B = \ell$. Then from (3.20), we get

$$n\log\alpha - \log 16 < 6.35 \times 10^{15} \cdot \log^2 b \cdot (1+\log \ell). \tag{3.22}$$

By Lemma 3.2, it is easy to see that $\ell < 5.51n$. So, $1 + \log\ell < 5.1\log n$. Using this with (3.22), we obtain

$$n < 6.8 \times 10^{16} \cdot \log^2 b \cdot \log n. \tag{3.23}$$

In all cases, we can consider $n < 6.8 \times 10^{16} \cdot \log^2 b \cdot \log n$. To get an upper bound of n in term of b, we need the following lemma due to Guzmán and Luca.

LEMMA 3.3 (Lemma 7 of [14]). *If* $l \geq 1$, $H > \left(4l^2\right)^l$ *and* $H > L/(\log L)^l$, *then*

$$L < 2^l H(\log H)^l.$$

We can apply Lemma 3.3 with

$$l = 1, \quad L = n \quad \text{and} \quad H = 6.8 \times 10^{16} \cdot \log^2 b.$$

Therefore, we obtain

$$n < 2 \times 6.8 \times 10^{16} \log^2 b \times (38.8 + 2\log\log b) < 7.5 \times 10^{18} \cdot \log^3 b$$

where we have used the fact that $38.8 + 2\log\log b < 55\log b$, for $b \geq 2$. Next, $\ell < 4.2 \times 10^{19} \cdot \log^3 b$. This completes the proof of Theorem 3.1.

Now, we will solve the Diophantine equation (3.1), for $2 \leq b \leq 10$. Therefore, we have the following result.

Corollary 3.4. *The only solution of the Diophantine equation* (1.1), *for* $2 \leq b \leq 10$, *is* $(n, b, d, \ell) = (2, 5, 2, 2)$, *where* $n \geq 2$. *Hence, this solution gives the following representation*

$$2\left(\frac{5^2-1}{4}\right) = 22_5 = F_2 \cdot P_2 \cdot B_2 \cdot J_2.$$

For the proof of Corollary 3.4, we obtain very large bounds for the variables whose reduction requires a variant of reduction method given by Dujella and Pethő [10] that we present as follows. In fact, we will use the improved version given by Bravo, Gómez, and Luca (see [7, Lemma 1]). For a real number x, $\|x\|$ is the distance from x to the nearest integer.

Lemma 3.5. *Let* M *be a positive integer,* p/q *be a convergent of the continued fraction expansion of the irrational number* τ *such that* $q > 6M$, *and* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *Furthermore, let*

$$\varepsilon := \|\mu q\| - M \cdot \|\tau q\|.$$

If $\varepsilon > 0$, *then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w} \tag{3.24}$$

in positive integers u, v *and* w *with*

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

Proof of Corollary 3.4. Since $2 \leq b \leq 10$, the bounds of n and ℓ become $n < 9.2 \times 10^{19}$ and $\ell < 5.2 \times 10^{20}$ according to Theorem 3.1. To lower these bounds, we return to inequality (3.17) by putting

$$\Lambda_1 := \ell \log b - n \log(2\alpha\beta\gamma) + \log\left(\frac{48d\sqrt{5}}{b-1}\right).$$

Inequality (3.17) can be written as

$$|\Gamma_1| = \left|e^{\Lambda_1} - 1\right| < \frac{16}{\alpha^n}.$$

For $n \geq 8$, we get $\left|e^{\Lambda_1} - 1\right| < \frac{16}{\alpha^n} < \frac{1}{2}$, which also implies that $\frac{1}{2} < e^{\Lambda_1} < \frac{3}{2}$.

- When $\Lambda_1 > 0$, then we get

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = |e^{\Lambda_1} - 1| < \frac{16}{\alpha^n}.$$

- When $\Lambda_1 < 0$, we get

$$0 < |\Lambda_1| < e^{|\Lambda_1|} - 1 = e^{-\Lambda_1}(1 - e^{\Lambda_1}) < \frac{32}{\alpha^n}.$$

In both cases, we have $0 < |\Lambda_1| < \dfrac{32}{\alpha^n}$, which implies

$$0 < \left| \ell \frac{\log b}{\log(2\alpha\beta\gamma)} - n + \frac{\log\left(16d\sqrt{5}/3\right)}{\log(2\alpha\beta\gamma)} \right| < 8.4 \cdot \alpha^{-n}. \tag{3.25}$$

Now, we apply Lemma 3.5 with

$$\tau := \frac{\log b}{\log(2\alpha\beta\gamma)}, \quad \mu := \frac{\log\left(16d\sqrt{5}/3\right)}{\log(2\alpha\beta\gamma)}, \quad A := 8.4, \quad B := \alpha,$$

and $w := n$. Note that $\ell < 5.2 \times 10^{20}$, so we can take $M := 5.2 \times 10^{20}$. We use Mathematica to do the computations. Notice that if the first convergent such that $q > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until we find the one that satisfies the conditions. Let q_t be the denominator of the t-th convergent of the continued fraction of τ. In fact, the results obtained are the following table.

b	2	3	4	5	6	7	8	9	10
q_t	q_{37}	q_{36}	q_{41}	q_{40}	q_{49}	q_{45}	q_{41}	q_{38}	q_{52}
$n \leq$	114	114	117	114	114	112	120	121	121
$\epsilon >$	0.3098	0.0394	0.0716	0.2422	0.0592	0.0733	0.0040	0.0017	0.0111

Therefore, we obtain $n \leq 121$, which is valid in all cases. Hence, it remains to check equation (1.1) for $2 \leq n \leq 121$ and $1 \leq \ell \leq 666$. A quick inspection using Maple reveals that the Diophantine equation (1.1) has only the solution mentioned in Corollary 3.4. This ends the proof of Corollary 3.4.

3.2. *On b-repdigits as sums of Fibonacci, Pell, Balancing and Jacobsthal numbers.* In this subsection, we will follow the method in Subsection 3.1. Our second main result is following result.

THEOREM 3.6. *Let $b \geq 2$ be an integer. Then, the Diophantine equation (1.2) has only finitely many solutions in integers (n, b, d, ℓ) such that $n, \ell \geq 1$ and $1 \leq d \leq b-1$. Moreover, we have*

$$n < 1.5 \times 10^{16} \log^3 b \quad \textit{and} \quad \ell < 4.5 \times 10^{16} \log^3 b.$$

For $n = 1$, it is easy to show that all solutions of equation (1.2) are of the form $(n, b, d, \ell) = (1, 3, 1, 2), (1, b, 4, 1)$, for $b \geq 5$. Now, we assume that $n \geq 2$. The next lemma relates the sizes of n and ℓ.

LEMMA 3.7. *All solutions of the Diophantine equation (1.2) satisfy*

$$(\ell - 1)\frac{\log b}{\log \gamma} + \frac{\log(\gamma/4)}{\log \gamma} < n < \ell \frac{\log b}{\log \alpha} + 2.$$

Proof. Combining inequalities (2.4), (2.7), (2.10), and (2.13), we have

$$\alpha^{n-2} < F_n + P_n + B_n + J_n = d\left(\frac{b^\ell - 1}{b-1}\right) < b^\ell. \tag{3.26}$$

Taking the logarithm of both sides of (3.26), we get

$$n\log\alpha < \ell\log b + 2\log\alpha. \tag{3.27}$$

Now, from (2.4), (2.7), (2.10), and (2.13), we obtain

$$\begin{aligned} b^{\ell-1} \le d\left(\frac{b^\ell - 1}{b-1}\right) &= F_n + P_n + B_n + J_n \\ &\le \alpha^{n-1} + \beta^{n-1} + \gamma^{n-1} + 2^{n-1} < 4\gamma^{n-1}. \end{aligned}$$

Taking the logarithm of both sides, we have

$$(\ell-1)\log b < \log 4 + (n-1)\log\gamma,$$

which leads to

$$(\ell-1)\log b + \log(\gamma/4) < n\log\gamma. \tag{3.28}$$

Combining (3.27) and (3.28), we obtain the desired inequalities. □

Now, we will complete the proof of Theorem 3.6. For this, inserting (2.5), (2.8), (2.11), and (2.12) in (1.2), we have

$$\frac{db^\ell}{b-1} - \frac{d}{b-1} = \frac{\alpha^n}{\sqrt{5}} + v + \frac{\beta^n}{2\sqrt{2}} + w + \frac{\gamma^n}{4\sqrt{2}} + r + \frac{2^n}{3} - \frac{(-1)^n}{3},$$

which leads to

$$\frac{\gamma^n}{4\sqrt{2}} - \frac{db^\ell}{b-1} = -\frac{d}{b-1} - \frac{\alpha^n}{\sqrt{5}} - v - \frac{\beta^n}{2\sqrt{2}} - w - r - \frac{2^n}{3} + \frac{(-1)^n}{3}. \tag{3.29}$$

Taking the absolute value of both sides of (3.29), for $n \ge 2$, we get

$$\begin{aligned} \left|\frac{\gamma^n}{4\sqrt{2}} - \frac{db^\ell}{b-1}\right| &< \frac{d}{b-1} + \frac{\alpha^n}{\sqrt{5}} + |v| + \frac{|\beta|^n}{2\sqrt{2}} + |w| + |r| + \frac{2^n}{3} + \frac{1}{3} \\ &< \frac{\alpha^n}{\sqrt{5}} + \frac{\beta^n}{2\sqrt{2}} + \frac{2^n}{3} + \frac{1}{\sqrt{5}} + \frac{1}{2\sqrt{2}} + \frac{1}{3} + \frac{1}{4\sqrt{2}} + 1 \\ &< \beta^n\left[\frac{2.32}{\beta^n} + \frac{1}{(\beta/\alpha)^n\sqrt{5}} + \frac{1}{3(\beta/2)^n} + \frac{1}{2\sqrt{2}}\right] = c\cdot\beta^n \end{aligned}$$

where

$$c = \frac{2.32}{\beta^n} + \frac{1}{(\beta/\alpha)^n\sqrt{5}} + \frac{1}{3(\beta/2)^n} + \frac{1}{2\sqrt{2}} < 1.2 \quad \text{for} \quad n \ge 2.$$

Hence, we obtain

$$\left|\frac{\gamma^n}{4\sqrt{2}} - \frac{db^\ell}{b-1}\right| < 1.2\cdot\beta^n. \tag{3.30}$$

Now, dividing both sides of (3.30) by $\dfrac{\gamma^n}{4\sqrt{2}}$, we get

$$\left|1 - b^{\ell} \cdot \gamma^{-n} \cdot \frac{4d\sqrt{2}}{b-1}\right| < \frac{4.8\sqrt{2}}{(\gamma/\beta)^n}. \tag{3.31}$$

Let

$$\Gamma_2 := b^{\ell} \cdot \gamma^{-n} \cdot \frac{4d\sqrt{2}}{b-1} - 1. \tag{3.32}$$

Next, we will apply Theorem 2.1 to Γ_2. First, we need to check that $\Gamma_2 \neq 0$. Note that if $\Gamma_2 = 0$, then we see that

$$\gamma^n = \frac{4db^{\ell}\sqrt{2}}{b-1},$$

which is false, since taking the norm of both sides in $\mathbb{L} = \mathbb{Q}(\sqrt{2})$ we arrive at

$$\pm 1 = -2\left(\frac{4db^{\ell}}{b-1}\right)^2,$$

a clear contradiction. We conclude that $\Gamma_2 \neq 0$. So, to apply Theorem 2.1 to (3.32), we take $s := 3$ and

$$(\gamma_1, b_1) := (b, \ell), \quad (\gamma_2, b_2) := (\gamma, -n), \quad (\gamma_3, b_3) := \left(\frac{4d\sqrt{2}}{b-1}, 1\right).$$

Thus, we see that $\mathbb{L} = \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3) = \mathbb{Q}(\sqrt{2})$ and then $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 2$. Note also that $h(\gamma_1) = \log b$, $h(\gamma_2) = (\log \gamma)/2$, and

$$\begin{aligned} h(\gamma_3) &\le h\left(\frac{d}{b-1}\right) + h(4) + h(\sqrt{2}) = \log\left(\max\{b-1, d\}\right) + \frac{1}{2}\log 32 \\ &= \log b + \frac{1}{2}\log 32. \end{aligned}$$

Let us take

$$A_1 = 2\log b, \quad A_2 = \log \gamma \quad \text{and} \quad A_3 := 2\log b + \log 32.$$

Since $n \ge 2$ and $B \ge \max\{|b_1|, |b_2|, |b_3|\}$, then we can take $B = \max\{n, \ell\}$. Hence, by Theorem 2.1, we get

$$\log|\Gamma_2| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2) \cdot (1 + \log B) \cdot A_1 A_2 A_3 \tag{3.33}$$

with

$$\begin{aligned} A_1A_2A_3 &= 2\log b \cdot \log \gamma \cdot (2\log b + \log 32) \\ &< 16 \cdot \log \gamma \cdot \log^2 b. \end{aligned} \tag{3.34}$$

In above inequality, we have used the fact that $\log b + \log 4\sqrt{2} < 4\log b$, for $b \geq 2$. Thus, from (3.31), (3.33), and (3.34), we obtain

$$n < 3.11 \times 10^{13} \cdot (1 + \log B) \cdot \log^2 b + 2.2. \tag{3.35}$$

Now, we study the following two cases according to the values of B.

Case a: $B = n$. Then, for $n \geq 2$, we have $1 + \log n < 2.5 \log n$. Thus, the inequality (3.35) leads to

$$n < 9.4 \times 10^{13} \cdot \log^2 b \cdot \log n.$$

Case b: $B = \ell$. We have

$$n < 3.11 \times 10^{13} \cdot (1 + \log \ell) \cdot \log^2 b + 2.2. \tag{3.36}$$

From Lemma 3.7, one can easily see that $\ell < 3n$, then inequality (3.36) implies

$$n < 1.6 \times 10^{14} \cdot \log^2 b \cdot \log n,$$

where we have used the fact that $1 + \log 3 + \log n < 5 \log n$. So, in all cases we conclude that $n < 1.6 \times 10^{14} \cdot \log^2 b \cdot \log n$ holds, for $n \geq 2$.

To obtain an upper bound of n in term of b, we will apply Lemma 3.3 with $l = 1$, $L = n$ and $H = 1.6 \times 10^{14} \cdot \log^2 b$. Thus, we obtain

$$n < 2 \cdot 1.6 \times 10^{14} \cdot \log^2 b \cdot (32.8 + 2 \log\log b).$$

As $32.8 + 2\log(\log b) < 46 \log b$ for $b \geq 2$, one can see that

$$n < 1.5 \times 10^{16} \cdot \log^3 b \quad \text{and} \quad \ell < 4.5 \times 10^{16} \cdot \log^3 b. \tag{3.37}$$

This completes the proof of Theorem 3.6.

Now, as an illustration we will solve equation (1.2), for $2 \leq b \leq 10$. Thus, we have the following result.

Corollary 3.8. *The only solutions of the Diophantine equation* (1.2) *with* $2 \leq b \leq 10$ *and* $n \geq 2$ *are* $(n, b, d, \ell) = (2, 4, 2, 2),\ (3, 8, 5, 2),\ (2, 9, 1, 2)$. *Their representations are*

$$2\left(\frac{4^2 - 1}{3}\right) = 22_4 = F_2 + P_2 + B_2 + J_2,$$
$$5\left(\frac{8^2 - 1}{7}\right) = 55_8 = F_3 + P_3 + B_3 + J_3,$$
$$1\left(\frac{9^2 - 1}{8}\right) = 11_9 = F_2 + P_2 + B_2 + J_2.$$

Proof. When $2 \leq b \leq 10$, Theorem 3.6 gives $n < 1.84 \times 10^{17}$ and $\ell < 5.5 \times 10^{17}$. To reduce the upper bounds for n and ℓ, we will apply Lemma 3.5. So, let

$$\Lambda_2 := \log(\Gamma_2 + 1) = \ell \log b - n \log \gamma - \log\left(\frac{4d\sqrt{2}}{9}\right).$$

From (3.31), we conclude that

$$|\Gamma_2| = \left|e^{\Lambda_2} - 1\right| < \frac{4.8\sqrt{2}}{(\gamma/\beta)^n}. \tag{3.38}$$

If $n \geq 3$, then $|e^{\Lambda_2} - 1| < \frac{4.8\sqrt{2}}{(\gamma/\beta)^n} < \frac{1}{2}$, which implies that $\frac{1}{2} < e^{\Lambda_2} < \frac{3}{2}$. For $\Lambda_2 > 0$, we have

$$0 < \Lambda_2 < e^{\Lambda_2} - 1 = |e^{\Lambda_2} - 1| < \frac{4.8\sqrt{2}}{(\gamma/\beta)^n}.$$

For $\Lambda_2 < 0$, we have

$$0 < |\Lambda_2| < e^{|\Lambda_2|} - 1 = e^{|\Lambda_2|}(1 - e^{-|\Lambda_2|}) < \frac{9.6\sqrt{2}}{(\gamma/\beta)^n}.$$

Therefore, in all cases we have $0 < |\Lambda_2| < \frac{9.6\sqrt{2}}{(\gamma/\beta)^n}$, which implies that

$$0 < \left|\ell\frac{\log b}{\log \gamma} - n - \frac{\log(4d\sqrt{2}/9)}{\log \gamma}\right| < 7.8 \cdot (\gamma/\beta)^{-n}. \tag{3.39}$$

Using (3.39) and Lemma 3.5, we can take $M := 5.5 \times 10^{17}$ because $\ell < 5.5 \times 10^{17}$. For the application of Lemma 3.5, we define the following quantities

$$\tau := \frac{\log b}{\log \gamma}, \quad \mu := -\frac{\log(4d\sqrt{2}/9)}{\log \gamma}, \quad A := 7.8, \quad B := \gamma/\beta \quad \text{and} \quad w := n.$$

We used Mathematica to find the results mentioned in the following table where q_t denotes the denominator of the t-th convergent of the continued fraction of τ such that $q_t > 6M$ and $\epsilon > 0$.

b	2	3	4	5	6	7	8	9	10
q_t	q_{38}	q_{37}	q_{38}	q_{31}	q_{23}	q_{43}	q_{42}	q_{34}	q_{38}
$n \leq$	54	52	55	55	56	55	55	54	56
$\epsilon >$	0.2104	0.2221	0.0291	0.0248	0.0499	0.0332	0.0980	0.0719	0.0109

So, we obtain $2 \leq n \leq 56$ and then $1 \leq \ell \leq 168$. To finish the proof, we use a simple routine written in Maple which reveals that the Diophantine equation (1.2) has only the solutions mentioned in the statement of Corollary 3.8 if $n \geq 2$ and $2 \leq b \leq 10$. This completes the proof of Corollary 3.8. □

Acknowledgements.

The authors are grateful to the referee for the useful comments/suggestions that help to improve the quality of the paper. The first and second authors are supported by Institut de Mathématiques et de Sciences Physiques (IMSP) de l'Université d'Abomey-Calavi. The third author is partially supported by Purdue University Northwest.

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O b-Rep znamenkama kao umnošcima ili zbrojevima Fibonaccijevih, Pellovih, balansirajućih i Jacobsthalovih brojeva

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Sažetak. Neka je $b \geq 2$ cijeli broj. U ovom radu proučavamo repoznamenke u bazi b koje se mogu izraziti kao zbrojevi ili umnošci Fibonaccijevih, Pellovih, Balansirajućih i Jacobsthalovih brojeva. Dokaz našeg glavnog teorema koristi donje granice za linearne oblike u logaritmima algebarskih brojeva i verziju Baker-Davenportove redukcijske metode.

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