from $\mathbb{Z}^{n+1}$, such that $v = Br$ is a vector from L whose norm is

$$\leq \Lambda := 2^{n/4} \cdot C^{n/(n+1)} = 2^{n/4}Q^n.$$

The components of the vector v are $v_1 = q$, $v_{i+1} = Cp_i - q\lfloor C\alpha_i \rceil$, $i = 1, \ldots, n$. We have

$$q^2 + \sum_{i=1}^{n}(Cp_i - q\lfloor C\alpha_i \rceil)^2 \leq \Lambda^2.$$

Thus, $q \leq \Lambda$ and $\max_{1\leq i\leq n} |Cp_i - q\lfloor C\alpha_i \rceil| \leq \Lambda$. Since $|Cp_i - q\lfloor C\alpha_i \rceil| \geq C|p_i - q\alpha_i| - q/2$, we obtain

$$|p_i - q\alpha_i| \leq C^{-1}(|Cp_i - q\lfloor C\alpha_i \rceil| + q/2).$$

Finally, let us use this simple fact: for real numbers x, y,

$$2x + y \leq \sqrt{5(x^2 + y^2)}. \tag{8.47}$$

Indeed, by squaring, we obtain $(x - 2y)^2 \geq 0$, which is obviously true. If we apply (8.47) to $x = |Cp_i - q\lfloor C\alpha_i \rceil|$, $y = q$, we obtain

$$|p_i - q\alpha_i| \leq C^{-1} \cdot \sqrt{5(|Cp_i - q\lfloor C\alpha_i \rceil|^2 + q^2)}/2 \leq \frac{\sqrt{5}}{2C}\Lambda = \frac{\sqrt{5}}{2}2^{n/4}Q^{-1}. \qquad \square$$

Example 8.9. Let $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{3}$, $\alpha_3 = \sqrt{5}$. Let us take $Q = 1000$ and apply the algorithm from Theorem 8.59. We form a matrix B as in the proof of the theorem and by using PARI, we calculate `qflll`(B). From the first column of the obtained matrix, we read numbers $q = 118452669$, $p_1 = 167517371$, $p_2 = 205166041$, $p_3 = 264868220$. We get $q < 1.2 \cdot Q^3$, $|q\sqrt{2} - p_1| < 0.91 \cdot Q^{-1}$, $|q\sqrt{3} - p_2| < 0.14 \cdot Q^{-1}$, $|q\sqrt{5} - p_3| < 0.29 \cdot Q^{-1}$,

$$\max\left(\left|\sqrt{2} - \frac{p_1}{q}\right|, \left|\sqrt{3} - \frac{p_2}{q}\right|, \left|\sqrt{5} - \frac{p_3}{q}\right|\right) < q^{-4/3}.$$

Hence, the obtained simultaneous rational approximations are even better than Theorem 8.59 guarantees, and they satisfy inequality (8.37) from the corollary of Dirichlet's theorem on simultaneous approximations without any additional factors.

8.10 Exercises

1. Find all solutions of the inequality

$$\left|\sqrt{3} - \frac{p}{q}\right| < \frac{1}{q^2},$$

where p and q are relatively prime positive integers and $q < 100$.

2. Determine the fraction p/q with the smallest possible denominator such that
$$\left|\log_2 3 - \frac{p}{q}\right| < \left|\log_2 3 - \frac{19}{12}\right|.$$

3. Prove that the number $\alpha = \sum_{n=0}^{\infty} 2^{-n^2}$ is irrational.

4. Let $\frac{h}{k}$ and $\frac{h'}{k'}$ be consecutive elements of Farey sequence $\mathcal{F}_n$. Prove that $k + k' > n$.

5. Let $\frac{h}{k}$ and $\frac{h'}{k'}$ run through all pairs of consecutive elements of Farey sequence $\mathcal{F}_n$, $n > 1$. Calculate
$$\min\left(\frac{h'}{k'} - \frac{h}{k}\right) \quad \text{and} \quad \max\left(\frac{h'}{k'} - \frac{h}{k}\right).$$

6. Let $\frac{h}{k}$, $\frac{h''}{k''}$, $\frac{h'}{k'}$ be three consecutive elements of $\mathcal{F}_n$. Prove that $\dfrac{k+k'}{k''}$ is a positive integer and that
$$\frac{k+k'}{k''} = \frac{h+h'}{h''} = \left\lfloor \frac{k+n}{k''} \right\rfloor.$$

7. Let s_n denote the n-th element of the sequence $1, 2, 3, 4, 6, 8, 9, 12, \ldots$ of positive integers of the form $2^a 3^b$, $a, b \in \mathbb{N} \cup \{0\}$, written in increasing order. Prove that
$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = 1.$$

8. Prove that denominators q_n in convergents of the continued fraction expansion of an irrational number α satisfy the inequality $q_n \geq F_n$, where F_n is the n-th Fibonacci number.

9. Calculate first four convergents $\frac{p_0}{q_0}$, $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$, $\frac{p_3}{q_3}$ in the continued fraction expansion of $\pi = 3.1415926\cdots$.

10. Expand the numbers $\frac{3}{17}$, $\frac{101}{11}$ and $\frac{1117}{7111}$ into simple continued fractions.

11. a) Let $a_0 \neq 0$. Prove that
$$\frac{p_n}{p_{n-1}} = [a_n, a_{n-1}, \ldots, a_1, a_0].$$

b) Let $[a_0, a_1, \ldots, a_n]$ be the expansion into simple finite continued fraction of a rational number $\frac{r}{s}$, where $\gcd(r, s) = 1$ and $r \geq s > 0$. Prove that this expansion is palindromic, i.e. $a_0 = a_n$, $a_1 = a_{n-1}$, $a_2 = a_{n-2}, \ldots$ if and only if $r \mid (s^2 + (-1)^{n-1})$.

12. On the set $\mathbb{R} \setminus \mathbb{Q}$, the relation $\cong$ is defined by: $\alpha \cong \beta$ if and only if there are integers a, b, c, d such that $ad - bc = \pm 1$ and

$$\beta = \frac{a\alpha + b}{c\alpha + d}.$$

Prove that $\cong$ is an equivalence relation.

13. a) Are the numbers $\sqrt{5}$ and $\dfrac{1+\sqrt{5}}{2}$ equivalent?

    b) Are the numbers $\sqrt{3}$ and $\dfrac{1+\sqrt{3}}{2}$ equivalent?

14. Determine a real number whose expansion into simple continued fraction is:

    a) $[11, \overline{2, 2, 22}]$,
    b) $[1, \overline{1, 4, 1, 1}]$,
    c) $[\overline{1, 3, 1, 3, 1, 15}]$,
    d) $[2, 1, 1, 5, \overline{3, 6}]$.

15. Find the simple continued fraction expansion of the numbers:

    a) $\sqrt{23}$,
    b) $\frac{2+\sqrt{5}}{3}$,
    c) $\frac{4-\sqrt{11}}{3}$.

16. Prove that in the algorithm for calculating the expansion into continued fraction of a quadratic irrationality, we have $s_n > 0$ for sufficiently large indices n.

17. Prove that for every $p, q \in \mathbb{N}$, $\left|\dfrac{\sqrt{2}}{2} - \dfrac{p}{q}\right| > \dfrac{1}{4q^2}$.

18. Find the smallest positive integer d such that the length of the period in the simple continued fraction expansion of $\sqrt{d}$ is equal to 10.

19. Let $k \geq 3$ be a positive integer. Find the continued fraction expansion of $\sqrt{k^2-2}$.

20. Let $k \geq 5$ be a positive integer. Find the continued fraction expansion of $\sqrt{k^2-4}$, depending on whether k is even or odd.

21. Let k be a positive integer. Determine the number α whose continued fraction expansion is $[k, \overline{k, 2k}]$.

22. Prove that the continued fraction expansion of the number $e = \sum_{i=0}^{\infty} \frac{1}{i!}$ is $[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \ldots, 2k, 1, 1, \ldots]$, i.e. $a_0 = 2$, $a_1 = 1$, $a_{3k-1} = 2k$, $a_{3k} = a_{3k+1} = 1$ for $k > 0$ (see [353, Chapter 1.8]).

23. Find a subset $\mathcal{R}_1$ of the plane E^2 which is convex and whose area is $\mu(\mathcal{R}_1) = 10$, but $\mathcal{R}_1$ does not contain any integer points.

24. Find a subset $\mathcal{R}_2$ of the plane E^2 which is symmetric with respect to the origin and whose area is $\mu(\mathcal{R}_2) = 10$, but $\mathcal{R}_2$ does not contain any integer points.

25. Sketch the subset $\mathcal{R}_3$ of the plane E^2 defined by

$$\mathcal{R}_3 = \{(x_1, x_2) \,:\, x_1, x_2 \in \mathbb{R},\ |x_1^2 - 2x_2^2| < 1\}.$$

Is the set $\mathcal{R}_3$ symmetric with respect to the origin, is it convex, what is its area, and how many integer points does it contain?

26. Let α_1 and α_2 be real numbers and n a positive integer. Prove that there are integers p_1, p_2, q such that

$$0 < q \leq n \quad \text{and} \quad \left(\alpha_1 - \frac{p_1}{q}\right)^2 + \left(\alpha_2 - \frac{p_2}{q}\right)^2 \leq \frac{4}{\pi n q^2}\,.$$

27. Find positive integers q, p_1, p_2 such that $q > 10^9$ and

$$\left|\sqrt{2} - \frac{p_1}{q}\right| < q^{-3/2}, \quad \left|\sqrt{3} - \frac{p_2}{q}\right| < q^{-3/2}.$$