Here, we used Euler's product formula (Theorem 7.15) and the fact that for σ in some neighbourhood of 1, we have

$$\zeta(\sigma) = \frac{1}{\sigma - 1} + g(\sigma),$$

where $g(\sigma)$ is an analytic function at $\sigma = 1$.

Hence, we have already obtained the main term in the estimate for χ_0. We, therefore, need to show that the remaining characters will not "damage" this estimate, i.e. we need to prove that $S(\sigma, \chi) = O(1)$ for $\sigma > 1$ and all Dirichlet characters χ modulo k such that $\chi \neq \chi_0$.

Similarly to the computations for $S(\sigma, \chi_0)$, we obtain

$$\begin{aligned} S(\sigma, \chi) &= \sum_p \frac{\chi(p)}{p^\sigma} = \sum_p \sum_{m \geq 1} \frac{\chi(p)^m}{mp^{m\sigma}} + O(1) \\ &= \sum_p \ln\left(1 - \frac{\chi(p)}{p^\sigma}\right)^{-1} + O(1) \\ &= \ln(L(\sigma, \chi)) + O(1). \end{aligned}$$

For $\chi \neq \chi_0$, the function $L(s, \chi)$ is analytic for $\sigma > 0$, so $L(\sigma, \chi)$ is continuous for $\sigma > 1$ and

$$\lim_{\sigma \to 1} L(\sigma, \chi) = L(1, \chi).$$

Since, by Theorem 7.21, $L(1, \chi) \neq 0$, we obtain $S(\sigma, \chi) = O(1)$, which needed to be proved. □

Let us mention that in 2004, Green and Tao [201] proved that there are arbitrary long sequences of primes, which are consecutive elements of an arithmetic progression. For example, primes $3, 5, 7$ are three consecutive terms of the sequence $2n + 1$, while primes $251, 257, 263, 269$ are four consecutive terms of the sequence $6n + 245$.

7.6 Exercises

1. Compare numbers $\pi(x)$, $x/\ln(x)$ and $\mathrm{li}(x)$ for $x = 100, 1000, 10000$.

2. Let $p = 2n + 1$ be a prime number. Prove that

$$\binom{2n}{n} \equiv (-1)^n \pmod{p}.$$

3. Let p be a prime number. Prove that
$$\binom{2p}{p} \equiv 2 \pmod{p^2}.$$

4. Let $d_n = \operatorname{lcm}(1, 2, \ldots, n)$. Prove that $2^n < d_n < (13/4)^n$, for a large enough positive integer n.

5. Let n be a positive integer. Find the greatest common divisor of
$$\binom{2n}{1}, \binom{2n}{3}, \ldots, \binom{2n}{2n-1}.$$

6. Prove that for positive integers n and k, the number
$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k}$$
cannot be an integer.

7. Prove that there are no positive integers $k, m, n \geq 2$ such that $n! = m^k$.

8. Let k be an arbitrary positive integer. Prove that the set $\{1, 2, \ldots, 2k\}$ can be partitioned into k disjoint pairs of numbers such that the sum of numbers in each pair is a prime number.

9. Prove that each integer $n \geq 7$ is either a prime number or it can be expressed as a sum of several distinct prime numbers.

10. Prove that for every positive integer m, there is a positive integer n such that the interval $(n, 2n]$ contains exactly m prime numbers.

11. Let n be a positive integer. Let us denote by $D(n)$, the smallest positive integer k such that the numbers $1^2, 2^2, \ldots, n^2$ are incongruent modulo k. Check that $D(1) = 1$, $D(2) = 2$, $D(3) = 6$, $D(4) = 9$. Prove that, for $n \geq 5$, $D(n)$ is the smallest integer which is $\geq 2n$, and which is either a prime number or a double prime number.

12. Let $\mathcal{B}$ be a finite non-empty set of integers, $X = \max_{b \in \mathcal{B}} |b|$, and $\mathcal{S}$ a finite set, elements of which are powers of prime numbers. Assume that for each $q \in \mathcal{S}$, the set $\mathcal{B}$ contains a representative of at most $g(q)$ equivalence classes modulo q. Prove that
$$|\mathcal{B}| \leq \Big(\sum_{q \in \mathcal{S}} \Lambda(q) - \ln(2X)\Big) \Big/ \Big(\sum_{q \in \mathcal{S}} \frac{\Lambda(q)}{g(q)} - \ln(2X)\Big)$$

if the denominator is positive. This result is called Gallagher's larger sieve, by an analogy with Eratosten's sieve (see [86, Chapter 2.2]). Apply this result to the case when $\mathcal{B}$ is the set of all squares which are $\leq X$.

13. Prove that for a real number $s > 1$,

    a) $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$,

    b) $\displaystyle\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}$.

14. Let $s > 1$. Prove that

    a) $\displaystyle\sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$,

    b) $\displaystyle\sum_{n>x} \frac{1}{n^s} = O(x^{1-s})$.

15. Let $\alpha \geq 0$. Prove that $\displaystyle\sum_{n\geq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$.

16. Prove that

$$\sum_{n\leq x} \ln^2 n = x\ln^2 x - 2x\ln x + 2x + O(\ln^2 x).$$

17. Prove that there is constant c such that

$$\prod_{p\leq x}\Big(1-\frac{1}{p}\Big) = \frac{c}{\ln x} + O\Big(\frac{1}{\ln^2 x}\Big).$$

18. Prove that

$$\sum_{pq\leq x} \frac{1}{pq} = (\ln\ln x)^2 + O(\ln\ln x),$$

    where the sum runs through all pairs of prime numbers p, q such that $pq \leq x$.

19. By using the functional equation (7.17), compute $\zeta(0)$.

20. Prove that for $m = 4$, formula (7.19) can be written in the form

$$S_4(n) = \frac{(n-1)n(2n-1)(3n^2-3n-1)}{30}.$$

21. Determine all Dirichlet characters modulo 5.

22. Let $\chi \neq \chi_0$ be the Dirichlet character modulo 4.

    a) Prove: $\sum_{n \leq x} \chi(n)|\mu(n)| = O(\sqrt{x})$.

    b) Let $Q_1(x)$ be the number of square-free positive integers of the form $4k+1$ which are $\leq x$. Prove that

    $$Q_1(x) = \frac{2}{\pi^2}x + O(\sqrt{x}).$$

23. Prove that there are infinitely many primes p such that the numbers $p-100, p-99, \ldots, p-1, p+1, \ldots, p+99, p+100$ are all composite.

24. Let $a_1, \ldots, a_k$ be pairwise relatively prime positive integers, none of which is a perfect square. Prove that there are infinitely many prime numbers p such that a_i, for $i = 1, \ldots, k$, are quadratic non-residues modulo p.

25. Find five prime numbers which are consecutive elements of an arithmetic progression.