From $1 = h(1) = f(1)g(1) = f(1)$, it follows that $f(1) = f(1)f(1)$, so $mn > 1$. Due to the minimality, we have $f(ab) = f(a)f(b)$, for all a, b such that $\gcd(a, b) = 1$ and $1 \leq ab < mn$. Now,

$$\begin{aligned} h(mn) &= f(mn)g(1) + \sum_{\substack{a|m,\, b|n \\ ab<mn}} f(ab)g\left(\frac{mn}{ab}\right) \\ &= f(mn) + \sum_{\substack{a|m,\, b|n \\ ab<mn}} f(a)f(b)g\left(\frac{m}{a}\right)g\left(\frac{n}{b}\right) \\ &= f(mn) + (h(m)h(n) - f(m)f(n)). \end{aligned}$$

From this, we obtain

$$h(mn) - h(m)h(n) = f(mn) - f(m)f(n) \neq 0,$$

which is a contradiction to the assumption that h is multiplicative. □

6.5 Exercises

1. With how many zeros the following numbers end:

   a) $1111!$,

   b) $3333!$,

   c) $5555!$?

2. With how many zeros the following numbers end:

   a) $\binom{233}{24}$,

   b) $\binom{343}{49}$,

   c) $\binom{455}{34}$?

3. Prove that for any odd positive integer n,

$$\left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{3n}{4} \right\rfloor = \frac{3(n-1)}{2}.$$

4. Prove that for all real numbers x, y,

$$\lfloor x \rfloor + \lfloor y \rfloor + \lfloor x + y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor.$$

5. Let x and y be real numbers such that

$$\lfloor x+y\rfloor = \lfloor x\rfloor + \lfloor y\rfloor \quad \text{and} \quad \lfloor -x-y\rfloor = \lfloor -x\rfloor + \lfloor -y\rfloor .$$

Prove that at least one of the numbers x, y is an integer.

6. Let $m \geq 2$ be an integer. How many solutions in positive integers are there to the equation $\left\lfloor \dfrac{x}{m} \right\rfloor = \left\lfloor \dfrac{x}{m-1} \right\rfloor$?

7. Calculate $\displaystyle\sum_{k=1}^{n^2-1} \lfloor \sqrt{k} \rfloor$.

8. Prove that $\displaystyle\sum_{k\geq 1} 2^k \left\lfloor \frac{n}{2^k} + \frac{1}{2} \right\rfloor^2 = n(n+1)$.

9. Prove that for any positive integer n,

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{n} + \sqrt{n+2} \rfloor .$$

10. Prove that for any integer $n \geq 2$,

$$\lfloor \log_2 n\rfloor + \lfloor \log_3 n\rfloor + \cdots + \lfloor \log_n n\rfloor = \lfloor \sqrt{n}\rfloor + \lfloor \sqrt[3]{n}\rfloor + \cdots + \lfloor \sqrt[n]{n}\rfloor .$$

11. Express the numbers $\frac{8}{11}$, $\frac{13}{21}$ and $\frac{5}{121}$ in the form $\frac{1}{x_1} + \cdots + \frac{1}{x_k}$, where x_i are distinct positive integers.

12. Check that the equation $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ has a solution in positive integers x, y, z for $n = 2, 3, \ldots, 10$.

13. Prove that $\displaystyle\sum_{d|n} \frac{1}{d} = \frac{\sigma(n)}{n}$.

14. Is the function $F(n) = \varphi(n^2)$ multiplicative?

15. Determine all positive integers n such that

$$\varphi(n^2) + \varphi((n+1)^2) \geq 2n^2 .$$

16. Determine all positive integers n such that $n + \tau(n) = 2\varphi(n)$.

17. a) Prove that for every prime p, $p\sigma(p) \equiv 2 \pmod{\varphi(p)}$.

    b) Let n be a composite number such that $n\sigma(n) \equiv 2 \pmod{\varphi(n)}$. Prove that then $n = 4, 6$ or 22 ([394]).

18. Let $S : \mathbb{N} \to \mathbb{R}$ be the function defined by

$$S(n) = \begin{cases} n, & \text{if } n \text{ is the square of a positive integer,} \\ 0, & \text{otherwise.} \end{cases}$$

Is the function S multiplicative?

19. Let $f : \mathbb{N} \to \mathbb{R}$ be the function defined by $f(n) = \lfloor\sqrt{4n}\rfloor - \lfloor\sqrt{4n-1}\rfloor$. Is the function f multiplicative?

20. Let $\omega(n)$ denote the number of prime divisors of a positive integer n, i.e. $\omega(n) = \sum_{p|n} 1$. Is the function ω multiplicative?

21. Is the function $\lambda(n) = (-1)^{\omega(n)}$ multiplicative?

22. Prove $\displaystyle\sum_{d|n} \mu(d)\tau(d) = (-1)^{\omega(n)}$.

23. Prove $\displaystyle\sum_{d|n} |\mu(d)| = 2^{\omega(n)}$.

24. Let $s(n)$ denote the product of all prime divisors of n, with the agreement that $s(1) = 1$. Is the function s multiplicative?

    Prove $\displaystyle\sum_{d|n} \mu(d)\sigma(d) = (-1)^{\omega(n)} s(n)$.

25. Prove $\displaystyle\sum_{d|n} \frac{\mu^2(d)}{\varphi(d)} = \frac{n}{\varphi(n)}$.

26. Let

$$H(n) = \tau(n) \cdot \Big(\sum_{d|n} \frac{1}{d}\Big)^{-1}.$$

    a) Prove that H is a multiplicative function.

    b) Determine all positive integers n such that $H(n) \leq 2$.

27. Let $f : \mathbb{N} \to \mathbb{N}$ be a multiplicative and strictly increasing function such that $f(2) = 2$. Prove that $f(n) = n$, for any $n \in \mathbb{N}$.

28. Calculate:

    a) $\sum_{n\le 20}\tau(n)$,
    b) $\sum_{n\le 20}\sigma(n)$,
    c) $\sum_{n\le 20}\varphi(n)$.

29. Prove $\displaystyle\sum_{n\le x}\frac{\sigma(n)}{n}=\frac{\pi^2}{6}\,x+O(\ln x)$.

30. Prove that for any $\delta>0$, $\tau(n)=O(n^{\delta})$.

31. Let $t(x)=\displaystyle\sum_{n\le x}\tau(n)$. Prove

$$\sum_{n\le x}\frac{\tau(n)}{n}=\frac{t(x)}{x}+\int_1^x\frac{t(u)}{u^2}\,du.$$

By using this formula, prove the following estimate

$$\sum_{n\le x}\frac{\tau(n)}{n}=\frac{1}{2}\ln^2 x+O(\ln x).$$

32. Prove $\displaystyle\sum_{n\le x}\omega(n)=x\ln\ln x+O(x)$.

33. Prove $\displaystyle\sum_{n\le x}2^{\omega(n)}=\frac{6}{\pi^2}\,x\ln x+O(x)$.

34. Determine the function $u*u$.

35. Let f and g be completely multiplicative functions. Does the function $f*g$ have to be completely multiplicative?