number of integer points on $y^2 = (x + ab)(x + ac)(x + bc)$ depends on the rank of this elliptic curve.

Recently, in [161, 162], the sets which are at the same time $D(n_1)$ and $D(n_2)$-quadruples for $n_1 \neq n_2$ were found. For instance, $\{27, 115, 160, 1755\}$ is a $D(-2016)$ and $D(37296)$-quadruple, while $\{1458, 66248, 5000, 14112\}$ is a $D(16769025)$ and $D(406425600)$-quadruple (and also a $D(0)$-quadruple since all its elements are twice squares). Moreover, there are infinitely many quadruples with this property. In [144], it was shown that there are infinitely many (essentially different) $D(n)$-quintuples with square elements (so they are also $D(0)$-quintuples). One such example is a $D(480480^2)$-quintuple $\{225^2, 286^2, 819^2, 1408^2, 2548^2\}$.

16.8 Exercises

1. Find at least three right triangles with rational side lengths and the area equal to 6.

2. Determine all congruent numbers less than 30. For each such number n, find at least one right triangle with rational side lengths and the area equal to n.

3. Check that the triangle whose side lengths are

$$\frac{6803298487826435051217540}{411340519227716149383203}, \frac{411340519227716149383203}{21666555693714761309610},$$

$$\frac{224403517704336969924557513090674863160948472041}{8912332268928859588025535178967163570016480830}$$

is a right triangle with the area 157.

4. Prove that any positive integer n of the form $n = m(4m^2 + 1)$, $m \in \mathbb{N}$, is a congruent number.

5. Number 1729 is called the *Hardy-Ramanujan number*. It was named after an anecdote connected to the mathematicians G. H. Hardy and Srinivasa Ramanujan. While the Indian mathematician Ramanujan was in a hospital in London, his colleague Hardy came to visit him. Hardy mentioned that he arrived by a taxi cab number 1729, a rather uninteresting number. Ramanujan replied that he disagrees and said that 1729 is interesting because it is the smallest positive integer which

can be expressed as a sum of cubes of two positive integers in two different ways. Indeed,

$$1729 = 9^3 + 10^3 = 1^3 + 12^3.$$

Prove that for any positive integer M, there is a positive integer m such that the equation $x^3 + y^3 = m$ has at least M integer solutions.

6. Prove that the equation $y^2 = x^3 + 7$ does not have integer solutions.

7. Find an integer k ($k \neq 0$) such that the equation $y^2 = x^3 + k$ has at least 12 integer solutions (x, y).

8. Let $R(x) = c\prod_{i=1}^{s}(x-\rho_i)^{q_i}$, where $q_i \in \mathbb{Z}$, $c \neq 0$, be a rational function. Prove that
$$\frac{R'(x)}{R(x)} = \sum_{i=1}^{s} \frac{q_i}{x - \rho_i}.$$

9. Prove that Davenport's theorem 16.13 also holds if the condition that f and g are relatively prime is omitted.

10. Prove that from the abc conjecture, it follows that for any $\varepsilon > 0$ there is a constant $c(\varepsilon)$ such that for all positive integers x, y with $x^3 \neq y^2$,
$$|x^3 - y^2| \geq c(\varepsilon)x^{1/2-\varepsilon}$$
(Hall's conjecture).

11. Prove the following identity
$$(z^2 + 6z + 4)^3 - (z^2 + 1)(z^2 + 9z + 19)^2 = -27(2z + 11).$$
By using Pell's equation $z^2 - 5w^2 = -1$ and this identity, prove that there are infinitely many pairs of positive integers (x, y) such that
$$|x^3 - y^2| < \sqrt{x}$$
(Danilov, 1982).

12. Determine the coordinates of the points $P \pm S$ on the elliptic curve
$$y^2 = (ax + 1)(bx + 1)(cx + 1)$$
for P and S as in (16.17).

13. Let $ab+1=r^2$, $ac+1=s^2$, $bc+1=t^2$. Prove that the points $S=(1/abc, rst/abc)$, $R=((rs+rt+st+1)/abc, (r+s)(r+t)(s+t)/abc)$ on the elliptic curve $y^2=(ax+1)(bx+1)(cx+1)$ satisfy that $2R=S$.

14. Determine the rank of the elliptic curve
$$y^2=(x+1)(3x+1)(8x+1).$$

15. Find three distinct positive rational numbers x_1, x_2, x_3 such that the numbers x_i+1, $3x_i+1$ and $8x_i+1$ are squares of rational numbers $(i=1,2,3)$.

16. By using the construction from Proposition 16.14, extend the original rational quadruple $\{1/16, 33/16, 17/4, 105/16\}$ found by Diophantus to a rational Diophantine quintuple.

17. Prove that the elliptic curve induced by the rational Diophantine triple $\{3/4, -4/3, -7/12\}$ has torsion group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$.

18. Prove that the $D(-1)$-triple $\{1,5,10\}$ cannot be extended to a $D(-1)$-quadruple.

19. Find positive integers $1<n_2<n_3<n_4$ such that the $D(1)$-triple $\{4,12,420\}$ is also a $D(n_i)$-triple for $i=2,3,4$.

20. Find positive integers $n_1<n_2<n_3<n_4<n_5$ such that $\{6,48,720\}$ is a $D(n_i)$-triple for $i=1,2,3,4,5$.

21. Find positive integers $n_1\neq n_2$ such that $\{-1,7,64,119\}$ is a $D(n_i)$-quadruple for $i=1,2$.

22. Let p be an odd prime. Determine the number of Diophantine pairs and Diophantine triples in $\mathbb{F}_p$, depending on whether p is congruent to 1 or 3 modulo 4 (see [141]).