clarified) connection between the existence of $D(n)$-quadruples and representability of n as a difference of two squares in the considered ring. Note that the integers $n \equiv 2 \pmod 4$ from Theorem 14.12 are exactly those integers which cannot be represented as a difference of two squares of integers (Example 5.2). Let us also mention that Adžaga recently proved that in the ring of integers in an imaginary quadratic field there does not exist a $D(1)$-m-tuple for $m \geq 43$ (see [4]; in the proof, a generalization of the previously mentioned Bennett's result [34] to imaginary quadratic fields from [233] was used).

Different versions of the problem of Diophantus were also studied in the rings of polynomials, starting with the paper [237] by Jones. In [130], it was proved that every Diophantine quadruple in $\mathbb{Z}[x]$ is regular, while in [183], the same result was proved for Diophantine quadruples in $\mathbb{R}[x]$. On the other hand, it was shown in [139] that this statement does not hold in $\mathbb{C}[x]$ because for any $p \in \mathbb{C}[x]$,

$$\left\{\frac{\sqrt{-3}}{2}, -\frac{2\sqrt{-3}}{3}(p^2-1), \frac{-3+\sqrt{-3}}{3}p^2 + \frac{2\sqrt{-3}}{3}, \frac{3+\sqrt{-3}}{3}p^2 + \frac{2\sqrt{-3}}{3}\right\}$$

is a Diophantine quadruple which is not regular.

A short overview of results on Diophantine m-tuples can be found in the review paper [125] and on the web page [126], which also contains a complete list of literature on this topic, which, at the time of writing of this book, contains 458 titles.

14.7 Exercises

1. Find all integer solutions of the equation
$$x^4 - 2x^3y + 2x^2y^2 - 4xy^3 + 4y^4 = 1.$$

2. Find all integer solutions of the equation
$$(x^3 - 2x^2y + y^3)(x^2 - 3y^2) = 1.$$

3. Let $\alpha = \sqrt{2} + \sqrt{3}$. Determine the height $H(\alpha)$ and the Mahler measure $M(\alpha)$.

4. Find all solutions of the system of equations
$$y^2 - 2x^2 = 1, \quad z^2 - 3x^2 = 1$$
in integers x, y, z.

5. Find a cubic polynomial $a_3x^3+a_2x^2+a_1x+a_0$ with integer coefficients such that $|a_i| < 1000$, $i = 0, 1, 2, 3$, which has a root α such that $|\alpha - e| < 10^{-4}$, where $e = 2.718281828459...$ is the basis of the natural logarithm.

6. Find all solutions (in integers x_1, x_2, x_3) of the inequality

$$|x_1 \ln 2 + x_2 \ln 3 + x_3 \ln 7| \leq e^{-X},$$

where $X = \max(|x_1|, |x_2|, |x_3|) \leq X_0 = 10^{15}$.

7. Prove that for any complex number z, we have

$$|e^z - 1| \leq |z| \cdot e^{|z|}.$$

8. Let x and A be real numbers ≥ 3. Prove that $\frac{x}{\ln x} < A$ implies that $x < 2A \ln A$.

9. Prove that for any positive integer n, the set

$$\{F_{2n},\ F_{2n+4},\ 5F_{2n+2},\ 4L_{2n+1}F_{2n+2}L_{2n+3}\}$$

is a Diophantine quadruple.

10. Prove that there are infinitely many Diophantine triples $\{a, b, c\}$ such that $b = (a + c)/2$ (see [103]).

11. Let us denote by $D_m(N)$ the number of Diophantine m-tuples with elements $\leq N$. Prove that $D_2(N) = \frac{6}{\pi^2} N \ln N + O(N)$ and $D_3(N) = \frac{3}{\pi^2} N \ln N + O(N)$ (see [121]).

12. Prove that for any positive integer k, there is a Diophantine triple $\{a, b, c\}$ such that $a \equiv b \equiv c \equiv 0 \pmod{k}$ (see [164]).

13. Let a, b, c be odd integers such that $\{a, b, c\}$ is a $D(n)$-triple for an integer n. Prove that then $a \equiv b \equiv c \pmod{4}$.

14. Prove that there is no integer n and $D(n)$-quadruple $\{a, b, c, d\}$ such that $a \equiv 1 \pmod{4}$, $b \equiv 2 \pmod{4}$, $c \equiv 3 \pmod{4}$ (see [310]).

15. Prove that for any positive integer n, the set

$$\{F_{2n},\ F_{2n+6},\ 4F_{2n+2},\ 4F_{2n+1}F_{2n+3}F_{2n+4}\}$$

is a $D(4)$-quadruple.

16. Prove that for any positive integer $n \geq 2$, the set

$$\{1,\ n^4 - 3n^2 + 1,\ n^2(n^2 - 1),\ 4n^2(n^2 - 1)(n^2 - 2)\}$$

is a $D(n^2)$-quadruple.

17. Prove that for any positive integer n, the set

$$\{F_{2n-1}, F_{2n+1}, F_{2n+3}\}$$

is a $D(-1)$-triple. Prove that there is a positive integer d such that $F_{2n-1}d + 1$, $F_{2n+1}d + 1$ and $F_{2n+3}d + 1$ are perfect squares. Such sets all called $D(-1; 1)$-quadruples.

18. Let x_1, x_2, x_3 be rational numbers such that the denominator of

$$x_4 = \frac{8(x_3 - x_1 - x_2)(x_1 + x_3 - x_2)(x_2 + x_3 - x_1)}{(x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3)^2}$$

is not equal to 0. Prove that $x_1x_4 + 1$, $x_2x_4 + 1$ and $x_3x_4 + 1$ are squares of rational numbers.

19. Let $k \in \mathbb{Z} \setminus \{-1, 0\}$. Prove that

$$\{1, 9k^2 + 8k + 1, 9k^2 + 14k + 6, 36k^2 + 44k + 13\}$$

is a $D(4k + 3)$-quadruple.

20. Prove that for any positive integer k, the sets

$$\{k, 9k + 8, 16k + 14, 25k + 20\} \quad \text{and} \quad \{k, 4k + 2, 9k + 8, 25k + 20\}$$

are $D(10k + 9)$-quadruples.

21. Find one $D(256)$-quintuple and one $D(-255)$-quintuple. (Let us mention that it is not known whether there exists a $D(n)$-quintuple for an integer n such that $-255 < n < 0$ or $0 < n < 256$.)

22. Find three positive integers a, b, c such that $ab + 1$, $ac + 1$ and $bc + 1$ are cubes of positive integers.

23. Find positive integers a, b, c, x, y, z such that $ab + 1 = x^4$, $ac + 1 = y^4$ and $bc + 1 = z^4$.

24. Prove that the number $a+bi \in \mathbb{Z}[i]$ cannot be expressed as a difference of squares of two elements in $\mathbb{Z}[i]$ if and only if b is odd or $a \equiv b \equiv 2 \pmod{4}$.

25. Determine all elements in the ring $\mathbb{Z}[(1+\sqrt{-3})/2]$ which can be expressed as a difference of squares of two elements of that ring.

26. Prove that for any integer n, there is an infinite set of integers A_n with the property that for all $a, b \in A_n$, the number $ab+n$ is not square-free.