for sufficiently large a. Since the height of $P_{n,a}(x)$ is bounded from above by the product of a^3 and a number depending only on n, when we let that $a \to \infty$, we obtain

$$e(n) \geq \frac{2n-1}{3}. \qquad \square$$

Let us mention another variant of the problem of polynomial root separation. That is the question of how close can the absolute values of two real roots of a polynomial with integer coefficients be if those absolute values are distinct. In the paper [68], a complete answer to the question of the best possible separation exponent was obtained. On the one hand, an analogue of Mahler's theorem was proved, showing that

$$||\alpha| - |\beta|| \gg H(P)^{-n+1}.$$

On the other hand, explicit examples which demonstrate that the exponent $-n+1$ cannot be improved were constructed. For example, if $n \geq 4$ is even, then the polynomial

$$q_{n,a}(x) = x^n - (ax^2 - 1)(1 - x^{n-3})$$

has two real roots α, β, such that

$$\begin{aligned} \alpha &\approx -a^{-1/2} - \frac{1}{2}a^{-(n+1)/2} + \frac{1}{2}a^{-n+1}, \\ \beta &\approx a^{-1/2} + \frac{1}{2}a^{-(n+1)/2} + \frac{1}{2}a^{-n+1}, \end{aligned}$$

so that

$$||\alpha| - |\beta|| = |\alpha + \beta| \approx a^{-n+1} \ll H(P)^{-n+1}.$$

Results on other variants of the problem of polynomial root separation can be found in [38], [59], [67], [69], [107], [108], [151], [178, Chapter 18.1] and [340].

13.6 Exercises

1. Let $g(x)$ be the minimal polynomial of an algebraic number α, let m be a positive integer such that the coefficients of the polynomial $m \cdot g(x)$ are integers and let $\alpha^{(1)} = \alpha$, $\alpha^{(2)}$, ... , $\alpha^{(d)}$ be the roots of g. Prove that the constant $c(\alpha)$ in Liouville's theorem can be taken to be

$$c(\alpha) = m^{-1} \prod_{j=2}^{d} (1 + |\alpha| + |\alpha^{(j)}|)^{-1}.$$

2. Prove that the numbers $\sum_{n\geq 1} a_n 10^{-n!}$, with $a_n \in \{1,2\}$, are Liouville numbers.

3. Prove that, for all positive integers p, q,
$$\left|\frac{\sqrt{2}}{2} - \frac{p}{q}\right| > \frac{1}{4q^2}\,.$$

4. Prove that the statement of Roth's theorem holds for all $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

5. Let α be an algebraic number of degree $d \geq 2$ and $\kappa > 2$. Prove that there is a constant $c = c(\alpha, \kappa) > 0$ such that
$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^\kappa}$$
for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

6. Determine the coefficients of the polynomial $F\left(\begin{matrix}-3,\ \beta \\ \gamma\end{matrix}\,\middle|\, x\right)$.

7. Prove that the derivative of the hypergeometric function $F\left(\begin{matrix}\alpha,\ \beta \\ \gamma\end{matrix}\,\middle|\, x\right)$ is equal to
$$\frac{\alpha\beta}{\gamma} F\left(\begin{matrix}\alpha+1,\ \beta+1 \\ \gamma+1\end{matrix}\,\middle|\, x\right).$$

8. Prove that the hypergeometric function $F = F\left(\begin{matrix}\alpha,\ \beta \\ \gamma\end{matrix}\,\middle|\, x\right)$ satisfies the differential equation
$$x(x-1)F'' + ((1+\alpha+\beta)x - \gamma)F' + \alpha\beta F = 0\,.$$

9. Let a, b be positive integers such that $\frac{7}{8}a \leq b < a$. Prove that, for $r = 1, 2, 3, 4$,
$$\sum_{k=0}^{\lfloor r/2\rfloor} \binom{r+k}{k}^2 \binom{2r+k+1}{k}^{-1} \left(\frac{a-b}{a}\right)^k < \left(\frac{a}{b}\right)^r.$$

10. Prove that, for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$, we have
$$\left|\sqrt[3]{19} - \frac{p}{q}\right| > \frac{10^{-7}}{q^{2.56}}\,.$$

11. Let $P(x) = x^4 - 14x^3 + 56x^2 - 52x + 12$. Determine the Mahler measure $M(P)$.

12. Prove that the determinant of the Vandermonde matrix $V = (V_{i,j})$, $V_{i,j} = \alpha_i^{j-1}$, is equal to $\prod_{1 \le i < j \le n}(\alpha_j - \alpha_i)$.

13. Prove that the Catalan numbers satisfy the recurrence

$$c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}, \quad n \ge 0.$$

14. Prove that the polynomial $(x^3 + ax + 1)(x^2 - a^2 x - a)$ has two real roots whose difference is $\approx a^{-7}$.

15. Prove that the polynomial $(x^2 + x - 1)(x^2 + (1 + F_{n+1})x - (F_n + 1))$ has two real roots whose difference is $\approx \frac{1}{\sqrt{5}} F_{n+1}^{-2}$.

16. Polynomials

$$s_{n,a}(x) = (-1)^{n-1}(F_{n-1} p_{n,a}(x) - F_n x p_{n-1,a}(x))$$

are defined using the polynomials $p_{n,a}$ from Theorem 13.16. Prove that polynomials $s_{n,a}(x)$ are monic.

17. Prove that the polynomial $x^3 + ax^2 - 1$ has two real roots whose sum is $\approx -a^{-2}$.