By the inductive assumption, we have $h_k(x_1, \dots, x_{n-1}, 0) = 0$, which implies that the polynomial $h_k(x_1, \dots, x_n)$ is divisible by $x_n$, so due to symmetry, it is also divisible by $\sigma_n$. On the other hand, the degree of the polynomial $h_k$ is $\leq k < n$, so it cannot be divisible by $\sigma_n$, unless $h_k = 0$. □

Examples of the application of symmetric polynomials to solving symmetric equations and systems of symmetric equations can be found in [216, 334, 336].

## 11.6 Exercises

1. Let $A$ be an integral domain and $f \in A[x]$. Prove that $f$ is invertible in $A[x]$ if and only if $\deg f = 0$ and $f(0)$ is invertible in $A$.

2. Let $A$ be a unique factorization domain. Prove that a polynomial $f \in A[x]$ is primitive if and only if $\mathrm{cont}(f)$ is invertible in $A$.

3. Prove that for $n \geq 5$ there is no polynomial $P(x)$ of degree $n$ with integer coefficients with the property that $P(0) = 0$ and that $P(a_i) = n$ for $n$ distinct integers $a_i$, $i = 1, \dots, n$. Is there a polynomial with this property for $n = 4$?

4. Let $A$ be an integral domain and let $f, g, h \in A[x]$. Prove:

   a) $\mathrm{Res}(f, 0) = 0$,
   b) $\mathrm{Res}(f, g) = (-1)^{\deg f \cdot \deg g} \mathrm{Res}(g, f)$,
   c) $\mathrm{Res}(f, gh) = \mathrm{Res}(f, g) \mathrm{Res}(f, h)$.

5. Let $f(x) = x^3 + ax + b$. Prove that $\mathrm{Disc}(f) = -4a^3 - 27b^2$.

6. Prove that there is no polynomial $f(x) \in \mathbb{Z}[x]$, $\deg f \geq 1$ such that $f(n)$ is square-free for every $n \in \mathbb{Z}$.

7. Solve the equations:

   a) $6x^3 + 19x^2 + x - 6 = 0$,
   b) $12x^3 - 4x^2 - 25x + 12 = 0$,
   c) $18x^4 + 69x^3 + 41x^2 - 16x - 12 = 0$.

8. Prove that the equation $x^4 + ax + 1 = 0$ does not have rational solutions if $a \in \mathbb{Z}$, $a \neq \pm 2$.

9. Find all prime numbers $p$ for which the equation $px^3 - x + 2 = 0$ has at least one rational root.

10. Find all integers $m$ such that the polynomial $P_m(x) = x^4 - mx^3 - 6x^2 + mx + 1$ is reducible over $\mathbb{Q}$.

11. Prove that Eisenstein's criterion (Theorem 11.15) also holds if the condition that the polynomial $f$ is monic is replaced by the condition that the leading coefficient of the polynomial $f$ is not divisible by $p$.

12. Let $f \in \mathbb{Z}[x]$ be a polynomial of degree 7 and let $|f(a_i)| = 1$ for five distinct integers $a_1, a_2, a_3, a_4, a_5$. Prove that $f$ cannot be written as the product of two polynomials with integer coefficients of positive degree.

13. Let $n \geq 6$ and $a_1, \ldots, a_n$ be integers. Prove that the polynomial
$$f(x) = (x - a_1) \cdot \ldots \cdot (x - a_n) + 1$$
is irreducible in $\mathbb{Z}[x]$.

14. Let $a_1, a_2, a_3, a_4$ be integers. Find an example of a polynomial
$$f(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4) + 1$$
that is not irreducible over $\mathbb{Z}$.

15. Let $n$ be a positive integer. Prove that the polynomial
$$f(x) = (x^2 + 2)^n + 5x^{2n-1} + 100x^n + 25$$
is irreducible in $\mathbb{Z}[x]$.

16. Let the coefficients of a polynomial $f(x) = a_n x^n + \cdots + a_0 \in \mathbb{Z}[x]$ be such that $a_0$ is prime and $|a_0| > |a_1| + \cdots + |a_n|$. Prove that $f$ is irreducible in $\mathbb{Z}[x]$.

17. Express the polynomial
$$f(x) = x^8 + 4x^6 - 8x^2 - 2$$
as a composition of indecomposable polynomials.

18. Let

$$g_1(x) = x^3, \quad g_2(x) = \frac{x(x-12)}{x-3}, \quad g_3 = \frac{x(x+6)}{x-3},$$
$$h_1(x) = \frac{x^3(x+24)}{x-3}, \quad h_2(x) = \frac{x(x^2-6x+36)}{x^2+3x+9}.$$

Prove that $g_1 \circ g_2 \circ g_3 = h_1 \circ h_2$ and explain why this equality implies that the first Ritt's theorem does not hold for decompositions of rational functions (the degree of a rational function $f/g$ is defined as $\max(\deg f, \deg g)$).

19. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of the polynomial $x^4+5x^3+kx^2+3x+5$. Find the coefficient $k$ if it is known that $\alpha_1 \cdot \alpha_2 = 1$.

20. Let $\alpha_1, \alpha_2, \alpha_3$ be the roots of the polynomial $x^3 + 5x^2 + 7x + 11$. Find the polynomial of degree 3 whose roots are $\beta_1, \beta_2, \beta_3$, where

$$\beta_1 = \frac{\alpha_1+\alpha_2}{2}, \quad \beta_2 = \frac{\alpha_2+\alpha_3}{2}, \quad \beta_3 = \frac{\alpha_3+\alpha_1}{2}.$$

21. Solve the (symmetric) equation $3x^4 - 13x^3 + 16x^2 - 13x + 3 = 0$.

22. Solve the system of (symmetric) equations

$$\begin{aligned} x + y + z &= 2, \\ x^2 + y^2 + z^2 &= 14, \\ x^3 + y^3 + z^3 &= 20. \end{aligned}$$

23. Factorize the (symmetric) polynomial

$$f(x, y, z) = x^3 + y^3 + z^3 - 3xyz.$$