Hence, we can apply Theorem 10.39 for $n = 3$. We conclude that there are $c_1, c_2, c_3, c_4 \in \mathbb{Q}$, such that

$$a = a_1 c_1^2 + a_2 c_2^2, \quad a = -a_3 c_3^2 - a_4 c_4^2,$$

which needed to be proved. □

The proof of Theorem 10.39 for $n \geq 5$:
It is sufficient to consider forms

$$f(x_1, \ldots, x_n) = a_1 x_1^2 + \cdots + a_n x_n^2,$$

where a_i are relatively prime and square-free, and $a_1 > 0$, $a_n < 0$. Suppose that the statement holds for all forms with less than n variables. Let us denote

$$g(x_1, x_2) = a_1 x_1^2 + a_2 x_2^2, \quad h(x_3, \ldots, x_n) = -a_3 x_3^2 - \cdots - a_n x_n^2.$$

As in the case $n = 4$, by Dirichlet's theorem, we can find a positive integer a which is represented by g and by h in $\mathbb{R}$ and $\mathbb{Q}_p$, with the possible exception of one prime number q which does not divide the coefficients a_i. Then, from the properties of ternary quadratic forms, it follows that g represents a also in $\mathbb{Q}_q$. Since $n - 2 \geq 3$, the form h represents zero in $\mathbb{Q}_q$ (the Chevalley-Warning theorem and Hensel's lemma), so it also represents the number a. Now we can apply the inductive assumption to the forms $-ax_0^2 + g$ and $-ax_0^2 + h$, so we conclude that g and h represent a in $\mathbb{Q}$. □

Remark 10.3. It can be shown that for $n \geq 5$, any quadratic form f represents zero in $\mathbb{Q}_p$ for all primes p (see [311, Chapter 7]). Therefore, for $n \geq 5$, a sufficient condition for the solvability of the equation $f(x_1, \ldots, x_n) = 0$ in $\mathbb{Q}$ is that the equation has solutions in $\mathbb{R}$.

10.9 Exercises

1. Find all solutions of the equation $15x + 7y = 210$ in integers. Find all solutions of the same equation in positive integers.

2. In which ways can we pay the postage of 50 cents by using stamps of 3 and 5 cents?

3. Find all solutions of the equation $7x + 12y - 5z = 100$ in integers.

4. Find all positive integers x, y, z which satisfy the system of equations $2x + 3y + 5z = 201$, $2x + 5y + 7z = 315$.

5. Determine the Frobenius number $g(6, 9, 20)$.

6. Find all Pythagorean triples whose members are three consecutive elements of a geometric progression.

7. Find all Pythagorean triangles such that the length of one side is equal to:
a) 21, b) 34, c) 119, d) 2019.

8. Find all primitive Pythagorean triangles such that the length of one side is equal to:
a) 35, b) 85, c) 100, d) 125.

9. Let A, B, C and a, b, c be the lengths of the legs and the hypotenuse of two primitive Pythagorean triangles. Prove that it either exists an integer F such that
$$Aa + Bb + Cc = F^2$$
or an integer G such that
$$Aa + Bb + Cc = 2G^2.$$

10. Prove the formula $(F_nF_{n+3})^2 + (2F_{n+1}F_{n+2})^2 = (F_n^2 + 2F_{n+1}F_{n+2})^2$ (see [222]).

11. Prove that for any positive integer $n \geq 3$, there is a Pythagorean triple with one member equal to n.

12. Prove that in every Pythagorean triangle, the length of the radius of the inscribed circle is a positive integer.

13. Prove that there are infinitely many Pythagorean triangles such that the length of the radius of the circumscribed circle is a positive integer.

14. Find at least one right triangle whose lengths of sides are rational numbers and which has the property that its perimeter is equal to the square of a rational number, and the sum of perimeter and area is a cube of a rational number.

15. Let $n \neq 3$ be a positive integer. Prove that there are no positive integers x, y, z, which are three consecutive elements of an arithmetic progression, such that $x^n + y^n = z^n$ (this result was proved in [303]).

16. Find the smallest solutions in positive integers of the equations:

 a) $x^2 - 13y^2 = \pm 1$,
 b) $x^2 - 14y^2 = \pm 1$,
 c) $x^2 - 31y^2 = \pm 1$.

17. Find the fundamental solution of Pell's equation $x^2 - 991y^2 = 1$.

18. Prove that for an arbitrary $k \in \mathbb{Z}$, the equation $x^2 - (k^2 - 1)y^2 = -1$ does not have solutions in integers x and y.

19. Let m be an arbitrary positive integer. Prove that there are infinitely many solutions of Pell's equation $x^2 - dy^2 = 1$ which satisfy the additional condition that $y \equiv 0 \pmod{m}$.

20. Let (x_n, y_n) be the increasing sequence of solutions of Pell's equation $x^2 - dy^2 = 1$ in positive integers. Prove that for all positive integers m, n, we have:

$$\begin{aligned} x_{m+n} &= x_m x_n + dy_m y_n, \\ y_{m+n} &= x_m y_n + x_n y_m, \\ \frac{x_{2m}}{y_{2m}} &= \frac{1}{2}\left(\frac{x_m}{y_m} + \frac{dy_m}{x_m}\right). \end{aligned}$$

21. Determine all positive integers less than $1\,000$ such that the arithmetic mean of their square and their double value is a perfect square.

22. Find ten primitive Pythagorean triangles with the lengths of legs which are consecutive positive integers.

23. Prove that there are infinitely many positive integers n such that the sum of the first n positive integers is a perfect square. Find six smallest positive integers with this property.

24. Prove that the congruence $x^2 - 34y^2 \equiv -1 \pmod{p}$ has solutions for every prime number p. Find at least one positive integer d ($d \neq 34$ and d is not a perfect square) with the property that the congruence $x^2 - dy^2 \equiv -1 \pmod{p}$ has a solution for every prime number p, but that the equation $x^2 - dy^2 = -1$ does not have solutions in integers.

25. Prove that the solutions (x_n, y_n) of the equation $x^2 - dy^2 = 4$ satisfy the recurrences:

$$x_{n+2} = x_1 x_{n+1} - x_n,$$
$$y_{n+2} = x_1 y_{n+1} - y_n.$$

26. Let x, y be odd positive integers such that $x^2 - dy^2 = -4$. Prove: if $d \equiv 5 \pmod{16}$, then $x \equiv \pm y \pmod{8}$, and if $d \equiv 13 \pmod{16}$, then $x \equiv \pm 3y \pmod{8}$.

27. Let $k \geq 3$ be an odd integer. Find the fundamental solution of Pell's equation $x^2 - (k^2 + 4)y^2 = 1$.

28. Let p be an odd prime number and assume the equation $x^2 - dy^2 = p$ has a solution. Prove that all solutions of that equation are in the same class if and only if p divides d. Provide an example of a positive integer d and prime number p such that the equation $x^2 - dy^2 = p$ has precisely one (ambiguous) class of solutions.

29. Prove that the equation $x^2 - 82y^2 = 23$ does not have solutions in integers.

30. Determine all fundamental solutions of the equation $x^2 - (k^2 + 1)y^2 = k^2$ for $k = 3, 4, 5, 6, 7, 8$ (cf. [293]).

31. Let $k > 1$ be an odd integer. Prove that the equation $x^2 - (k^2 - 4)y^2 = 4k$ has no solutions in coprime integers (x, y) (see [283]).

32. Find a positive integer d such that the equation $x^2 - dy^2 = 65$ has precisely four classes of solutions.

33. Find ten primitive Pythagorean triangles such that the lengths of their legs differ by 7.

34. Is there a primitive Pythagorean triangle such that the lengths of its legs differ by 10?

35. Let m and n be positive integers less than $2\,000$ which satisfy the equation $(m^2 - mn - n^2)^2 = 1$. Determine the largest possible value of the expression $m^2 + n^2$.

36. Prove that for any positive integer n, the following identity holds

$$L_{n+1}^2 - L_{n+1}L_n - L_n^2 = 5 \cdot (-1)^n,$$

and prove that all solutions of the Diophantine equation

$$(u^2 - uv - v^2)^2 = 25$$

are given by $u = L_{n+1}$, $v = L_n$ (here L_n denotes the n-th Lucas number).

37. Prove that all solutions of the Diophantine equation $F_n = 2x^2$ are given by $n = 0, \pm 3$ or 6, i.e. $x = 0, \pm 1$ or ± 2.

38. Solve the following Diophantine equations:

    a) $y^2 = 5x^4 + 4$,
    b) $y^2 = 5x^4 - 4$,
    c) $y^2 = 5x^4 + 1$,
    d) $y^2 = 5x^4 - 1$.

39. Examine whether the equation $x^2 - 5y^2 - 91z^2 = 0$ has non-trivial integer solutions.

40. Find an equation of the form $ax^2 + by^2 + cz^2 = 0$, abc square-free, which is equivalent to the equation

$$3x^2 + 5y^2 + 7z^2 + 9xy + 11yz + 13zx = 0$$

and examine whether that equation has non-trivial integer solutions.

41. Determine the smallest positive integer c with the following three properties:

    1) c is square-free,
    2) c is relatively prime both to 7 and to 15,
    3) the equation $-7x^2 + 15y^2 + cz^2 = 0$ has a non-trivial solution.

42. Find at least one non-trivial integer solution of the equation

$$2003x^2 - 3001y^2 - 4091z^2 = 0.$$

43. Let a, b, c be rational numbers different from zero. Prove that if the equation $ax^2 + by^2 + cz^2 = 0$ has a non-trivial rational solution, then for any rational number α, the equation $ax^2 + by^2 + cz^2 = \alpha$ has a rational solution.

44. Find the parametric formulas which give all solutions of the equation

$$x^2 + 2y^2 = z^2$$

in relatively prime integers.

45. Let a be an odd integer. Prove that the congruence

$$x^2 \equiv a \pmod{2^k}$$

has solutions for every positive integer k if and only if $a \equiv 1 \pmod 8$.

46. Find all solutions of the congruences:

    a) $x^2 \equiv 41 \pmod{125}$,
    b) $x^2 \equiv 41 \pmod{128}$.

47. Prove that the congruence $x^2 + 23y^2 \equiv 41 \pmod m$ has solutions for every positive integer m, but that the equation $x^2 + 23y^2 = 41$ does not have solutions in integers.

48. Prove that the equation

$$(x^2 - 13y^2)(x^2 - 17y^2)(x^2 - 221y^2) = 0$$

has non-trivial solutions in $\mathbb{R}$ and $\mathbb{Q}_p$ for every p, but does not have any solution in $\mathbb{Q}$.