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★**Diophantine m-tuples and elliptic curves.**

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This book explores the main historical and contemporary results, along with open problems, related to Diophantine m-tuples. An integer (or rational) Diophantine m-tuple is a set of m positive integers (or rational numbers) with the property that the product of any two distinct elements, increased by 1, is a perfect square.

The first chapter introduces the foundational contributions of Diophantus, Fermat, and Euler, who were pioneers in studying this topic. It provides a comprehensive overview of key definitions, significant results, and unresolved questions discussed throughout the book. Additionally, it examines several generalizations of the concept of Diophantine m-tuples.

The second chapter delves into the connections between Diophantine m-tuples and elliptic curves, with a particular focus on problems and algorithms concerning the torsion group and rank.

The connections between Diophantine m-tuples and elliptic curves are examined in detail in the third chapter. The chapter demonstrates how certain classical results, such as the extension of quadruples to rational quintuples, can be interpreted through the lens of elliptic curves. The chapter also highlights the crucial role elliptic curves play in constructing infinite families of rational Diophantine sextuples. Furthermore, the chapter discusses the application of Diophantine m-tuples in the construction of elliptic curves with large rank and specified torsion groups, showcasing their importance in advancing research in this field.

The fourth chapter focuses on general methods for finding integer points on elliptic curves and their application in determining all integer points on elliptic curves arising from Diophantine triples. Additionally, the chapter presents a proof of an absolute upper bound for the size of Diophantine tuples. It also outlines the proof of the non-existence of Diophantine quintuples, offering significant insights into this long-standing problem.

The last chapter explores m-tuples with the property $D(n)$. These are sets of m distinct nonzero integers (or rational numbers) $\{a_1, \ldots, a_m\}$ such that $a_i a_j + n$, where n is a fixed integer (or rational number), is a perfect square for all $1 \le i < j \le m$. The chapter examines the existence of integer $D(n)$-quadruples and rational $D(n)$-quintuples, as well as their connection to the distribution of ranks in families of twists of certain elliptic curves. It also includes a detailed study of $D(n)$-triples, quadruples, and quintuples for several specific values of n.

This book provides the essential motivational material for those new to the field of Diophantine m-tuples, effectively guiding readers to the forefront of current research. Graduates and researchers in number theory will find it an invaluable resource. Moreover, the book is engaging, well written, and includes numerous insightful exercises, making it an excellent tool for deepening understanding and fostering further exploration of this fascinating subject.

Dimitrios Poulakis