TRIPLES, QUADRUPLES AND QUINTUPLES WHICH ARE $D(n)$-SETS FOR SEVERAL n'S

ANDREJ DUJELLA

Dedicated to Professor Michel Waldschmidt on the occasion of his 75th birthday.

Abstract. For an integer n, a set of distinct nonzero integers $\{a_1, a_2, ..., a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$, is called a Diophantine m-tuple with the property $D(n)$ or simply a $D(n)$-set. $D(1)$-sets are also called Diophantine m-tuples. The first Diophantine quadruple, the set $\{1, 3, 8, 120\}$ was found by Fermat. He, Togbé and Ziegler proved in 2019 that there does not exist a Diophantine quintuple. On the other hand, it is known that there exist infinitely many rational Diophantine sextuples. When considering $D(n)$-sets, usually an integer n is fixed in advance. However, we may ask if a set can have the property $D(n)$ for several different n's. For example, $\{8, 21, 55\}$ is a $D(1)$-triple and $D(4321)$-triple. In joint work with Adžaga, Kreso and Tadić, we presented several families of Diophantine triples, which are $D(n)$-sets for two distinct n's with $n \neq 1$. In joint work with Petričević we proved that there are infinitely many (essentially different) quadruples which are simultaneously $D(n_1)$-quadruples and $D(n_2)$-quadruples with $n_1 \neq n_2$. Moreover, the elements in some of these quadruples are squares, so they are also $D(0)$-quadruples. E.g. $\{54^2, 100^2, 168^2, 364^2\}$ is a $D(8190^2)$, $D(40320^2)$ and $D(0)$-quadruple. In recent joint work with Kazalicki and Petričević, we considered $D(n)$-quintuples with square elements (so they are also $D(0)$-quintuples). We proved that there are infinitely many such quintuples. One example is a $D(480480^2)$-quintuple $\{225^2, 286^2, 819^2, 1408^2, 2548^2\}$. In this survey paper, we describe methods used in constructions of mentioned triples, quadruples and quintuples.

1. Introduction

A *Diophantine m-tuple* is a set of m distinct positive integers with the property that the product of any two of its distinct elements plus 1 is a perfect square. The first example of a Diophantine quadruple was found by Fermat, and it was the set $\{1, 3, 8, 120\}$,

A *rational Diophantine m-tuple* is a set of m distinct nonzero rational numbers with the property that the product of any two of its distinct elements plus 1 is the square of a rational number. Euler was able to extend Fermat's quadruple to the rational quintuple $\{1, 3, 8, 120, \frac{777480}{8288641}\}$.

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The ancient Greek mathematician Diophantus of Alexandria was the first one who studied sets with this property. In the fourth part of his book *Arithmetica* [31], Exercise no. 20 states:

> *Find four numbers (for Diophantus, this meant positive rational numbers) such that the product of any two among them, increased by* 1 *gives a square.*

Let us describe how Diophantus solved this exercise (of course, by using contemporary mathematical notation). Two numbers with the required property can be obtained by taking $a = x$ and $b = x + 2$, so $ab + 1 = (x + 1)^2$. A pair $\{a, b\}$ such that $ab + 1 = r^2$, can be extended to a triple by taking $c = a + b + 2r$. Indeed, then $ac + 1 = (a + r)^2$, $bc + 1 = (b + r)^2$. In this manner, we obtain $c = 4x + 4$. Now, we apply the same construction to the pair $\{a, c\}$ and the equality $ac + 1 = (2x + 1)^2$. We obtain $d = x + (4x + 4) + 2(2x + 1) = 9x + 6$. In this manner, we obtain the set $\{a, b, c, d\}$ which satisfies five out of six conditions from the definition of a Diophantine quadruple. The only missing condition is that $bd + 1$ is a square. Hence, it remains to find a rational solution of the equation $9x^2 + 24x + 13 = y^2$. Diophantus knew how to solve the equations of the form $\alpha^2x^2 + \beta x + \gamma = y^2$, by putting $y = \alpha x + t$. Thus, he searched for a solution in the form $y = 3x + t$, and after the substitution, he would obtain a linear equation in variable x. He did not search for a general solution to the equation; instead, he would introduce a concrete value and obtain one solution. In this case, he took $y = 3x - 4$ and obtained the equation $48x = 3$ and the solution $x = \frac{1}{16}$. Thus, he found the first example of what we call nowadays the rational Diophantine quadruple $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$.

It is natural to ask how large can these sets be. This question was recently completely solved in the integer case. On the one hand, it is easy to show that there are infinitely many integer Diophantine quadruples. Namely, there are parametric families of Diophantine quadruples involving polynomials and Fibonacci numbers, such as $\{k, k+2, 4k+4, 16k^3+48k^2+44k+12\}$ and $\{F_{2k}, F_{2k+2}, F_{2k+4}, 4F_{2k+1}F_{2k+2}F_{2k+3}\}$ for $k \geq 1$. On the other hand, recently, He, Togbé and Ziegler [30] proved that there is no Diophantine quintuple, and so they solved a long-standing open problem. Previously, Dujella [12] proved in 2004 that there is no Diophantine sextuple and that there are at most finitely many Diophantine quintuples. Let us mention that Baker and Davenport [4] in 1969 obtained the first important result concerning this problem. Using linear forms in logarithms of algebraic numbers and a reduction method based on continued fractions, they proved that if d is a positive integer such that $\{1, 3, 8, d\}$ is a Diophantine quadruple, then d has to be 120, so that Fermat's set $\{1, 3, 8, 120\}$ cannot be extended to a Diophantine quintuple. So they answered the question raised by Denton [9], Gardner [26] and van Lint [35].

It is known that any Diophantine triple $\{a, b, c\}$ can be extended to a Diophantine quadruple $\{a, b, c, d\}$. Indeed, if $ab+1 = r^2$, $ac+1 = s^2$, $bc+1 = t^2$, where $r, s, t \in \mathbb{N}$, then we may take $d = a+b+c+2abc+2rst$, and then $ad+1 = (at+rs)^2$, $bd+1 = (bs+rt)^2$, $cd+1 = (cr+st)^2$. This was shown by Arkin, Hoggatt and Strauss [3] in 1979, and in the special case, when $c = a+b+2r$, this was already known to Euler. Quadruples of this form are called *regular*. In other words, a Diophantine quadruple $\{a, b, c, d\}$ is regular if and only if $(a+b-c-d)^2 = 4(ab+1)(cd+1)$. The conjecture that all Diophantine quadruples are regular is still open. Hence, the conjecture is that for each Diophantine triple $\{a, b, c\}$, there is a unique positive integer d, such that $d > \max(a, b, c)$ and that $\{a, b, c, d\}$ is a Diophantine quadruple. Let us mention that Cipu, Fujita and Miyazaki [8] proved that any fixed Diophantine triple $\{a, b, c\}$ can be extended to a Diophantine quadruple with an element d such that $d > \max(a, b, c)$ in at most eight ways.

In the rational case, the question is entirely open, and we do not even have a widely accepted conjecture on how many elements a rational Diophantine m-tuple can contain. In particular, no absolute upper bound for the size of rational Diophantine m-tuples is known. Euler proved that there are infinitely many rational Diophantine quintuples. However, the question of the existence of rational Diophantine sextuples remained open for more than two centuries. In 1999, Gibbs [27] found the first rational sextuple $\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\}$, while in 2017, Dujella, Kazalicki, Mikić and Szikszai [18] proved that there exist infinitely many rational Diophantine sextuples. Alternative constructions of infinite families of rational Diophantine sextuples are given in [17, 19, 21]. All these constructions use various connections between rational Diophantine m-tuples and elliptic curves. Such connections can be used in the construction of high rank elliptic curves with given torsion group (see e.g. [22, 23]). No example of a rational Diophantine septuple is known and whether there is such septuple is an open problem. Let us mention that recently Stoll [38] proved that Euler's extension of Fermat's quadruple to the rational quintuple by the fifth number $\frac{777480}{8288641}$ is unique. In particular, this means that this quintuple cannot be extended to a rational Diophantine sextuple.

There are several natural generalizations of the notion of Diophantine m-tuples. The generalization which is the most relevant for the present paper is replacing number 1 in the condition "$ab + 1$ is a square" by a fixed integer or rational n. For a (nonzero) integer n, a set of m distinct nonzero integers $\{a_1, a_2, \ldots, a_m\}$ such that $a_i a_j + n$ is a perfect square for all $1 \leq i < j \leq m$, is called a *Diophantine m-tuple with the property $D(n)$* or a *$D(n)$-m-tuple* or simply a *$D(n)$-set*. Note that a Diophantine m-tuple is a $D(1)$-set. If elements of such an m-tuple are nonzero rationals, we call such set a *rational $D(n)$-m-tuple*. Usually, $n = 0$ is excluded from the definition of $D(n)$-m-tuples since it is trivial to see that there are infinite $D(0)$-sets (we may take any number of perfects squares or squares multiplied with the same number), but when we combine this condition with other nontrivial conditions, such as we will do in Sections 4 and 5, interesting problems may appear so in that context it make sense to allow $n = 0$ in the definition.

Again, we may ask how large such sets can be. Apart from the case $n = 1$, the most studied cases are $n = 4$ and $n = -1$. Bliznac Trebješanin and Filipin proved that there is no $D(4)$-quintuple [5], while Bonciocat, Cipu and Mignotte proved that there is no $D(-1)$-quadruple. It is easy to show that there are no $D(n)$-quadruples if $n \equiv 2 \pmod 4$. This result was proved independently in 1985 by Brown, Gupta and Singh, and Mohanty and Ramasamy [7, 28, 36]. On the other hand, it was shown in [11] that if $n \not\equiv 2 \pmod 4$ and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there is at least one $D(n)$-quadruple. The mentioned result of Bonciocat, Cipu and Mignotte solves the cases $n = -1$ and $n = -4$, while for other elements of the set S, the question of the existence of $D(n)$-quadruples is still open.

Concerning analogous questions in the rational case (when we usually write q instead of n), it is known that for any rational number q there exist infinitely many rational $D(q)$-quadruples. This result was probably already known to Diophantus, who in the fifth part of *Arithmetica* gave explicit examples of rational $D(q)$-quadruples for $q = 5$ and $q = -6$, but his method easily generalizes to arbitrary

q. Concerning quintuples, Dujella and Fuchs [16] proved that for infinitely many square-free numbers q there are infinitely many rational $D(q)$-quintuples. Assuming the Parity Conjecture for twists of certain elliptic curves, they showed that the density of $q \in \mathbb{Q}$ such that there exist infinitely many rational $D(q)$-quintuples is at least $1/2$; the density bound is recently improved to at least $49171/49335 \approx 0.995$ by Dražić [10].

A brief survey on Diophantine m-tuples and their generalizations can be found in [13] (see also [14]). Last edition of the well-known book *Unsolved Problems in Number Theory* by Richard Guy [29] contains a new section devoted to Diophantine m-tuples, and the problem of the existence of Diophantine quintuples is mentioned in 2001 by Michel Waldschmidt [39] as one of the important problems at the end of the second millennium. The author's web page [15] contains the full list of references related to Diophantine m-tuples. At the moment, it contains 484 references (38 references before 1990; 95 references before 2000; 245 references before 2010).

2. Triples which are $D(n)$-sets for several n's

When considering $D(n)$-m-tuples, usually an integer n is fixed in advance. However, we may ask if the same set can at the same time have the property $D(n)$ for several different n's. This question was raised by A. Kihel and O. Kihel [33] in 2001. For example, $\{8, 21, 55\}$ is at the same time a $D(1)$-triple and $D(4321)$-triple, while $\{1, 8, 120\}$ is a $D(1)$-triple and D(721)-triple (see [40]).

In this section, we will sketch a joint work with N. Adžaga, D. Kreso and P. Tadić [1, 2], where we proved that there are infinitely many triples $\{a, b, c\}$ which are at the same time $D(1)$, $D(n_2)$ and $D(n_3)$ triples for $1 < n_2 < n_3$.

The construction uses integer points on elliptic curves related to Diophantine triples. Let $\{a, b, c\}$ be a Diophantine triple and let $ab + 1 = r^2$, $ac + 1 = s^2$, $bc + 1 = t^2$. We are interested in integer solutions x of the system of equations

$$x + ab = \square, \quad x + ac = \square, \quad x + bc = \square. \tag{1}$$

Consider the corresponding elliptic curve obtained by multiplying three conditions in (1):

$$E: \quad y^2 = (x + ab)(x + ac)(x + bc).$$

Since E has only finitely many integer points, there are only finitely many n's such that $\{a, b, c\}$ is a $D(n)$-set. Note that $E(\mathbb{Q})$ has several obvious rational points:

$$A = (-bc, 0), \quad B = (-ca, 0), \quad C = (-ab, 0), \quad P = (0, abc), \quad S = (1, rst).$$

By 2-descent (see [34, 4.2, p. 85]), for $T \in E(\mathbb{Q})$, we have that $x = x(T)$ is a rational solution of the system (1) if and only if $T \in 2E(\mathbb{Q})$. Hence, we are interested in points in $2E(\mathbb{Q}) \cap \mathbb{Z}^2$. One such point is point S, which corresponds to $x = 1$. Indeed, $S = 2R$, where

$$R = (rs + rt + st + 1, (r + s)(r + t)(s + t)) \in E(\mathbb{Q}).$$

Points A, B, C have order 2, so they are not of interest here. In our search for Diophantine triples, which are $D(n)$-sets for several n's, we are thus led to look for triples $\{a, b, c\}$ for which $2kP + \ell S \in \mathbb{Z}^2$ for some small integers k and ℓ. In general, we may expect that the points P and S are two independent points of infinite order.

However, if $c = a + b \pm 2r$, where $ab + 1 = r^2$ (such triples are called *regular*), then $2P = \pm S$. We have

$$x(2P) = \frac{1}{4}(a + b + c)^2 - ab - ac - bc,$$

which lead us to the following result

Proposition 1. *Let a, b, c be nonzero integers such that $a + b + c$ is even. Then $\{a, b, c\}$ is a $D(n)$-set for*

$$n = \frac{1}{4}(a + b + c)^2 - ab - ac - bc,$$

provided $n \neq 0$. Furthermore, $n = 0$ is equivalent to $c = a + b \pm 2\sqrt{ab}$ (and thus impossible if $\{a, b, c\}$ is a $D(1)$-triple), while $n = 1$ is equivalent to $c = a + b \pm 2\sqrt{ab + 1}$.

Any Diophantine triple $\{a, b, c\}$ such that $a+b+c$ is even and $c \neq a+b\pm 2\sqrt{ab + 1}$ is also a $D(n)$-set for some $n \neq 1$.

A computer search shows that for Diophantine triples $\{a, b, c\}$ with $a, b \leq 1000$, $c \leq 1000000$, the corresponding points $S - 2P$ and $4P$ never have integer coordinates, while the point $S+2P = 2(R+P)$ has integer coordinates for 14 triples in the considered range, e.g. for $\{4, 12, 420\}$, $\{4, 420, 14280\}$, $\{12, 24, 2380\}$, $\{24, 40, 7812\}$, $\{40, 60, 19404\}$. We will show that there are infinitely many such examples.

We first note that all the examples above satisfy an additional condition that $x(S + 2P) = a + b + c$. A straightforward calculation shows that the condition $x(S + 2P) = a + b + c$ is equivalent to $q_1q_2q_3 = 0$, where

$$\begin{aligned}
q_1 &= -4 + a^2 - 2ab + b^2 - 2ac - 2bc + c^2,\\
q_2 &= a^2 - 4a - 2ac - 4c + c^2 - 2ab - 4b - 8abc - 2bc + b^2,\\
q_3 &= -4a - 4b - 4c - 2ab - 2ac - 2bc - 4abc + a^2 + b^2 + c^2\\
&\quad - 2a^2b - 2a^2c - 2ab^2 - 2ac^2 - 2b^2c - 2bc^2 - 2a^2b^2\\
&\quad + 2a^3 + 2b^3 + 2c^3 + a^4 + b^4 + c^4 - 2a^2c^2 - 2b^2c^2.
\end{aligned}$$

The condition $q_1 = 0$ is equivalent to $c = a + b \pm 2\sqrt{ab + 1}$, but in that case $x(2P) = 1$, so in this way, we do not get a Diophantine triple which is also a $D(n)$-set for two distinct n's with $n \neq 1$. The equation $q_3 = 0$ has no solutions in Diophantine triples $\{a, b, c\}$.

Thus, the only interesting condition for us is $q_2 = 0$. It is equivalent to

$$c = 2 + a + b + 4ab \pm 2\sqrt{(2a + 1)(2b + 1)(ab + 1)},$$

and this is exactly the condition that $\{2, a, b, c\}$ is a regular Diophantine quadruple.

It can be verified that for such triples $n_2 = x(S + 2P)$ and $n_3 = x(2P)$ satisfy $n_2 \neq n_3$, $n_1 \neq 1$, $n_3 \neq 1$.

Thus we obtain the following result:

Theorem 2. *Let $\{2, a, b, c\}$ be a regular Diophantine quadruple. Then the Diophantine triple $\{a, b, c\}$ is also a $D(n)$-set for two distinct n's with $n \neq 1$.*

From Theorem 2, we can get explicit infinite families of Diophantine triples which are also $D(n)$-sets for two distinct n's with $n \neq 1$. One example of such an infinite family of triples is

$$a = 2(i + 1)i, \quad b = 2(i + 2)(i + 1), \quad c = 4(2i^2 + 4i + 1)(2i + 3)(2i + 1),$$

with

$$n_2 = 32i^4+128i^3+172i^2+88i+16,$$
$$n_3 = 256i^8+2048i^7+6720i^6+11648i^5+11456i^4+6400i^3+1932i^2+280i+16,$$

for an arbitrary positive integer i. Other families can be found in [1].

By a computer search, we found 7 example of triples $\{a,b,c\}$ which are $D(n)$-sets for $n_1 = 1 < n_2 < n_3 < n_4$:

$\{a,b,c\}$	n_2, n_3, n_4
$\{4, 12, 420\}$	$436, 3796, 40756$
$\{10, 44, 21252\}$	$825841, 6921721, 112338361$
$\{4, 420, 14280\}$	$14704, 950896, 47995504$
$\{40, 60, 19404\}$	$19504, 3680161, 93158704$
$\{78, 308, 7304220\}$	$242805865, 4770226465, 13336497750865$
$\{4, 485112, 16479540\}$	$16964656, 2007609136, 63955397832496$
$\{15, 528, 32760\}$	$66609, 5369841, 15984081$

However, the question of whether there are infinitely many such triples remains open.

If we omit the condition $1 \in N$, then the size of a set N for which there exists a triple $\{a,b,c\}$ of nonzero integers, which is a $D(n)$-set for all $n \in N$ can be arbitrarily large. Indeed, take any triple $\{a,b,c\}$ such that the induced elliptic curve $E(\mathbb{Q})$ has positive rank. Then there are infinitely many rational points on E. For an arbitrarily large positive integer m, we may choose m distinct rational points $R_1,\dots,R_m \in 2E(\mathbb{Q})$, so that we have

$$x(R_i) + ab = \square, \quad x(R_i) + ac = \square, \quad x(R_i) + bc = \square.$$

We do so by taking points of the form $2m_1P_1+2m_2P_2+\cdots+2m_rP_r$, where $P_1,\dots,P_r$ are the generators of $E(\mathbb{Q})$. We then let $z \in \mathbb{Z} \setminus \{0\}$ be such that $z^2x(R_i) \in \mathbb{Z}$ for all $i = 1,2,\dots,m$. Then the triple $\{az,bz,cz\}$ is a $D(n)$-set for $n = x(R_i)z^2$ for all $i = 1,2,\dots,m$.

Example 1. Consider the Diophantine triple $\{1,8,120\}$, whose induced elliptic curve $E(\mathbb{Q})$ has rank 3. Following the procedure described above, we find points $R_1,\dots,R_5 \in 2E(\mathbb{Q})$ such that

$$x(R_1) = 1, \quad x(R_2) = 721, \quad x(R_3) = 12289/4, x(R_4) = 769/9, \quad x(R_5) = 1921/36.$$

We then let $z = 6$. It follows that the triple $\{az,bz,cz\} = \{6,48,720\}$ is a $D(n)$-set for $n = 36, 1921, 3076, 25956, 110601$.

3. Diophantine quadruples with the properties $D(n_1)$ and $D(n_2)$

In a joint paper with V. Petričević [24], we took one step forward and asked if there is any set of four distinct nonzero integers, which is a $D(n_i)$-quadruple for two distinct (nonzero) integers n_1 and n_2.

If $\{a,b,c,d\}$ is a $D(n_1)$ and $D(n_2)$-quadruple and u is a nonzero rational such that au, bu, cu, du, n_1u^2 and n_2u^2 are integers, then $\{au,bu,cu,du\}$ is a $D(n_1u^2)$ and $D(n_2u^2)$-quadruples. We will say that these two quadruples are equivalent. This is the main result of [24]:

Theorem 3. *There are infinitely many nonequivalent sets of four distinct nonzero integers $\{a, b, c, d\}$ with the property that there exist two distinct nonzero integers n_1 and n_2 such that $\{a, b, c, d\}$ a $D(n_1)$-quadruple and a $D(n_2)$-quadruple.*

Since before our work, no such example was known, we started with extensive experiments. The experiments start with a computational search for $D(n_1)$-quadruples, where $-500\,000 \leq n_1 \leq 500\,000$. For a fixed nonzero integer n_1, we considered divisors of integers of form $m^2 - n_1$ in the range $m \leq 333\,333$ (the details of the algorithm are described in [24]). For a $D(n_1)$-quadruple $\{a, b, c, d\}$, we searched for integer points on the hyperelliptic curve $y^2 = (ab + x)(ac + x)(ad + x)(bc+x)(bd+x)(cd+x)$ with $|x| =\leq 10^8$, $x \neq n_1$, using Stoll's program `ratpoints` [37], and then checked whether x satisfies the condition for n_2.

In that way, we found 26 examples of quadruples, which are simultaneously $D(n_1)$ and $D(n_2)$-quadruples for $n_1 \neq n_2$. Looking for certain patterns among these examples, we noted 8 quadruples $\{a, b, c, d\}$ in which $a/b = -1/7$. Here we list these quadruples:

$\{a, b, c, d\}$	$\{n_1, n_2\}$	$\{a, b, c, d\}$	$\{n_1, n_2\}$
-189, -133, 27, 32	6192, 8352	-3, 21, 2597, 3132	11512, 80152
-27, 28, 189, 493	13752, 61272	-1, 7, 64, 119	128, 848
-27, 189, 4189, 6364	194328, 1325304	-1, 7, 4484, 4879	6248, 43688
-3, 21, 1152, 1517	5392, 37312	-1, 7, 22532, 23407	30632, 214376

Looking for additional patterns within these quadruples, we noted that they contain regular triples. Namely, if $AB + n = R^2$, then $\{A, B, A + B + 2R\}$ and $\{A, B, A + B - 2R\}$ are $D(n)$-triples. Indeed, $A(A + B \pm 2R) + n = (A \pm R)^2$, $B(A + B \pm 2R) + n = (B \pm R)^2$. Let $cd + n_1 = r^2$ and $cd + n_2 = s^2$. If $c + d - 2r = 7$ and $c+d-2s = -1$, then $\{7, c, d\}$ is a $D(n_1)$-triple and $\{-1, c, d\}$ is a $D(n_2)$-triple. The remaining six conditions from the definition of $D(n_i)$-quadruples can be satisfied parametrically and we obtain rational quadruples of the form $\{-1, 7, c, d\}$ depending on one rational parameter u. By taking $u = v/w$ and getting rid of denominators, we obtain the following result.

Proposition 4. *Let v and w be coprime integers and*

$$v/w \notin \{0, 1, -1, 2, -2, 3, 4, -5, 7, -7, 7/2, -7/2, 7/3, 7/4, -7/5, \infty\}.$$

Then the set

$$\begin{gathered}\{-(-v^2 + 7w^2)^2, 7(-v^2 + 7w^2)^2, -(-2v^2 + vw + 7w^2)(2v^2 - 3vw + 7w^2),\\ (v^2 - 3vw + 14w^2)(-v^2 - vw + 14w^2)\}\end{gathered} \tag{2}$$

is a $D(n_1)$-quadruple and a $D(n_2)$-quadruple for

$$n_1 = 4(-v^2 + 7w^2)^2(2v^4 - v^3w - 20v^2w^2 - 7vw^3 + 98w^4),$$
$$n_2 = 4(-v^2 + 7w^2)^2(2v^2 - 7vw + 14w^2)(v^2 + 7w^2).$$

It is clear that if we take v and w to be solutions of the Pell equation $v^2-7w^2 = 1$, the formula (2) gives integer quadruples of the form $\{-1, 7, c, d\}$. Furthermore, if we take v and w to be solutions of the Pellian equation $v^2 - 7w^2 = 2$ (which has infinitely many solutions since $(u, v) = (3, 1)$ is a solution), then all elements of (2) will be divisible by 4, while corresponding n_1 and n_2 will be divisible by 16, so by dividing all elements of the quadruple by the common factor 4, we obtain again an integer quadruple of the form $\{-1, 7, c, d\}$. Here are some examples obtained in that way.

$\{a, b, c, d\}$	$\{n_1, n_2\}$
-1, 7, 119, 64	128, 848
-1, 7, 4879, 4484	6248, 43688
-1, 7, 23407, 22532	30632, 214376
-1, 7, 1191959, 1185664	1585088, 11095568
-1, 7, 5840864, 5826919	7778528, 54449648
-1, 7, 302003332, 301903007	402604232, 2818229576
-1, 7, 1481896324, 1481674079	1975713608, 13829995208
-1, 7, 76695715424, 76694116519	102259887968, 715819215728
-1, 7, 376369378007, 376365836032	501823476032, 3512764332176
-1, 7, 19480219444399, 19480193962244	25973608937768, 181815262564328
-1, 7, 95595918622159, 95595862172804	127461187196648, 892228310376488

4. Doubly regular Diophantine quadruples

Motivated by the results of the previous section, we may ask is it possible to obtain (infinitely many) quadruples with positive elements which are simultaneously $D(n_1)$ and $D(n_2)$-quadruples for $n_1 \neq n_2$ (note that in the quadruples $\{a, b, c, d\}$ from the previous section the elements a and b are always of the opposite sign). Furthermore, we may ask is it possible to find quadruple with are $D(n)$-quadruples for three distinct n's. Affirmative answers to these questions were given in joint work with V. Petričević [25]. However, to obtain the affirmative answer to the second question, we had to allow that one of the n is equal to 0. More precisely, we proved the following result.

Theorem 5. *There are infinitely many nonequivalent sets of four distinct nonzero integers $\{a, b, c, d\}$ which are regular $D(n_1)$ and $D(n_2)$-quadruples for distinct nonzero squares n_1 and n_2. Moreover, we may take that all elements of these sets are perfect squares, so they are also $D(0)$-quadruples.*

In [25], it is shown how to obtain explicit formulas for quadruples satisfying the conditions from Theorem 5. Here we give one of them.

Let t be an integer such that $t \neq 0, \pm 1, \pm 2$, and let

$$\begin{aligned}
a &= (t-1)^2(t-2)^2(t+2)^2(3t^6 - 2t^5 - 13t^4 + 8t^3 + 16t^2 - 16)^2 \\
&\quad \times (5t^6 - 6t^5 - 27t^4 + 40t^3 + 32t^2 - 64t + 16)^2, \\
b &= 64t^2(t-1)^2(t-2)^2(t+2)^2(t^3 - t^2 - 3t + 4)^2(t^2 - 2)^2 \\
&\quad \times (t^3 - t^2 - 2t + 4)^2(2t^4 - t^3 - 7t^2 + 4t + 4)^2, \\
c &= t^2(t-1)^2(t^2-3)^2(t^6 - 6t^5 - 3t^4 + 28t^3 - 8t^2 - 32t + 16)^2 \\
&\quad \times (4t^7 - 5t^6 - 26t^5 + 39t^4 + 48t^3 - 88t^2 - 16t + 48)^2, \\
d &= (t+1)^2(t^3 - t^2 - 3t + 4)^2(t^6 + 2t^5 - 7t^4 + 8t^2 - 16t + 16)^2 \\
&\quad \times (4t^7 - 7t^6 - 22t^5 + 49t^4 + 20t^3 - 88t^2 + 32t + 16)^2.
\end{aligned}$$

Then $\{a,b,c,d\}$ is a $D(n_1)$, $D(n_2)$ and $D(n_3)$-quadruple, where

$$\begin{aligned}
n_1 &= 16t^2(t+1)^2(t-2)^4(t+2)^4(t-1)^6(t^2-3)^2\\
&\quad\times(t^3-t^2-2t+4)^2(t^3-t^2-3t+4)^2(2t^4-t^3-7t^2+4t+4)^2\\
&\quad\times(3t^6-2t^5-13t^4+8t^3+16t^2-16)^2\\
&\quad\times(5t^6-6t^5-27t^4+40t^3+32t^2-64t+16)^2,\\
n_2 &= 4t^2(t^2-2)^2(t^3-t^2-3t+4)^2(t^6+2t^5-7t^4+8t^2-16t+16)^2\\
&\quad\times(t^6-6t^5-3t^4+28t^3-8t^2-32t+16)^2\\
&\quad\times(4t^7-5t^6-26t^5+39t^4+48t^3-88t^2-16t+48)^2\\
&\quad\times(4t^7-7t^6-22t^5+49t^4+20t^3-88t^2+32t+16)^2,\\
n_3 &= 0.
\end{aligned}$$

For example, by taking $t=3$ in these formulas,

$$\{1066758050, 7214407200, 8024417928, 44219811272\}$$

is a $D(90467582183447040000)$, $D(30185892484109116209)$ and $D(0)$-quadruple.

Main idea is to construct so-called *doubly regular quadruples*. A rational $D(1)$ Diophantine quadruple $\{a,b,c,d\}$ is called *regular* is it satisfies the equation

$$(a+b-c-d)^2 = 4(ab+1)(cd+1). \tag{3}$$

A quadruple $\{a,b,c,d\}$ is called doubly regular if it is a rational $D(1)$ and $D(x^2)$-quadruple for $x^2\neq 1$, such that $\{a,b,c,d\}$ and $\{\frac{a}{x},\frac{b}{x},\frac{c}{x},\frac{d}{x}\}$ are both regular rational $D(1)$-quadruples.

The regularity condition for $\{a/x,b/x,c/x,d/x\}$ implies the following quartic equation in x:

$$4x^4+(-a^2+2ab+2ad-b^2+2bc+2ac-c^2+2cd-d^2+2bd)x^2+4abcd=0. \tag{4}$$

By inserting the regularity condition for $\{a,b,c,d\}$, i.e. (3), in the x^2-term of (4), we obtain

$$4(x^2-1)(x^2-abcd)=0.$$

Since we are interested in solutions with $x^2\neq 1$, we get that $x^2=abcd$. Then $ab+x^2=ab(1+cd)=\square$ implies that ab is a square (and analogously, ac, ad, bc, bd and cd are squares, so $\{a,b,c,d\}$ is also a $D(0)$-quadruple).

We use a parametrization of rational $D(1)$-triples due to L. Lasić [32]:

$$\begin{aligned}
a &= \frac{2t_1(1+t_1t_2(1+t_2t_3))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},\\
b &= \frac{2t_2(1+t_2t_3(1+t_3t_1))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},\\
c &= \frac{2t_3(1+t_3t_1(1+t_1t_2))}{(-1+t_1t_2t_3)(1+t_1t_2t_3)},
\end{aligned}$$

modified by the following substitutions:

$$\begin{aligned}
t_1 &= \frac{k}{t_2t_3},\\
t_2 &= m-\frac{1}{t_3}.
\end{aligned}$$

After computing d from the regularity equation, the remaining condition that $abcd$ is a perfect square can be expressed in terms of an elliptic curve over $\mathbb{Q}(t)$ with positive rank. One of the points of infinite order on that curve gives the above-mentioned parametric family of quadruples with the required property.

Somewhat simpler examples can be found by a brute force search for parameters k, m, t_3 with small numerators and denominators which satisfy the condition that $abcd$ is a perfect square. The simplest example obtained in that way is $\{1458, 66248, 5000, 14112\}$ which is $D(16769025)$, $D(406425600)$ and $D(0)$-quadruple. By multiplying all elements of this quadruple by 2, we obtain the quadruple

$$\{54^2, 364^2, 100^2, 168^2\}$$

consisting of four perfect squares.

5. $D(n)$-quintuples with square elements

In this section, we will describe the joint work with M. Kazalicki and V. Petričević [20], where the previous results on triples and quadruples are partially extended to quintuples. Thus, we are interested in the question whether it is possible that the same set of five nonzero integers is simultaneously a $D(n_1)$ and $D(n_2)$-quintuple for $n_1 \neq n_2$. In order to provide an affirmative answer to this question, we will allow again that one of the integers n_1 and n_2 is equal to 0. It will remain an open problem whether there is an example with nonzero n_1 and n_2.

Note that if $\{a, b, c, d, e\}$ is a $D(n_1)$-quintuple, and u a nonzero rational, then $\{ua, ub, uc, ud, ue\}$ is a $D(n_1u^2)$-quintuple and we say that these two quintuples are equivalent.

Theorem 6. *There are infinitely many nonequivalent quintuples that have $D(n_1)$ property for some $n_1 \in \mathbb{N}$ such that all the elements in the quintuple are perfect squares. In particular, there are infinitely many nonequivalent integer quintuples that are simultaneously $D(n_1)$-quintuples and $D(0)$-quintuples.*

To prove Theorem 6, it suffice to show that there are infinitely many rational Diophantine quintuples with the property that the product of any two of its elements is a perfect square. Namely, it is clear that every rational Diophantine quintuple is equivalent to a $D(u^2)$-quintuple where u is a multiple of the common denominator of the elements in the quintuple.

By performing some experiments, we obtained an example of such rational Diophantine quintuple

$$\left\{\frac{3375}{32032}, \frac{143}{840}, \frac{2457}{1760}, \frac{5632}{1365}, \frac{4459}{330}\right\}. \tag{5}$$

And by multiplying all elements of (5) by 480480, we obtain the $D(480480^2)$-quintuple

$$\{225^2, 286^2, 819^2, 1408^2, 2548^2\}$$

with the desired property.

By analyzing the quintuple (5), we noted that it contains the subquadruple $\{\frac{3375}{32032}, \frac{143}{840}, \frac{5632}{1365}, \frac{4459}{330}\}$ with the property that the product of its elements is equal to 1, and two regular subquadruples $\{\frac{3375}{32032}, \frac{143}{840}, \frac{2457}{1760}, \frac{5632}{1365}\}$ and $\{\frac{3375}{32032}, \frac{2457}{1760}, \frac{5632}{1365}, \frac{4459}{330}\}$.

Motivated by this example, we say that a rational Diophantine quintuple $\{a, b, c, d, e\}$ is *exotic* if $abcd = 1$, the quadruples $\{a, b, d, e\}$ and $\{a, c, d, e\}$ are regular, and if the product of any two of its elements is a perfect square. We will show that there are infinitely many exotic quintuples.

Our starting point is the following characterization of rational Diophantine quadruples $\{a,b,c,d\}$ such that $abcd=1$.

Proposition 7. *Let $\{a,b,c,d\}$ be a rational Diophantine quadruple with $abcd=1$. Then there exist $r,s,t\in\mathbb{Q}$ such that*

$$a=xyz,\quad b=\frac{x}{yz},\quad c=\frac{y}{xz},\quad d=\frac{z}{xy},$$

where $x=\frac{t^2-1}{2t}, y=\frac{s^2-1}{2s}$ and $z=\frac{r^2-1}{2r}$. In particular, the product of any two elements of the quadruple is a perfect square.

Proof. From $ab+1=ab+abcd=ab(1+cd)$ it follows that ab is a perfect square, and similarly for other products. Set $ab=x^2, ac=y^2$ and $ad=z^2$, with $x,y,z\in\mathbb{Q}$. It follows $a^2=\frac{ab\cdot ac}{bc}=\frac{x^2y^2}{1/z^2}$, hence $a=xyz$ and similarly $b=\frac{x}{yz}$, $c=\frac{y}{xz}$ and $d=\frac{z}{xy}$ (with the appropriate choice of signs). Since x^2+1 is a perfect square, there is $t\in\mathbb{Q}$ such that $x=\frac{t^2-1}{2t}$, and similarly for y and z. $\square$

To extend quadruple $\{a,b,c,d\}$, given in the terms of $r,s,t\in\mathbb{Q}$ as in Proposition 7), to an exotic quintuple, it is enough that triples $\{a,b,d\}$ and $\{a,c,d\}$ have a common regular extension e and that ae is a perfect square.

It can be shown (see [20] for details) that the regularity conditions lead to

$$s=\frac{-1+r^2+t+r^2t}{-1-r^2-t+r^2t}.$$

It remains to satisfy the condition that ae is a square. This condition leads to considering several genus 0 curves. One of them is

$$r^2t^2+3r^2-t^2+2t-1=0,$$

with a parametric solution

$$(r,t)=\left(-\frac{2u+1}{u^2+u+1},\frac{u^2+4u+1}{(u-1)(u+1)}\right).$$

Then the condition that ae is a square gives the quartic

$$v^2=-48\left(u^2-3u-1\right)\left(u^2+5u+3\right),$$

which has a rational point $(u,v)=(0,12)$, and thus it is birationally equivalent to the elliptic curve

$$y^2+xy+y=x^3-x^2-41x+96,$$

with rank 1 (a generator is the point $P=(2,3)$) and torsion subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. The point P (the same holds for points $P+T$, where T is a torsion point) does not give a solution since the corresponds values of u are zeros of denominators appearing in the expression for a,b,c,d. The point $2P=(9,15)$ on the elliptic curve gives the point $(u,v)=(3,36)$ on the quartic, which corresponds to $(r,s,t)=(-\frac{7}{13},-\frac{7}{8},\frac{11}{4})$ and gives (a permutation of) our starting $D(1)$-quintuple

$$\left\{\frac{3375}{32032},\frac{4459}{330},\frac{143}{840},\frac{5632}{1365},\frac{2457}{1760}\right\}.$$

The points $2P+T$, where T is a torsion point, give equivalent solutions (with permuted elements or elements multiplied by -1). The point $3P=(-\frac{262}{49},\frac{4782}{343})$

on the elliptic curve gives the point $(u,v) = (-\frac{28}{37}, -\frac{5916}{1369})$ on the quartic which corresponds to $(r,s,t) = (\frac{703}{1117}, -\frac{703}{585}, \frac{1991}{585})$ and gives the $D(1)$-quintuple

$$\left\{\frac{126066472448}{914609323485}, \frac{388078111459}{22127635530}, \frac{131212873}{529696440}, \frac{398601435375}{238691175968}, \frac{190854470299}{39338018720}\right\}.$$

It is clear that this construction provides infinitely many quintuples with the desired properties by taking multiples $mP = (x,y)$, $m \geq 2$, computing the corresponding u's from $u = \frac{-3x+18}{-y+3x-15}$ and inserting it in formulas for a, b, c, d from Proposition 7 and e from the regularity conditions:

$$\begin{aligned}
a &= \frac{(u+2)^3(u^2-2u-2)u^3}{2(2u+1)(u-1)(u+1)(u^2+u+1)(u^2+4u+1)},\\
b &= \frac{8(u^2+u+1)(2u+1)^3}{u(u-1)(u+2)(u+1)(u^2-2u-2)(u^2+4u+1)},\\
c &= \frac{(u^2+u+1)(u^2-2u-2)(u^2+4u+1)}{2u(u-1)(u+2)(u+1)(2u+1)},\\
d &= \frac{(u+1)^3(u^2+4u+1)(u-1)^3}{2u(u+2)(u^2-2u-2)(2u+1)(u^2+u+1)},\\
e &= -\frac{3(2u^7+7u^6-17u^5-60u^4-85u^3-64u^2-23u-3}{2u(u^7+4u^6-6u^5-32u^4-17u^3+24u^2+22u+4)}.
\end{aligned}$$

While we have found infinitely many rational Diophantine quintuples with $D(0)$ property, i.e. the products of any of two of their elements are perfect squares, it remains an open question whether there exists a rational Diophantine quintuple with square elements. On the other hand, there are infinitely many rational Diophantine quadruples with square elements, for example, the following two parametric family has this property

$$\begin{aligned}
a &= \frac{3^2(s-1)^2(s+1)^2v^2}{2^2(2s^3-2s+v^2)^2},\\
b &= \frac{v^2(-4s^3+4s+v^2)^2}{2^2(s+1)^2(s-1)^2(-s^3+s+v^2)^2},\\
c &= \frac{(2s^3-2s+v^2)^2}{3^2v^2s^2},\\
d &= \frac{4^2(-s^3+s+v^2)^2s^2}{v^2(-4s^3+4s+v^2)^2}.
\end{aligned}$$

This family is obtained by taking $t = 1/(r-1)$ in the notation of Proposition 7, so it satisfies $abcd = 1$. We have also found an example of a rational Diophantine quadruple $\{a,b,c,d\}$ with square elements for which $abcd \neq 1$:

$$\left\{\left(\frac{18}{77}\right)^2, \left(\frac{55}{96}\right)^2, \left(\frac{56}{15}\right)^2, \left(\frac{340}{77}\right)^2\right\}.$$

Here we can mention the result from [19] where it is shown that there exist infinitely many rational Diophantine sextuples such that denominators of all the elements (in the lowest terms) in the sextuples are perfect squares. An example of such sextuple is

$$\left\{\frac{75}{8^2}, -\frac{3325}{64^2}, -\frac{12288}{125^2}, \frac{123}{10^2}, \frac{3498523}{2260^2}, \frac{698523}{2260^2}\right\}.$$

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(Andrej Dujella) Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia

Email address: `duje@math.hr`