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# On the size of sets whose elements have perfect power $n$-shifted products

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*Dedicated to Professors K. Győry and A. Sárközy on their 70th birthdays and Professors A. Pethő and J. Pintz on their 60th birthdays*

**Abstract.** We show that the size of sets $\mathcal{A}$ having the property that with some non-zero integer $n$, $a_1a_2 + n$ is a perfect power for any distinct $a_1, a_2 \in \mathcal{A}$, cannot be bounded by an absolute constant. We give a much more precise statement as well, showing that such a set $\mathcal{A}$ can be relatively large. We further prove that under the *abc*-conjecture a bound for the size of $\mathcal{A}$ depending on $n$ can already be given. Extending a result of Bugeaud and Dujella, we also derive an explicit upper bound for the size of $\mathcal{A}$ when the shifted products $a_1a_2 + n$ are $k$-th powers with some fixed $k \geq 2$. The latter result plays an important role in some of our proofs, too.

## 1. Introduction

A set $\mathcal{A} = \{a_1, \ldots, a_m\}$ of positive integers is called a Diophantine $m$-tuple, if for any $1 \leq i < j \leq m$ we have $a_ia_j + 1 = x_{ij}^2$ for an integer $x_{ij}$. The

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history and theory of Diophantine $m$-tuples is very rich. Diophantus found the set $\{1/16, 33/16, 17/4, 105/16\}$ of four positive rationals with the above property. However, the first Diophantine quadruple, $\{1, 3, 8, 120\}$, was found by Fermat (see [5]). A folklore conjecture is that there does not exist a Diophantine quintuple. The first important result concerning this conjecture was proved in 1969 by Baker and Davenport [1]. They proved that if $d$ is a positive integer such that $\{1, 3, 8, d\}$ forms a Diophantine quadruple, then $d = 120$. Hence, the triple $\{1, 3, 8\}$ cannot be extended to a Diophantine quintuple. In 1998, Dujella and Pethő [13] proved that the pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple. In 2004, Dujella [8] proved that there does not exist a Diophantine sextuple and there are only finitely many Diophantine quintuples (recently Fujita [15] showed that there are at most $10^{276}$ Diophantine quintuples). An overview of classical and recent results and the complete list of references on Diophantine $m$-tuples can be found on web page [10]. As a generalization of Diophantine $m$-tuples one can consider sets $\mathcal{A}$ of positive integers such that for any $a, b \in \mathcal{A}$ with $a \neq b$ we have $ab + n = x_{ab}^2$, where $n$ is a fixed non-zero integer. Such sets are referred to as $D(n)$-$m$-tuples. E.g. the set $\{99, 315, 9920, 32768, 44460, 19534284\}$, found by Gibbs [17] is a $D(2985984)$-sextuple. Define

$$M_n = \sup\{|\mathcal{A}| : \mathcal{A} \text{ is a } D(n)\text{-tuple}\}.$$

It is easy to prove that $M_n = 3$ for $n \equiv 2 \pmod 4$ (see e.g. [2]). By the Lang conjecture on varieties of general type, we expect that there exists an absolute constant $C$ such that $M_n < C$ for all non-zero integers $n$. However, the best known general result of this shape is $M_n \leq 31$ for $|n| \leq 400$, $M_n < 15.476 \log |n|$ for $|n| > 400$ (see [7, 9]). Furthermore, Dujella and Luca [12] proved that $M_p < 3 \cdot 2^{168}$ holds for all primes $p$. It is known that $4 \leq M_1 \leq 5$ [8], $4 \leq M_4 \leq 5$ [16] and $3 \leq M_{-1} \leq 4$ [11].

As an alternative, but also natural generalization of Diophantine $m$-tuples, Bugeaud and Dujella [3] considered sets $\mathcal{A}$ of positive integers with the property that $ab + 1 = x_{ab}^k$ whenever $a, b$ are distinct elements of $\mathcal{A}$ and $k$ is an integer with $k \geq 2$. Such sets are called $k$-th power Diophantine tuples. Examples of such triples for $k = 3$ and $k = 4$ are given, respectively, by $\{2, 171, 25326\}$ and $\{1352, 8539880, 9768370\}$. Let

$$E_k = \sup\{|\mathcal{A}| : \mathcal{A} \text{ is a } k\text{-th power Diophantine tuple}\}.$$

In [3, Corollary 4] absolute upper bounds for the numbers $E_k$, $k \geq 3$ were obtained. More precisely, it was proved that $E_3 \leq 7$, $E_4 \leq 5$, $E_5 \leq 5$, $E_k \leq 4$ for $6 \leq k \leq 176$, and $E_k \leq 3$ for $k \geq 177$.

As a further generalization, in this paper we consider sets $\mathcal{A}$ of positive integers such that for any distinct elements $a, b$ of $\mathcal{A}$, $ab + n$ is a perfect power, where $n$ is some fixed non-zero integer. That is, writing $\mathcal{A} = \{a_1, a_2, \dots\}$ we have

$$a_i a_j + n = x_{ij}^{k_{ij}} \tag{1}$$

for some integers $x_{ij}$ and $k_{ij}$ with $k_{ij} \geq 2$, and here the exponents $k_{ij}$ can of course be different. The case $n = 1$ of this problem has already been studied by several authors, see e.g. Gyarmati [19], Gyarmati, Sárközy, Stewart [20], Bugeaud, Gyarmati [4], Dietmann, Elsholtz, Gyarmati, Simonovits [6], Luca [22], Gyarmati, Stewart [21]. The main direction of research concerns finding an upper bound for the size of sets $\mathcal{A} \subseteq \{1, 2, \dots, N\}$ such that $ab + 1$ is a perfect power for all $a \neq b$ in $\mathcal{A}$. The best known result of that type is due to Stewart [24], who proved that $|\mathcal{A}| \ll (\log N)^{2/3}(\log\log N)^{1/3}$. Further, Luca [22] proved that if $\mathcal{A}$ satisfies (1) with $n = 1$, then assuming the *abc*-conjecture the number of elements $|\mathcal{A}|$ of $\mathcal{A}$ can be bounded by an absolute constant.

We show that this is not true in case of arbitrary $n$ (Theorem 1). We also give a much more precise statement (Theorem 2), which shows that such sets can be relatively large. Further, we prove that assuming the *abc*-conjecture we already have $|\mathcal{A}| < C(n)$, where $C(n)$ is a constant depending only on $n$. In view of our construction in the proof of Theorem 2, the dependence of $C(n)$ on $n$ is necessary. To prove this result we extend a theorem of Bugeaud and Dujella [3] concerning shifted products which are $k$-th powers (Theorem 3). Assuming the *abc*-conjecture we obtain a bound in terms of $n$ for all but one $a_i$, provided that the exponents $k_{ij}$ in $a_i a_j + n = x_{ij}^{k_{ij}}$ are sufficiently large (Lemma 1). Then following the approach of Luca [22], we use Ramsey theory to prove the bound $|\mathcal{A}| < C(n)$ (Theorem 4).

## 2. Main results

Our first theorem shows that the size of sets with the property (1) cannot be bounded by an absolute constant.

**Theorem 1.** *For any $K \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ and a set $\mathcal{A} \subseteq \mathbb{N}$ such that $|\mathcal{A}| \geq K$ and $ab + n$ is a perfect power for any distinct $a, b \in \mathcal{A}$.*

As one can easily see, Theorem 1 is a simple and immediate consequence of the following, much more precise statement.

**Theorem 2.** *Let $x \ge e^{e^e}$, and take*

$$K := \left\lfloor \left( \frac{\log\log x}{2\log\log\log x} \right)^{1/3} \right\rfloor. \tag{2}$$

*Then there exists a set $\mathcal{A}_K = \{a_1, \ldots, a_K\}$ with elements all in $[1, x]$, as well as an integer $n_K$ also in $[1, x]$, such that $a_i a_j + n_K = x_{ij}^{k_{ij}}$ for $1 \le i < j \le K$ with some integers $x_{ij}$, where the exponents $k_{ij}$ are the first $\binom{K}{2}$ primes.*

**Remark 1.** The condition $x \ge e^{e^e} = 3814279.105\ldots$ is meant to insure that $\log\log\log x > 1$. If $x > e^{e^{68}}$, then the above number $K$ is $\ge 2$. For smaller values of $x$ the statement is empty. However, obviously, $K \to \infty$ as $x \to \infty$.

**Remark 2.** Let $f(x)$ be the maximum $K$ such that there exists $\mathcal{A}_K \subseteq [1, x] \cap \mathbb{N}$ with $K$ elements and some $n \le x$ such that $aa' + n$ is a perfect power for all $a \ne a'$ in $\mathcal{A}_K$. A natural question is to find sharp upper and lower bounds on $f(x)$. It is clear that $f(x)$ is at least as large as the bound shown at (2) and it is easy to see that $f(x) \le x^{2/3+o(1)}$ as $x \to \infty$. Indeed, let $\mathcal{A}_K$ be a maximal example (with $K = f(x)$). Let $\mathcal{A}_1 = \{a \in \mathcal{A}_K : aa' + n$ is a square for all $a' \in \mathcal{A}_K \backslash \{a\}\}$. It is clear that elements in $\mathcal{A}_1$ participate in every maximal $D(n)$-tuple in $\mathcal{A}_K$, so the cardinality of $\mathcal{A}_1$ is $O(\log|n|) = O(\log x)$ (see [7, 9]). On the other hand, for each $a \in \mathcal{A}_K \backslash \mathcal{A}_1$ there is an $a'$ in $\mathcal{A}_K$ such that $aa' + n$ is a perfect power $u^k$ of exponent $k \ge 3$. Since $aa' + n = u^k \le 2x^2$, the number of such perfect powers is $O(x^{2/3})$. Given one such perfect power $u^k$, $a$ is a divisor of $u^k - n$, a positive integer $\le x^2$, so which has at most $x^{o(1)}$ divisors as $x \to \infty$. This indeed shows that $f(x) \le x^{2/3+o(1)}$ as $x \to \infty$, which is a nontrivial upper bound. To derive sharp upper and lower bounds for $f(x)$ we leave as an open problem.

The next result is an extension of a theorem of Bugeaud and Dujella [3].

**Theorem 3.** *Let $k$ and $n$ be integers with $k \ge 2$ and $n \ne 0$, and let $\mathcal{A} \subseteq \mathbb{Z}$ such that $ab + n$ is a $k$-th power for all distinct $a, b \in \mathcal{A}$. Then we have $|\mathcal{A}| \le C_1(k, n)$, where $C_1(k, n)$ is a constant depending only on $k$ and $n$. In particular, if $k = 2$ (or more generally, if $k$ is even), we may take $C_1(k, n) = 31 + 15.476 \log|n|$, if $k = 3$, we may take $C_1(k, n) = 2|n|^{17} + 6$, while for $k \ge 5$ we may take $C_1(k, n) = 2|n|^5 + 3$.*

**Corollary 1.** *Let $k$ and $n$ be integers with $k \ge 2$ and $n \ne 0$, and let $\mathcal{A} \subseteq \mathbb{Z}$ such that $ab+n$ is a $k$-th power for all distinct $a, b \in \mathcal{A}$. Then we have $|\mathcal{A}| \le C_2(n)$, where $C_2(n)$ is a constant depending only on $n$. We may take $C_2(n) = 2|n|^{17} + 31$.*

Our next result proves that assuming the *abc*-conjecture, the size of the sets $\mathcal{A}$ considered in Theorem 1, i.e. with the property that the products of distinct elements of $\mathcal{A}$ shifted by some fixed nonzero integer $n$ are perfect powers, can already be bounded in terms of $n$.

**Theorem 4.** *Let $n$ be a non-zero integer, and suppose that the abc-conjecture is valid. Then there exists a constant $C_3(n)$ depending only on $n$ with the following property. If $\mathcal{A} \subseteq \mathbb{Z}$ such that $ab + n$ is a perfect power for any distinct $a, b \in \mathcal{A}$ then $|\mathcal{A}| < C_3(n)$ holds.*

**Remark 3.** The above theorem extends Theorem 1.4 of Luca [22], where the case $n = 1$ is handled.

**Remark 4.** In view of the set $\mathcal{A} = \{2^\alpha \ : \ \alpha \geq 1\}$ it is necessary to assume that $n \neq 0$ in Theorem 4.

## 3. Lemmas and auxiliary results

We shall need the *abc*-conjecture. We use the same version of the conjecture as in [22]. For any positive integer $t$ write $N(t)$ for the radical of $t$, i.e. $N(t) = \prod_{p|t} p$.

**The *abc*-conjecture.** Let $\varepsilon > 0$ and $a, b, c$ be non-zero integers with $\gcd(a, b, c) = 1$ and $a + b = c$. Then

$$\max\{|a|, |b|, |c|\} \ll N(abc)^{1+\varepsilon}$$

where the implied constant depends only on $\varepsilon$.

The next lemma plays an important part in the proof of Theorem 4. It is in fact a simple extension of results of Luca [22] to the case where we shift our products by $n$, rather than just by 1.

**Lemma 1.** *Suppose that the set $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5\}$ has the following properties*

(1) *The elements of $\mathcal{A}$ are distinct non-zero integers with $|a_1| \leq |a_2| \leq |a_3| \leq |a_4| \leq |a_5|$,*
(2) *$a_i a_j + n = x_{ij}^{k_{ij}}$ with $k_{ij} \geq 3205$ for $1 \leq i < j \leq 5$.*

*If the abc-conjecture holds then we have*

$$|a_2| \leq c_0 |n|^3,$$

*where $c_0$ is an absolute constant.*

PROOF. In the proof below, the Vinogradov symbol always implies a constant depending only on $\varepsilon$. Since at the appropriate point of the proof we choose a concrete value for $\varepsilon$, in fact Vinogradov symbols imply an absolute constant. We shall follow the method in [22].

First put $u := x_{15}$, $v := x_{25}$, $k := k_{15}$ and $l := k_{25}$, and consider the identities

$$a_1a_5 + n = u^k, \quad a_2a_5 + n = v^l.$$

By eliminating the first terms of the above identities we get the equality

$$a_2u^k - a_1v^l = n(a_2 - a_1).$$

Putting $d := \gcd(a_2u^k, a_1v^l)$ we get

$$\frac{a_2u^k}{d} - \frac{a_1v^l}{d} = \frac{n(a_2 - a_1)}{d}. \tag{3}$$

By applying the $abc$-conjecture to equation (3) we obtain

$$\left|\frac{a_2u^k}{d}\right| \ll N(a_1a_2u^kv^l(a_2 - a_1)n)^{1+\varepsilon} \ll (2|a_2|^3 \cdot |n| \cdot |u| \cdot |v|)^{1+\varepsilon}. \tag{4}$$

However,

$$|u| \leq (2|na_1a_5|)^{\frac{1}{k}}, \quad |v| \leq (2|na_2a_5|)^{\frac{1}{l}}. \tag{5}$$

Thus combining (4), (5) and $|a_1| \leq |a_2|$ we get

$$\left|\frac{a_2u^k}{d}\right| \ll \left((2|n|)^{1+\frac{1}{k}+\frac{1}{l}} \cdot |a_2|^{3+\frac{1}{k}+\frac{1}{l}} \cdot |a_5|^{\frac{1}{k}+\frac{1}{l}}\right)^{1+\varepsilon}. \tag{6}$$

Choosing $\varepsilon := 0.1$, by $k, l > 11$ we infer

$$\left(\frac{1}{k} + \frac{1}{l}\right) \cdot (1 + \varepsilon) \leq \frac{1}{5}, \quad \left(3 + \frac{1}{k} + \frac{1}{l}\right) \cdot (1 + \varepsilon) \leq 4. \tag{7}$$

Moreover, since $d \mid (a_2 - a_1)n$, we get $d \leq 2|na_2|$. Hence, using

$$|a_5| \leq |a_1a_5| = |u^k - n| \leq 2|nu^k|$$

together with (6) and (7), we deduce

$$|a_5| \leq 2|nu^k| = \left|\frac{a_2u^k}{d}\right| \cdot \left|\frac{2nd}{a_2}\right| \leq \left|\frac{a_2u^k}{d}\right| \cdot 4n^2 \ll |na_2|^4 \cdot |a_5|^{1/5}.$$

This yields

$$|a_5|^{4/5} \ll |na_2|^4,$$

and we conclude

$$|a_5| \ll |na_2|^5. \tag{8}$$

In the sequel we consider the elements $0 < |a_1| \leq |a_2| \leq |a_3| \leq |a_4|$ and we use the following notations: $x_1 := x_{12}, x_2 := x_{23}, x_3 := x_{34}, x_4 := x_{41}$ and $k_1 := k_{12}, k_2 := k_{23}, k_3 := k_{34}, k_4 := k_{41}$. Further, suppose that $k > k_0$, where $k_0$ will be specified later. With these notations we have

$$\begin{aligned} a_1a_2 &= x_1^{k_1} - n, \qquad & a_3a_4 &= x_3^{k_3} - n, \\ a_2a_3 &= x_2^{k_2} - n, \qquad & a_4a_1 &= x_4^{k_4} - n. \end{aligned} \tag{9}$$

By (9) we clearly have

$$(x_1^{k_1} - n)(x_3^{k_3} - n) - (x_2^{k_2} - n)(x_4^{k_4} - n) = 0,$$

which yields

$$x_1^{k_1}x_3^{k_3} - x_2^{k_2}x_4^{k_4} = n(x_1^{k_1} + x_3^{k_3} - x_2^{k_2} - x_4^{k_4}). \tag{10}$$

In (10) neither the left nor the right hand side can be zero. Indeed, $x_1^{k_1} + x_3^{k_3} - x_2^{k_2} - x_4^{k_4} = 0$ would lead to $a_1a_2 + n + a_3a_4 + n - a_2a_3 - n - a_4a_1 - n = 0$, and this would mean $(a_1 - a_3)(a_2 - a_4) = 0$, which cannot happen since $\mathcal{A}$ contains distinct elements.

Put $D := \gcd(x_1^{k_1}x_3^{k_3}, x_2^{k_2}x_4^{k_4})$. Then by (10) we have

$$\frac{x_1^{k_1}x_3^{k_3}}{D} - \frac{x_2^{k_2}x_4^{k_4}}{D} = \frac{n(x_1^{k_1} + x_3^{k_3} - x_2^{k_2} - x_4^{k_4})}{D}. \tag{11}$$

Here we use again the *abc*-conjecture to infer

$$\left|\frac{x_1^{k_1}x_3^{k_3}}{D}\right| \ll \left|x_1x_2x_3x_4\frac{n(x_1^{k_1} + x_3^{k_3} - x_2^{k_2} - x_4^{k_4})}{D}\right|^{1+\varepsilon}. \tag{12}$$

For $i = 1, 2, 4$ with the appropriate $j$ we clearly have

$$|x_i^{k_i}| = |a_ia_j + n| \leq 2|n| \cdot |a_ia_j| \leq 2|n| \cdot |a_3a_4| = 2|n| \cdot |x_3^{k_3} - n| \leq 4n^2|x_3|^{k_3}.$$

This together with (12) proves that

$$|x_1^{k_1}x_3^{k_3}| \ll \left|n^3x_1x_2x_3x_4x_3^{k_3}\right|^{1+\varepsilon}. \tag{13}$$

Similarly to (5), using (9) we get the estimates

$$\begin{aligned} |x_1| &\le (2|na_1a_2|)^{1/k_1} \qquad & |x_3| &\le (2|na_3a_4|)^{1/k_3} \\ |x_2| &\le (2|na_2a_3|)^{1/k_2} \qquad & |x_4| &\le (2|na_4a_1|)^{1/k_4} \end{aligned} \tag{14}$$

and combining these with (13) we have

$$|x_1^{k_1}x_3^{k_3}| \ll \left| n^3(na_1a_2)^{\frac{1}{k_1}}(na_2a_3)^{\frac{1}{k_2}}(na_3a_4)^{\frac{1}{k_3}}(na_4a_1)^{\frac{1}{k_4}}\right|^{1+\varepsilon}|x_3|^{k_3(1+\varepsilon)}. \tag{15}$$

Using that $k_i > k_0$ and $|a_1| \le |a_2| \le |a_3| \le |a_4|$, (13) leads to the estimate

$$|x_1^{k_1}| \ll \left(|n|^{3+4/k_0}|a_4|^{8/k_0}\right)^{1+\varepsilon}|x_3|^{k_3\varepsilon}. \tag{16}$$

Now using again (14) for $|x_3|$, we have

$$\begin{aligned} |a_1|^2 \le |a_1a_2| \le 2|n||x_1|^{k_1} &\ll |n|\left(|n|^{3+4/k_0}|a_4|^{8/k_0}\right)^{1+\varepsilon}|x_3|^{k_3\varepsilon} \\ &\ll |n|^{1+(3+\frac{4}{k_0})(1+\varepsilon)}|a_4|^{\frac{8}{k_0}(1+\varepsilon)}(|na_3a_4|)^{\varepsilon}. \end{aligned}$$

This yields

$$|a_1|^2 \ll |n|^{(4+\frac{4}{k_0})(1+\varepsilon)} \cdot |a_4|^{\frac{8}{k_0}+(2+\frac{8}{k_0})\varepsilon}. \tag{17}$$

Now choose $\varepsilon = \frac{1}{1000}$ and $k_0 := 2000$, so that $\frac{8}{k_0} + \left(2+\frac{8}{k_0}\right)\varepsilon < \frac{1}{100}$. Thus we get

$$|a_1|^2 \ll |n|^5 \cdot |a_4|^{\frac{1}{100}}, \tag{18}$$

i.e.

$$|a_1|^{200} \ll |n|^{500} \cdot |a_4|. \tag{19}$$

Since $0 < |a_1| \le |a_2| \le |a_3| \le |a_4| \le |a_5|$ we also have

$$|a_2|^{200} \ll |n|^{500} \cdot |a_5|. \tag{20}$$

Now (20) and (8) together show that

$$|a_2|^{200} \ll |n|^{500} \cdot |a_5| \ll |n|^{505}|a_2^5|,$$

which proves the estimate

$$|a_2| \ll |n|^3.$$

□

## 4. Proof of Theorem 2

Proof of Theorem 2. We construct inductively for every $K \geq 2$ a set $\mathcal{A}_K = \{a_1, \ldots, a_K\}$ with $a_1 < \cdots < a_K$ and a positive integer $n_K$ such that

$$a_i a_j + n_K = x_{ij}^{k_{ij}}$$

for $1 \leq i < j \leq K$, where the exponents $k_{ij}$ are the first $t(K) := \binom{K}{2}$ primes. When $K = 2$, we take $\mathcal{A}_2 = \{1, 3\}$ and $n_2 = 1$. Let $T_K = \max\{n_K, a_K^2\}$, and choose an integer $a_{K+1}$ with $\sqrt{2T_K} > a_{K+1} > \sqrt{T_K}$. Observe that $a_{K+1} > a_K$. Let

$$m_K := \prod_{i=1}^{K} (a_i a_{K+1} + n_K).$$

Clearly,

$$m_K < ((\sqrt{2}+1)T_K)^K < T_K^{2K}.$$

Let $\mathcal{P}_K$ be the set of prime factors of $m_K$. Let $p_i$ be the $i$th prime. For a positive integer $m$ and a prime $q$ we write $\nu_q(m)$ for the exponent of $q$ in the factorization of $m$. For each prime $p \in \mathcal{P}_K$, consider the following system of congruences

$$\begin{cases} \alpha_p & \equiv & 0 & \pmod{p_i} & \text{for} & 1 \leq i \leq t(K), \\ \alpha_p & \equiv & -\nu_p(a_j a_{K+1} + n_K) & \pmod{p_{t(K)+j}} & \text{for} & 1 \leq j \leq K. \end{cases} \tag{21}$$

Let $\alpha_p$ be the first positive integer in the above progression. Clearly,

$$\alpha_p \leq \prod_{i \leq t(K+1)} p_i < 4^{p_{t(K+1)}} < 4^{2K(K+1)\log K} < e^{3(K+1)^2 \log(K+1)}.$$

In the above inequalities, we used the Erdős lemma, i.e. the fact that $\prod_{p \leq x} p < 4^x$ holds for all $x \geq 1$, as well as the inequality $p_n < 2n \log n$ holding for all positive integers $n \geq 3$ (see estimate (3.13) in [23]), which we may apply with $n = t(K+1)$ since $t(K+1) \geq t(3) = 3$ for $K \geq 2$.

Put $\beta_p := \alpha_p/2$. Since $\alpha_p$ is even by the first of the above congruences (21), $\beta_p$ is an integer. Put

$$u_K := \prod_{p \in \mathcal{P}_K} p^{\beta_p}.$$

A simple calculation gives

$$u_K < m_K^{\max\{\alpha_p : p \in \mathcal{P}_K\}} < T_K^{e^{4(K+1)^2 \log(K+1)}}. \tag{22}$$

Put $n_{K+1} := u_K^2 n_K$, and observe that $n_{K+1} \le u_K^2 T_K$. Set $a_i^* := u_K a_i$ for $i = 1, \dots, K+1$. Then we obviously have $a_1^* < \dots < a_{K+1}^*$, and by the choice of $a_{K+1}$, also $(a_{K+1}^*)^2 < 2u_K^2 T_K$. Further, by the construction of our numbers, one can easily check that $a_i^* a_j^* + n_{K+1} = u_K^2(a_i a_j + n_K)$ is a perfect power of exponent $k_{ij}$ for all $1 \le i < j \le K+1$, and moreover the exponents $k_{ij}$ can be chosen to be exactly the $t(K+1)$ primes $p_1, \dots, p_{t(K+1)}$.

Let $T_{K+1} = \max\{n_{K+1}, (a_{K+1}^*)^2\}$. Then combining the above upper bounds for $n_{K+1}$ and $(a_{K+1}^*)^2$ with (22), we obtain

$$T_{K+1} < 2u_K^2 T_K < T_K^{2+2e^{4(K+1)^2\log(K+1)}} < T_K^{e^{5(K+1)^2\log(K+1)}}$$

for all $K \ge 2$. Hence by induction, using that $T_2 = 9$, by a simple calculation we get that $T_K < e^{e^{6K^3 \log K}}$ holds for all $K \ge 2$. Now we would like to choose a positive integer $x$ such that $\mathcal{A}_K$ and $n_K$ are all contained in $[1, x]$. Then it suffices that

$$e^{e^{6K^3 \log K}} \le x,$$

giving $6K^3 \log K \le \log\log x$. This yields $K^3 \log(K^3) \le (\log\log x)/2$. This is fulfilled with

$$K := \left\lfloor \left( \frac{\log\log x}{2\log\log\log x} \right)^{1/3} \right\rfloor,$$

and the statement follows. □

## 5. Proofs of Theorems 3 and 4

In the proof of Theorem 3 we follow [3]. In particular, we use the following result of Evertse [14, Theorem 2.1].

**Lemma 2.** *If $a, b$ and $k$ are positive integers with $k \ge 3$ and $c$ is a positive real number, then there is at most one positive integral solution $(x, y)$ to the inequality*

$$|ax^k - by^k| \le c$$

*with* $\gcd(x, y) = 1$ *and*

$$\max\{|ax^k|, |by^k|\} > \beta_k c^{\alpha_k},$$

*where $\alpha_k$ and $\beta_k$ are effectively computable positive constants satisfying*

$$\alpha_3 = 9, \quad \alpha_k = \max\left\{ \frac{3k-2}{2(k-3)}, \frac{2(k-1)}{k-2} \right\} \qquad \text{for } k \ge 4$$

*and*

$$\beta_3 = 1152.2, \quad \beta_4 = 98.53, \quad \beta_k < k^2 \qquad \text{for } k \geq 5.$$

Note that in [3], in the application of Lemma 2, the condition $\gcd(x, y) = 1$ was omitted. However, all corresponding inequalities from the proofs in [3] hold with safe margins, except for $k = 4, 5$, so that this omission has not significant influence to validity of the final results. In particular, in the result from [3, Corollary 4] cited in the introduction, only $E_5 \leq 4$ should be replaced by $E_5 \leq 5$.

Proof of Theorem 3. By the results from [7, 9] cited in the introduction, we may assume that $k$ is odd and $k \geq 3$.

Consider first the case $k \geq 5$. Let $\{a_1, a_2, \ldots, a_m\}$ be a $k$th-power $D(n)$-$m$-tuple, and $0 < a_1 < a_2 < \cdots < a_m$. For $i \geq 3$ we have

$$a_1 a_i + n = x_i^k, \quad a_2 a_i + n = y_i^k,$$

i.e.

$$a_2 x_i^k - a_1 y_i^k = n(a_2 - a_1). \tag{23}$$

Let $d_i = \gcd(x_i, y_i)$ and write $x_i = d_i x_i'$. Note that $d_i^k \leq |n|(a_2 - a_1)$. We apply Lemma 2 to the Thue inequality

$$|a_2 x^k - a_1 y^k| \leq |n|(a_2 - a_1). \tag{24}$$

By Lemma 2, there is only one very large primitive solution to (24). It may correspond to $a_m$, but certainly not to $a_i$ for $i < m$. Thus we have

$$a_1 a_{m-1} < 2|n| x_{m-1}^k = 2|n| x_{m-1}'^k d_{m-1}^k \leq 2n^2 a_2 x_{m-1}'^k < 2n^2 \cdot k^2 \cdot (|n| a_2)^{13/4},$$

i.e.

$$a_{m-1} < 2k^2 |n|^{21/4} a_2^{13/4}. \tag{25}$$

Assume now that at least four $a_i$'s are larger than $2|n|^5$, i.e. $a_{m-3} > 2|n|^5$. In order to obtain a lower bound for $a_{m-1}$, we first consider the case $n > 0$. Then we have

$$(a_1 a_{m-2} + n)(a_2 a_{m-1} + n) > (a_2 a_{m-2} + n)(a_1 a_{m-1} + n),$$

which implies

$$(a_1 a_{m-2} + n)(a_2 a_{m-1} + n) \geq (((a_2 a_{m-2} + n)(a_1 a_{m-1} + n))^{1/k} + 1)^k,$$

$$na_2a_{m-1} \geq k(a_1a_2a_{m-2}a_{m-1})^{(k-1)/k},$$

and finally

$$a_{m-1} > k^k a_1^{k-1} a_{m-2}^{k-2} n^{-k}. \tag{26}$$

Assume now that $n < 0$. Then

$$(a_1a_{m-2} + n)(a_2a_{m-1} + n) < (a_2a_{m-2} + n)(a_1a_{m-1} + n),$$

which implies

$$(a_2a_{m-2} + n)(a_1a_{m-1} + n) \geq ((a_1a_{m-2} + n)(a_2a_{m-1} + n)^{1/k} + 1)^k,$$

$$|n|a_2a_{m-1} \geq k(4a_1a_2a_{m-2}a_{m-1}/9)^{(k-1)/k}, \tag{27}$$

(here we use that $a_{m-2} \geq 2|n|^5 + 1 \geq 3|n|$) and finally

$$a_{m-1} > (9/4)^{1-k} k^k a_1^{k-1} a_{m-2}^{k-2} |n|^{-k}. \tag{28}$$

From (26) and (28) in both cases we get

$$a_{m-1} > 2k^2 a_{m-2}^{k-2} |n|^{-k}. \tag{29}$$

By the same arguments we get $a_{m-2} > 2k^2 a_{m-3}^{k-2} |n|^{-k}$. Therefore,

$$a_{m-1} > (2k^2)^{k-1} a_{m-2}^{(k-2)^2} |n|^{-k(k-1)}. \tag{30}$$

Comparing (25) with (30), we get $a_{m-3}^{(k-2)^2-13/4} < |n|^{k^2-k+21/4}$. Now we use the assumption that $a_{m-3} > 2|n|^5$. We get $4k^2 - 19k - 3/2 < 0$, and $k < 5$, a contradiction. Hence, at most three $a_i$'s are greater than $2|n|^5$, which shows that $m \leq 2|n|^5 + 3$, as claimed.

It remains to consider the case $k = 3$. In that case the above approach needs some modifications because the exponent of $a_{m-2}$ in (28), i.e. $k-2$, is not greater than 1. The bound for $m$ will also be considerably weaker. Assume that at least seven $a_i$'s are larger than $2|n|^{17}$, i.e. $a_{m-6} > 2|n|^{17}$. We take a closer look at (27), which for $k = 3$ gives

$$a_2a_{m-1} > 5a_1^2 a_{m-2}^2 |n|^{-3} \tag{31}$$

and analogously

$$a_3a_{m-1} > 5a_2^2 a_{m-2}^2 |n|^{-3}. \tag{32}$$

We claim that

$$a_{m-1} > 5|n|^{-3} a_{m-2}^{5/3}. \tag{33}$$

Indeed, if $a_{m-1} \leq 5|n|^{-3} a_{m-2}^{5/3}$, then (31) and (32) imply $a_2 > a_1^2 a_{m-2}^{1/3}$ and $a_3 > a_2^2 a_{m-2}^{1/3}$. But this leads to $a_3 > a_1^4 a_{m-2} \geq a_{m-2}$, a contradiction. By iterating (33) five times, we obtain

$$a_{m-1} > (5|n|^{-3})^{1441/81} a_{m-6}^{3125/243}. \tag{34}$$

On the other hand, an application of Lemma 2 to (24) for $k = 3$ gives

$$a_{m-1} < 2305|n|^{11} a_2^9. \tag{35}$$

Comparing (35) with (34) we get

$$a_{m-6}^{938/243} < |n|^{1738/27}. \tag{36}$$

The assumption that $a_{m-6} > 2|n|^{17}$, combined with (36), leads to a contradiction. Hence, $m \leq 2|n|^{17} + 6$, as we claimed. $\square$

Proof of Theorem 4. The proof goes along the same lines as the corresponding one in [22, Theorem 1.4]. However, for the convenience of the reader we give the details. We may assume that $\mathcal{A} \subseteq \mathbb{N}$, since the bound for subsets of $\mathbb{Z}$ can be obtained by doubling the bound for subsets of $\mathbb{N}$. Let $\mathcal{A}' = \{a \in \mathcal{A} : a > c_0|n|^3\}$, where $c_0$ is defined in Lemma 1. By Lemma 1, in the set $\mathcal{A}'$ there does not exist a subset of five elements such that $a_i a_j + n = x_{ij}^{k_{ij}}$ with $k_{ij} \geq 3205$ for all distinct $i$ and $j$. Let $t = \pi(3205) = 453$ and let $p_i$ be the $i$th prime. We let $G$ be the graph whose vertices are the elements of $\mathcal{A}'$. We color the edges of $G$ with the $t+1$ colors $p_1, \ldots, p_t, \infty$ in such a way that if $a, b \in \mathcal{A}'$, then we assign to the edge $ab$ the color $p_i$, $i \in \{1, \ldots, t\}$ if $p_i$ is the smallest prime for which there exist an integer $x$ such that $ab + 1 = x^{p_i}$. If such $p_i$ does not exist, we assign the color $\infty$ to the edge $ab$.

We finish the proof by using the existence of Ramsey numbers. The Ramsey number $R(n_1, \ldots, n_s)$ is the smallest positive integer $R$ such that no matter how we color the edges of the complete graph with $R$ vertices with the colors $1, 2, \ldots, s$, there exist a color $i$ and a complete monochromatic subgraph with $n_i$ vertices colored with color $i$ (see e.g. [18]). For given non-zero integer $n$, consider the following well-defined positive integer

$$R(n) = R(C_1(2,n), C_1(3,n), C_1(5,n), \ldots, C_1(3203,n), 5),$$

where the quantities $C_1(k,n)$ are defined in Theorem 3. We claim that $|\mathcal{A}'| < R(n)$, and therefore $|\mathcal{A}| < c_0 n^3 + R(n)$, which will complete the proof of Theorem 4. Indeed, if $|\mathcal{A}'| \geq R(n)$, then either there exist a prime number $p \leq 3203$ and at least $C_1(p,n)$ elements of $\mathcal{A}'$ such that the product of any two of them plus $n$ is a $p$th power, contradicting Theorem 3, or there exist at least five elements of $\mathcal{A}'$ such that the product of any two of them plus $n$ is a $k$th power with some $k \geq 3205$, contradicting Lemma 1. □

## References


[1] A. Baker, H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J. Math. Oxford Ser. (2)* **20** (1969), 129–137.

[2] E. Brown, Sets in which $xy + k$ is always a square, *Math. Comp.* **45** (1985), 613–620.

[3] Y. Bugeaud, A. Dujella, On a problem of Diophantus for higher powers, *Math. Proc. Cambridge Phil. Soc.* **135** (2003), 1–10.

[4] Y. Bugeaud, K. Gyarmati, On generalizations of a problem of Diophantus, *Illinois J. Math* **148** (2004), 1105–1115.

[5] L. E. Dickson, History of the Theory of Numbers, Vol. 2, *Chelsea*, *New York*, 1966, 513–520.

[6] R. Dietmann, C. Elsholtz, K. Gyarmati, M. Simonovits, Shifted products that are coprime pure powers, *J. Combin. Theory Ser. A* **111** (2005), 24–36.

[7] A. Dujella, On the size of Diophantine $m$-tuples, *Math. Proc. Cambridge Phil. Soc.* **132** (2002), 23–33.

[8] A. Dujella, There are only finitely many Diophantine quintuples, *J. Reine Angew. Math.* **566** (2004), 183–214.

[9] A. Dujella, Bounds for the size of sets with the property $D(n)$, *Glas. Mat. Ser. III* **39** (2004), 199–205.

[10] A. Dujella, Diophantine $m$-tuples, http://web.math.hr/~duje/dtuples.html.

[11] A. Dujella, C. Fuchs, Complete solution of a problem of Diophantus and Euler, *J. London Math. Soc.* **71** (2005), 33–52.

[12] A. Dujella, F. Luca, Diophantine m-tuples for primes, *Intern. Math. Research Notices* **47** (2005), 2913–2940.

[13] A. Dujella, A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser. (2)* **49** (1998), 291–306.

[14] J. H. Evertse, Upper Bounds for the Numbers of Solutions of Diophantine Equations, MCT 168, *Mathematisch Centrum*, *Amsterdam*, 1983.

[15] Y. Fujita, The number of Diophantine quintuples, *Glas. Mat. Ser. III* **45** (2010), 15–29.

[16] A. Filipin, There does not exist a $D(4)$-sextuple, *J. Number Theory* **128** (2008), 1555–1565.

[17] P. Gibbs, Some rational Diophantine sextuples, *Glas. Mat. Ser. III* **41** (2006), 195–203.

[18] R.L. Graham, B.L. Rothschild, J.H. Spencer, Ramsey Theory, *John Wiley & Sons*, *New York*, 1980.

[19] K. Gyarmati, On a problem of Diophantus, *Acta Arith.* **97** (2001), 53–65.

[20] K. Gyarmati, A. Sárközy, C. L. Stewart, On shifted products which are powers, *Mathematika* **49** (2002), 227–230.

[21] K. Gyarmati, C. L. Stewart, On powers in shifted products, *Glas. Mat. Ser. III* **42** (2007), 273–279.

[22] F. Luca, On shifted products which are powers, *Glas. Mat. Ser. III* **40** (2005), 13–20.

[23] J. B. Rosser, L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962), 64–94.

[24] C. L. Stewart, On sets of integers whose shifted products are powers, *J. Combin. Theory Ser. A* **115** (2008), 662–673.

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