Root separation for reducible monic quartics

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Abstract

We study root separation for reducible monic integer polynomials of degree four. If $\mathrm{H}(P)$ is the height and $\mathrm{sep}(P)$ the minimal distance between two distinct roots of a separable integer polynomial $P(x)$, and $\mathrm{sep}(P) = \mathrm{H}(P)^{-e(P)}$, we show that $\limsup e(P) = 2$, where limsup is taken over all reducible monic integer polynomials $P(x)$ of degree 4.

1 Introduction

The height $\mathrm{H}(P)$ of an integer polynomial $P(x)$ is the maximum of the absolute values of its coefficients. For an integer polynomial $P(x)$ of degree $d \geq 2$ and with distinct roots $\alpha_1, \ldots, \alpha_d$, we set

$$\mathrm{sep}(P) := \min_{1 \leq i < j \leq d} |\alpha_i - \alpha_j|$$

and define $e(P)$ by

$$\mathrm{sep}(P) := \mathrm{H}(P)^{-e(P)}.$$

For an infinite set S of integer polynomials containing polynomials of arbitrary large height, we define

$$e(S) = \limsup_{P(X) \in S,\, \mathrm{H}(P) \to +\infty} e(P).$$

In this note we will be concerned with reducible monic polynomials of degree four with integer coefficients. Therefore, we introduce notation $\mathcal{RM}_d$

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for the set of all reducible monic polynomials of degree d with integer coefficients.

First, we briefly summarize what is known about bounds on $e(S)$ if S is some class of integer polynomials of small degree. A classical result of Mahler [5] asserts that if S contains only polynomials of degree d, then $e(S) \leq d-1$.

The case of quadratic polynomials is almost trivial and won't be discussed further:

$d=2$	general	monic
irreducible	$e=1$	$e=0$
reducible	$e=1$	$e=0$

For cubic polynomials, the case of general (i.e. nonmonic) polynomials was first solved by Evertse [4] and later Schönhage [6] gave an easier constructive proof. In the monic case Bugeaud and Mignotte [3] proved the lower bound $e(\mathcal{M}_3) \geq \frac{3}{2}$, where $\mathcal{M}_3$ is the set of monic cubic polynomials with integer coefficients. They also showed that $e(\mathcal{M}_3) = \frac{3}{2}$ is equivalent to Hall conjecture. Proving that $e(\mathcal{RM}_3) = 1$ is not hard when we notice that a polynomial from this set is a product of a linear and a quadratic polynomial, both monic and with integer coefficients because of Gauss's Lemma. In the next table we summarize known results for $d=3$:

$d=3$	general	monic
irreducible	$e=2$	$e \geq \frac{3}{2}$
reducible	$e=2$	$e=1$

Until now no exact values when $d=4$ were known, just the lower bounds given in the following table:

$d=4$	general	monic
irreducible	$e \geq \frac{13}{6}$	$e \geq \frac{3}{2}$
reducible	$e \geq \frac{7}{3}$	$e \geq 2$

The bound for nonmonic irreducible case arises from a general construction by Bugeaud and Dujella [2] which in this special case gives $e((\overline{P}_{4,n}(x))_{n\in\mathbb{N}}) = \frac{13}{6}$, where

$$\overline{P}_{4,n}(x) = (20n^4-2)x^4 + (16n^5+4n)x^3 + (16n^6+4n^2)x^2 + 8n^3x + 1.$$

For nonmonic reducible polynomials, a recent unpublished result by Bugeaud and Dujella, shows that the sequence

$$\widetilde{P}_{4,n}(x) = \big((2n+1)x^3 + (2n-1)x^2 + (n-1)x - 1\big)\big((n^2+3n+1)x - (n+2)\big)$$

gives $e \geq e((\widetilde{P}_{4,n}(x))_{n\in\mathbb{N}}) = \frac{7}{3}$. The bound for monic irreducible polynomials $e \geq \frac{3}{2}$ is deduced by looking at the sequence

$$\widehat{P}_{4,n}(x) = (x^2 - nx + 1)^2 - 2(nx-1)^2, \quad n \in \mathbb{N}$$

(see Bugeaud and Mignotte [3]). Finally, for reducible monic polynomials, it follows from a general case discussed in [3] that $e(\mathcal{RM}_4) \geq 2$. While the proof from [3] is nonconstructive, in Section 2 we establish the same inequality by exhibiting a set $S \subseteq \mathcal{RM}_4$ such that $e(S) = 2$. In Section 3 we prove that $e(\mathcal{RM}_4) \leq 2$. By putting together the results from Sections 2 and 3, we obtain the main result of this paper, which gives the first exact value in the above table for $d = 4$.

Theorem 1 *It holds that $e(\mathcal{RM}_4) = 2$.*

Furthermore, in Section 4, we show that if the coefficients of polynomials in the sequence $S = (P_n(x))_{n\in\mathbb{N}} \subseteq \mathcal{RM}_4$ grow polynomially in n, we must have a strict inequality $e(S) < 2$. But we also show that we can choose such a sequence so that $e(S)$ is arbitrarily close to 2. More precisely, we prove the following theorem.

Theorem 2 *If $S = (P_n(x))_{n\in\mathbb{N}} \subseteq \mathcal{RM}_4$ is a sequence of polynomials whose coefficients are polynomials in n, then $e(S) < 2$. For any $\varepsilon > 0$, there is a a sequence of polynomials $S = (P_n(x))_{n\in\mathbb{N}} \subseteq \mathcal{RM}_4$ whose coefficients are polynomials in n such that $e(S) > 2 - \varepsilon$.*

A survey of results on separation of roots for integer polynomials of general degree can be found in the paper by Bugeaud and Mignotte [3] (see also [2]).

2 The constructive proof of $e(\mathcal{RM}_4) \geq 2$

We want to find a sequence of polynomials $S = (P_n(x))_{n\in\mathbb{N}} \subseteq \mathcal{RM}_4$ such that $e(S) = 2$. We look at integer polynomials of the type

$$P(x) = (x^2 + rx + s)(x^2 + ax + b),$$

where r and s are fixed while a and b depend on them and on n such that one root of the polynomial in the first bracket is very close to a root of the polynomial in the second bracket.

Choose r and s such that the roots λ_1, λ_2 of the polynomial $R(x) = x^2 + rx + s \in \mathbb{Z}[x]$ satisfy $\lambda = \lambda_1 > 1 > \lambda_2 > 0$. Also, let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integers that satisfies the recurrence $a_{n+2} + ra_{n+1} + sa_n = 0$ whose characteristic polynomial is $R(x)$. Hence,

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n = c_1\lambda^n + c_2\frac{s}{\lambda^n},$$

for some constants c_1, c_2.

Assume that $\lambda + \varepsilon$ is a root of the polynomial $x^2 + ax + b \in \mathbb{Z}[x]$. Then we have

$$(\lambda+\varepsilon)^2 + a(\lambda+\varepsilon) + b = 0$$
$$\varepsilon^2 + (2\lambda + a)\varepsilon + (a-r)\lambda + (b-s) = 0.$$

Therefore $2\varepsilon = -(2\lambda+a) \pm \sqrt{(2\lambda+a)^2 - 4\big((a-r)\lambda + (b-s)\big)}$. If we have

$$2\lambda + a > 0 \qquad \text{and} \qquad |4\big((a-r)\lambda+(b-s)\big)| < (2\lambda+a)^2, \tag{1}$$

then we get a smaller $|\varepsilon|$ for the $+$ sign, so

$$\begin{aligned} |2\varepsilon| &= \left|\frac{4\big((a-r)\lambda+(b-s)\big)}{-(2\lambda+a) - \sqrt{(2\lambda+a)^2 - 4\big((a-r)\lambda+(b-s)\big)}}\right| \\ &\asymp \left|\frac{(a-r)\lambda+(b-s)}{2\lambda+a}\right| \end{aligned} \tag{2}$$

(here $M \asymp N$ stands for $M \ll N$ and $N \ll M$, where the implicit constants depend only on r and s). At this point we see that by choosing

$$a - r = a_n, \quad r \le -1, \quad b - s = -a_{n+1}, \quad s = 1,$$

conditions on λ_1, λ_2, $(a_n)_{n\in\mathbb{N}}$ and inequalities (1) are fulfilled, while from (2) we have

$$\begin{aligned} \operatorname{sep}(P_n) = |\varepsilon| &\asymp \left|\frac{a_n\lambda - a_{n+1}}{2\lambda + a_n + r}\right| = \left|\frac{c_1\lambda^{n+1} + \frac{c_2}{\lambda^{n-1}} - c_1\lambda^{n+1} - \frac{c_2}{\lambda^{n+1}}}{2\lambda + c_1\lambda^n + \frac{c_2}{\lambda^n} + r}\right| \\ &\asymp \frac{1}{\lambda^{2n}} \asymp \max\{1, |a|, |b|\}^{-2} \asymp \mathrm{H}(P_n)^{-2} \end{aligned}$$

and thus

$$e((P_n)_{n\in\mathbb{N}}) = 2,$$

where

$$P_n(x) = (x^2 + rx + 1)\big(x^2 + (r + a_n)x + (1 - a_{n+1})\big).$$

This shows that $e(\mathcal{RM}_4) \geq 2$.

Note that we could have taken $s = -1$ before and if we were trying to approach the smaller root i.e. λ_2, we would get a similar family of polynomials

$$P_n(x) = (x^2 + rx - 1)\big(x^2 + (r - a_{n+1})x - (a_n + 1)\big),$$

and after substitution $x \mapsto -x$, we would get

$$P_n(x) = (x^2 - rx - 1)\big(x^2 + (-r + a_{n+1})x - (a_n + 1)\big).$$

In case of $a_1 = 1$, $a_2 = 1$, $r = -1$, the above polynomial is

$$P_n(x) = (x^2 + x - 1)\big(x^2 + (1 + F_{n+1})x - (F_n + 1)\big)$$

where $(F_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence. This last sequence of polynomials, which was first obtained by numerical experiments, was the motivating factor for this study.

3 The proof of $e(\mathcal{RM}_4) \leq 2$

Let us prove that $e(\mathcal{RM}_4) \leq 2$. In other words, the best separation of roots we can get in the case of a *reducible* separable monic quartic polynomial $P(x) \in \mathbb{Z}[x]$ is $\asymp \big(\mathrm{H}(P)\big)^{-2}$. (All the constants implied in $\asymp, \ll, \gg$ in this section are absolute.)

We have to look at two cases: when the polynomial has a cubic irreducible factor and when the polynomial has a quadratic irreducible factor. Because of Gauss's Lemma all the divisors in $\mathbb{Q}[x]$ of $P(x)$ will actually be from $\mathbb{Z}[x]$. Therefore, the case when $P(x)$ is a product of linear factors is trivial.

If we have $P(x) = (x-k)(x^3+ax^2+bx+c)$, where $a, b, c, k \in \mathbb{Z}$, then by the result of Mahler we know that the roots of $Q(x) = x^3 + ax^2 + bx + c$ can be no closer than $\asymp (\max\{1, |a|, |b|, |c|\})^{-2}$. Because of Gelfond's Lemma (see e.g. [1, p. 221]), we have

$$\frac{1}{16}\max\{1,|k|\}\max\{1,|a|,|b|,|c|\} \leq \mathrm{H}(P) \leq 16\max\{1,|k|\}\max\{1,|a|,|b|,|c|\}, \tag{3}$$

so $\mathrm{sep}(Q) \gg \mathrm{H}(P)^{-2}$. There only remains to check whether we can have a root of $Q(x)$ close to k. Let us take $Q(k+\varepsilon) = (k+\varepsilon)^3 + a(k+\varepsilon)^2 +$

$b(k+\varepsilon)+c=0$ where without loss of generality we can suppose $|\varepsilon|<1$. It is obvious that $|k+\varepsilon|<|a|+|b|+|c|+1$ must hold, otherwise we get a contradiction. Thus, from (3) we get $|k| \ll \mathrm{H}(P)^{1/2}$. Since $P(x)$ does not have multiple roots and $Q(x) \in \mathbb{Z}[x]$ we have

$$1 \leq |Q(k)| = |Q(k+\varepsilon) - Q(k)| = |Q'(t)| \cdot |\varepsilon|,$$

where $t \in (k, k+\varepsilon) \subset \langle k-1, k+1 \rangle$. But, using (3) and $|k| \ll \mathrm{H}(P)^{1/2}$, we get

$$|Q'(t)| = |3t^2 + 2at + b| \leq 3(|k|+1)^2 + 2|a|(|k|+1) + |b| \ll \mathrm{H}(P).$$

Finally, we arrive at $|\varepsilon| \geq 1/|Q'(t)| \gg \mathrm{H}(P)^{-1}$.

If $P(x) = Q_1(x)Q_2(x)$, where $Q_1(x), Q_2(x) \in \mathbb{Z}[x]$ are two quadratic polynomials, then we have from Gelfond's Lemma

$$\frac{1}{16}\,\mathrm{H}(Q_1)\,\mathrm{H}(Q_2) \leq \mathrm{H}(P) \leq 16\,\mathrm{H}(Q_1)\,\mathrm{H}(Q_2). \tag{4}$$

Since for quadratic polynomials we have $\mathrm{sep}(Q_i) \gg H(Q_i)^{-1}$, we only have to check the proximity of the roots α and β of $Q_1(x)$ and $Q_2(x)$, respectively. Theorem A.1 from [1, p. 223] states that in our separable case

$$|\alpha - \beta| \geq 2^{-1}3^{-5/2} \cdot \mathrm{H}(Q_1)^{-2}\,\mathrm{H}(Q_2)^{-2} \cdot \max\{1, |\alpha|\}\max\{1, |\beta|\} \gg \mathrm{H}(P)^{-2}.$$

Hence, we proved that $e(\mathcal{RM}_4) \leq 2$, which concludes the proof of Theorem 1.

4 Polynomial growth of coefficients

In Section 2 we exhibited a family of reducible monic polynomials $P_n(x)$ whose coefficients grow exponentially in n such that $\mathrm{sep}(P_n) \asymp \mathrm{H}(P_n)^{-2}$.

We will show that this is not possible if the coefficients grow polynomially. More precisely, let $P_n(x) = P(n,x) \in \mathbb{Z}[n,x]$ be a polynomial which is monic of degree 4 in x and such that for every positive integer n', polynomial $P_{n'}(x) \in \mathbb{Z}[x]$ is reducible. This is the exact meaning of conditions in the first statement of Theorem 2. Hilbert's Irreducibility Theorem (see e.g. Zannier [7]) implies that

$$P_n(x) = Q_{n,1}(x)Q_{n,2}(x),$$

where $Q_{n,1}(x)$ and $Q_{n,2}(x)$ are monic polynomials in x whose coefficients are integer polynomials in n. Note that because of the previous section, the case of a reducible monic polynomial with a linear factor is not very interesting. Therefore, we will assume that $Q_{n,1}(x)$ and $Q_{n,2}(x)$ are irreducible quadratic polynomials in x without common roots, so

$$Q_{n,1}(x) = x^2 + r(n)x + s(n), \quad Q_{n,2}(x) = x^2 + a(n)x + b(n),$$

where $r(n), s(n), a(n), b(n) \in \mathbb{Z}[n]$. For the sake of simplicity, we will most often omit n. As already mentioned, we can assume that the closest roots of P are a root of Q_1 and a root of Q_2. So, without loss of generality, let us take

$$2\,\mathrm{sep}(P) = 2\varepsilon = -r + \sqrt{r^2 - 4s} + a + \sqrt{a^2 - 4b}.$$

After some manipulation we get that ε satisfies the following equality

$$\begin{aligned}&\varepsilon^4 - 2(a-r)\varepsilon^3 + (r^2 + a^2 - 3ra + 2s + 2b)\varepsilon^2 \\ &\quad - (a-r)(-ra + 2s + 2b)\varepsilon + (s^2 + b^2 - rsa - rab - 2bs + sa^2 + br^2) = 0.\end{aligned} \tag{5}$$

Notice that the last term is just the resultant $\mathrm{Res}_x(Q_1, Q_2)$ of polynomials Q_1 and Q_2:

$$\mathrm{Res}(Q_1, Q_2) = \mathrm{Res}(Q_1, Q_2 - Q_1) = (b-s)^2 + (a-r)(as - br).$$

Let us suppose that $\varepsilon \ll \mathrm{H}^{-2}$, where by Gelfond's Lemma $\mathrm{H} = \mathrm{H}(P) \asymp \mathrm{H}(Q_1)\,\mathrm{H}(Q_2)$. It can be mentioned here that all the constants in $\mathcal{O}$, $\ll$, $\gg$, $\asymp$ in the first part of this section depend at most on the coefficients of r, s, a, b. Since $P(x)$ is a separable integer polynomial, it follows that $\mathrm{Res}(Q_1, Q_2)$ is an integer polynomial in n and $|\mathrm{Res}(Q_1, Q_2)| \geq 1$. Now we get from (5) and (4) that

$$\mathrm{H}^{-2} \gg \varepsilon \gg \frac{|\mathrm{Res}(Q_1, Q_2)|}{|\underbrace{\varepsilon^3}_{\mathcal{O}(\mathrm{H}^{-6})} - \underbrace{2(a-r)\varepsilon^2}_{\mathcal{O}(\mathrm{H}^{-3})} + \underbrace{(\underbrace{r^2 + a^2}_{\mathcal{O}(\mathrm{H}^2)} - \underbrace{3ra + 2s + 2b}_{\mathcal{O}(\mathrm{H})})\varepsilon}_{\mathcal{O}(1)} - (a-r)(-ra + 2s + 2b)|}$$

and

$$\mathrm{H}^{-2} \gg \varepsilon \gg \frac{|\mathrm{Res}(Q_1, Q_2)|}{|\mathcal{O}(1) - \underbrace{2as + 2rb}_{\mathcal{O}(\mathrm{H})} + ra^2 - r^2 a + 2rs - 2ab|}. \tag{6}$$

Because of Gelfond's Lemma, $|r|, |s|, |a|, |b| \ll \mathrm{H}$ and $|ar| \ll \mathrm{H}$ which implies that $|a| \ll \mathrm{H}^{1/2}$ or $|r| \ll \mathrm{H}^{1/2}$. Without loss of generality we can suppose that $|a| \ll \mathrm{H}^{1/2}$. Thus we get $|ra^2| = |ra| \cdot |a| \ll \mathrm{H}^{3/2}$ and $|ab| = |a| \cdot |b| \ll \mathrm{H}^{3/2}$. We also have $|-r^2a + 2rs| = |r| \cdot |ra - 2s| = |r|\mathcal{O}(\mathrm{H})$ so the inequality (6) becomes

$$\mathrm{H}^{-2} \gg \varepsilon \gg \frac{1}{\max\{\mathcal{O}(\mathrm{H}^{3/2}), |r|\mathcal{O}(\mathrm{H})\}}.$$

It implies that $|r| \gg \mathrm{H}$, so from $|r| \ll \mathrm{H}$, we get $|r| \asymp \mathrm{H}$. Also, $|\operatorname{Res}(Q_1, Q_2)| = \mathcal{O}(1)$. Since r, s, a, b are polynomials in n and $|ra| \ll \mathrm{H}$, $|rb| \ll \mathrm{H}$, we conclude that a and b are constants.

If we now have $\deg_n s < \deg_n r$ then

$$\deg_n \operatorname{Res}(Q_1, Q_2) = \deg_n \big((b-s)^2 + (a-r)(as-br)\big) \geq \deg_n r + \deg_n s,$$

so $|\operatorname{Res}(Q_1, Q_2)| \gg \mathrm{H}$, which leads to a contradiction. Therefore, $\deg_n s = \deg_n r$ and hence $|s| \asymp |r| \asymp \mathrm{H} \to \infty$.

The leading coefficient of $\operatorname{Res}(Q_1, Q_2)$ as a polynomial in n, i.e. the coefficient that belongs to the monomial of degree $2\deg_n r = 2\deg_n s$, is the leading coefficient of $s^2 - ars + br^2$, i.e. $k_s^2 - ak_rk_s + bk_r^2$, where k_s, k_r are leading coefficients of s and r, respectively. If it were 0, then $-k_s/k_r \in \mathbb{Q}$ would be a root of $x^2 + ax + b$ which is impossible, since by our assumption this polynomial is irreducible. Thus $\deg_n \operatorname{Res}(Q_1, Q_2) = 2\deg_n r \geq 2$ and this is in contradiction with the condition $|\operatorname{Res}(Q_1, Q_2)| = \mathcal{O}(1)$.

We conclude that $\operatorname{sep}(P_n) \ll \mathrm{H}(P_n)^{-2}$ cannot hold in this case, and this proves the first statement of Theorem 2.

Although the previous result of this section shows that we cannot have a family of reducible monic quartic integer polynomials with polynomial growth of coefficients that has the best possible exponent for root separation in this case, i.e. -2, we can still construct families with the exponent as close to -2 as we like. The construction that follows is similar to the one in Section 2.

We look at the family of polynomials $P_{k,n}(x)$ indexed with $n \in \mathbb{N}$ in variable x. As before, we will usually omit n and write simply $P_k(x)$. We define

$$\begin{aligned} P_k(x) &= \underbrace{(x^2 + nx + 1)}_{Q_k(x)} \underbrace{(x^2 + nx + 1 + A_{k+1}x + A_k)}_{R_k(x)} \\ &= (x^2 + \underbrace{n}_{r} x + \underbrace{1}_{s})\big(x^2 + \underbrace{(A_{k+1} + n)}_{a} + \underbrace{(A_k + 1)}_{b}\big), \end{aligned}$$

where $\big(A_k(n)\big)_{k \in \mathbb{N}_0}$ is defined recursively by

$$A_0(n) = 1, \ A_1(n) = n, \quad A_{k+1}(n) = nA_k(n) - A_{k-1}(n) \text{ for } n \geq 2.$$

It is easy to see that $\deg_n A_k = k$, so we get (implied constants are absolute from now on)

$$\mathrm{H}(P_k) \asymp n^{k+2}.$$

Let us look at the resultant:

$$\begin{aligned}
\mathrm{Res}_x(Q_k, R_k) &= (b-s)^2 - r(b-s)(a-r) + s(a-r)^2 \\
&= A_k^2 - nA_kA_{k+1} + A_{k+1}^2 \\
&= A_k^2 + A_{k+1}(A_{k+1} - nA_k) \\
&= A_k^2 - A_{k+1}A_{k-1} \\
&= A_k^2 - (nA_k - A_{k-1})A_{k-1} \\
&= A_k(A_k - nA_{k-1}) + A_{k-1}^2 \\
&= A_{k-1}^2 - A_kA_{k-2} \\
&= \ldots = A_1^2 - A_2A_0 = n^2 - (n^2-1)\cdot 1 = 1.
\end{aligned}$$

The roots of $Q_k(x)$ are

$$\alpha_1 = \frac{-n-\sqrt{n^2-4}}{2}, \quad \alpha_2 = \frac{-n+\sqrt{n^2-4}}{2},$$

and the roots of $R_k(x)$ are

$$\begin{aligned}
\beta_1 &= \frac{-(A_{k+1}+n) - \sqrt{(A_{k+1}+n)^2 - 4(A_k+1)}}{2}, \\
\beta_2 &= \frac{-(A_{k+1}+n) + \sqrt{(A_{k+1}+n)^2 - 4(A_k+1)}}{2}.
\end{aligned}$$

Therefore,

$$\alpha_1 \asymp -n, \quad \alpha_2 \asymp -\frac{1}{n}, \qquad \beta_1 \asymp -n^{k+1}, \quad \beta_2 = \frac{A_k+1}{\beta_1} \asymp \frac{-1}{n},$$

so we have

$$1 = \mathrm{Res}(Q_k, R_k) = 1^2 1^2 \underbrace{|\alpha_1 - \alpha_2|}_{\asymp n} \underbrace{|\alpha_1 - \beta_1|}_{\asymp n^{k+1}} \underbrace{|\alpha_2 - \beta_1|}_{\asymp n^{k+1}} \mathrm{sep}(P_k),$$

and it follows that

$$\mathrm{sep}(P_k) \asymp n^{-2k-3} = n^{-2(k+2)} n \asymp \mathrm{H}(P_k)^{-2+\frac{1}{k+2}}.$$

Hence, we proved the last statement of Theorem 2.

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