FIBONACCI POWER MEANS

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ABSTRACT. Starting from an inequality for partial sums of integer powers of the Fibonacci sequence, a model of Fibonacci means is developed, possessing a multitude of interesting properties expressed through inequalities.

1. Introduction

The Fibonacci sequence is defined recursively by

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-2} + F_{n-1}, \quad n \in \mathbb{N}.$$

The next theorem can be found in [9].

Theorem 1.1. *Let n be a positive integer and ℓ be an integer. Then,*

$$\left(F_1^\ell + F_2^\ell + \ldots + F_n^\ell\right)\left(\frac{1}{F_1^{\ell-4}} + \frac{1}{F_2^{\ell-4}} + \cdots + \frac{1}{F_n^{\ell-4}}\right) \geq F_n^2 F_{n+1}^2. \tag{1.1}$$

In the following remarks, we outline the structure of this paper.

(i) A direct examination of the proof in the paper [9] shows that $\ell \in \mathbb{N}$ can be replaced by any real number. In the works [1] and [2], extensions of the results (1.1) to real numbers are also provided, but the proof techniques and the direction of generalization differ from our approach.

(ii) Let $u \in \mathbb{R}$. By using the arithmetic-harmonic inequality

$$\frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} \frac{w_i}{x_i}} \leq \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} \tag{1.2}$$

using the substitutions

$$w_i = F_i^2, \; x_i = F_i^u, \quad i = 1, \ldots, n, \tag{1.3}$$

and using identity (see [7, p. 12])

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1} \tag{1.4}$$

we obtain

$$F_n^2 F_{n+1}^2 \leq \sum_{i=1}^{n} F_i^{2-u} \sum_{i=1}^{n} F_i^{2+u}, \; u \in \mathbb{R}. \tag{1.5}$$

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(iii) The arithmetic and harmonic means belong to the class of power means, which are defined as follows (see [10, p. 108]):

$$M_n^{[r]}(\mathbf{x},\mathbf{w}) = \begin{cases} \left(\frac{\sum_{i=1}^n w_i x_i^r}{W_n}\right)^{\frac{1}{r}}, & r \neq 0, \\ \prod_{i=1}^n x_i^{w_i/W_n}, & r = 0, \\ \min\limits_{1\le i\le n} x_i, & r = -\infty, \\ \max\limits_{1\le i\le n} x_i, & r = \infty, \end{cases} \tag{1.6}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ stand for strictly positive $n-$tuples and $W_n = \sum_{i=1}^n w_i$.
A key property of power means is their comparability

$$M_n^{[r]}(\mathbf{x},\mathbf{w}) \le M_n^{[s]}(\mathbf{x},\mathbf{w}), \ -\infty \le r < s \le \infty. \tag{1.7}$$

Thus, if we choose the harmonic and arithmetic means by taking $r = -1$, $u = 1$ in (1.7), with the substitutions (1.3), we obtain (1.5). If we apply (1.7) for the case $-\infty < -1 < 0 < 1 < \infty$, with substitutions (1.3), we obtain the following refinement of (1.5):

$$\min\{F_1^u, F_n^u\} \le \frac{F_n F_{n+1}}{\sum_{i=1}^n F_i^{2-u}} \le \prod_{i=1}^n F_i^{\frac{sF_i^2}{F_n F_{n+1}}} \le \frac{\sum_{i=1}^n F_i^{u+2}}{F_n F_{n+1}} \le \max\{F_1^u, F_n^u\} \tag{1.8}$$

for all $n \in \mathbb{N}$, $u \in \mathbb{R}$.

2. Fibonacci means

Construction. Solving the problem from the previous section can be structured into the following model.

Definition 2.1. *Let $n \in \mathbb{N}$, $u \in \mathbb{R}$ and let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ stand for a strictly positive $n-$tuple. The Fibonacci power mean is defined by:*

$$F_n^{[r]}(\mathbf{w};u) = \begin{cases} \left(\frac{\sum_{i=1}^n w_i F_i^{ru}}{W_n}\right)^{\frac{1}{r}}, & r \neq 0 \\ \prod_{i=1}^n F_i^{uw_i/W_n}, & r = 0 \\ \min\{F_1^u, F_n^u\}, & r = -\infty \\ \max\{F_1^u, F_n^u\}, & r = \infty, \end{cases} \tag{2.1}$$

where $W_n = \sum_{i=1}^n w_i$ and where F_i denotes the i^{th} Fibonacci number.

The comparability property

$$F_n^{[r]}(\mathbf{w};u) \le F_n^{[s]}(\mathbf{w};u), -\infty \le r < s \le \infty \tag{2.2}$$

holds for all $n \in \mathbb{N}$, $u, s \in \mathbb{R}$, as this property is inherited from (1.7).
Other interesting relationships among power means that apply to our adapted Fibonacci means can also be utilized here.

Theorem 2.2. *Let $n \in \mathbb{N}$, $u \in \mathbb{R}$, $m = \min\{F_1^u, F_n^u\}$, $M = \max\{F_1^u, F_n^u\}$. If $0 < r < s$ or $r < 0 < s$, then*

$$(M^r - m^r)\left[F_n^{[s]}(\mathbf{w};u)\right]^s - (M^s - m^s)\left[F_n^{[r]}(\mathbf{w};u)\right]^r \le M^r m^s - M^s m^r. \tag{2.3}$$

If $r < s < 0$, then (2.3) is reversed.

Proof. See [4, p. 195] or [10, p. 109]. □

Corollary 2.3. *Let* $n \in \mathbb{N},\ u \in \mathbb{R}\ , m = \min\{F_1^u, F_n^u\},\ M = \max\{F_1^u, F_n^u\}$*. If* $0 < r < s$ *or* $r < 0 < s$*, then*

$$(M^r - m^r)\sum_{i=1}^{n}\binom{n}{i}2^iF_i^{su+1} - (M^s - m^s)\sum_{i=1}^{n}\binom{n}{i}2^iF_i^{ru+1} \le F_{3n}(M^r m^s - M^s m^r). \tag{2.4}$$

If $r < s < 0$*, then* (2.4) *is reversed.*

Proof. From identity (see [7, p. 56])

$$\sum_{i=0}^{n}\binom{n}{i}2^iF_i = F_{3n} \text{ we set } w_i = \binom{n}{i}2^iF_i,\ i = 0, 1, \dots, n,\ W_n = F_{3n} \tag{2.5}$$

and put in Theorem 2.2. □

The second result is about the ratio of the Fibonacci means.

Theorem 2.4. *Let* $u \in \mathbb{R}\ , m = \min\{F_1^u, F_n^u\},\ M = \max\{F_1^u, F_n^u\},\ \delta = M/m$*. Then for any* $-\infty < r < s < \infty$

$$F_n^{[s]}(\mathbf{w}; u) \le \Gamma_{r,s}(\delta) \cdot F_n^{[r]}(\mathbf{w}; u), \tag{2.6}$$

where

$$\Gamma_{r,s}(\delta) = \left(\frac{s-r}{\delta^s - \delta^r}\right)^{\frac{1}{r}-\frac{1}{s}} \left(\frac{\delta^{s-1}}{s}\right)^{\frac{1}{r}} \left(\frac{r}{\delta^{r-1}}\right)^{\frac{1}{s}}, \quad rs \ne 0,$$

$$\Gamma_{0,s}(\delta) = \left(\frac{\delta^{s/(\delta^s-1)}}{e\log\left(\delta^{s/(\delta^s-1)}\right)}\right)^{\frac{1}{s}} = \lim_{x\to 0^-}\Gamma_{r,s}(\delta),$$

$$\Gamma_{r,0}(\delta) = 1/\Gamma_{0,r}(\delta) = \lim_{s\to 0+}\Gamma_{r,s}(\delta).$$

Proof. See [4, p. 198] or [10, p. 110]. □

The next theorem is about estimation of the difference of the Fibonacci means.

Theorem 2.5. *Let* $u \in \mathbb{R},\ m = \min\{F_1^u, F_n^u\},\ M = \max\{F_1^u, F_n^u\}$ *and let* $-\infty < r < s < \infty$*. Then for* $n > 1$

$$F_n^{[s]}(\mathbf{w}; u) - F_n^{[r]}(\mathbf{w}; u) \le h(y), \tag{2.7}$$

where

$$h(y) = \begin{cases} (\theta M^s + (1-\theta)m^s)^{1/s} - (\theta M^r + (1-\theta)m^r)^{1/r}, & r \cdot s \ne 0, \\ (\theta M^s + (1-\theta)m^s)^{1/s} - M^\theta m^{1-\theta}, & r = 0, \\ M^\theta m^{1-\theta} - (\theta M^r + (1-\theta)m^r)^{1/r}, & s = 0, \end{cases}$$

and where

$$\theta = \begin{cases} \frac{y-m^s}{M^s-m^s}, & s \ne 0 \\ \frac{y-m^r}{M^r-m^r}, & s = 0. \end{cases}$$

Proof. See [4, p. 206-207] or [10, p. 111]. □

The approach in constructing these means is to select weights $\mathbf{w} = (w_1, w_2, \dots, w_n)$ that allow for a straightforward expression of the sum $W_n = \sum_{i=1}^n w_i$. With this in mind, we can make use of the following list of identities for Fibonacci numbers, similar to (1.3)-(1.4) and (2.5).

$$\begin{aligned}
\text{For}\quad i=1,\dots,n,\quad & w_i = F_i,\quad W_n = F_{n+2}-2, && [7,\text{ p. }11]\\
& w_i = F_{2i-1},\quad W_n = F_{2n}, && [7,\text{ p. }11]\\
& w_i = F_{2i},\quad W_n = F_{2n+1}-1, && [7,\text{ p. }11]\\
& w_i = iF_i,\quad W_n = F_{n+2}-F_{n+3}+2, && [7,\text{ p. }11]\\
& w_i = F_iF_{i+1},\quad W_n = F_{n+1}^2 - \frac{1}{2}\left[1+(-1)^n\right], && (\text{T. Koshy, 1998})\ [8]\\
& w_i = F_iF_{3i},\quad W_n = F_nF_{n+1}F_{2n+1}, && (\text{K. G. Recke, 1969})\ [11]\\
& w_i = F_{4i-2},\quad W_n = F_{2n}^2, && [7,\text{ p. }61]\\
& w_i = \binom{n}{i}F_i,\quad W_n = F_{2n}, && [7,\text{ p. }61]\\
\text{for}\quad i=1,\dots,2n+1,\quad & w_i = \binom{2n+1}{i}F_i^2,\quad W_{2n+1} = 5^nF_{2n+1}. && [7,\text{ p. }56]
\end{aligned}$$

Lesser known identities that can be used in our construction are

$$\text{for}\quad i=1,\dots,n,\quad w_i = \arctan\left(\frac{1}{F_{2i+1}}\right),\quad W_n = \frac{\pi}{4} - \arctan\left(\frac{1}{F_{2n+2}}\right), \qquad [7,\text{ p. }116] \tag{2.8}$$

$$\text{for}\quad i=1,\dots,n,\quad w_i = \binom{n}{i}\alpha^{3i-2n},\quad W_n = 2^n, \qquad (\text{H. Freitag, 1975})\ [6]$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $F_n = \frac{1}{\sqrt{5}}(\alpha^n - (\alpha-\sqrt{5})^n)$.

Lucas numbers and means. Lucas numbers are closely related to Fibonacci numbers and are defined as

$$L_n = F_{n+1} + F_{n-1},\quad n \geq 1,$$

and $L_0 := 2$.
The following identities about Lucas numbers can be included in our construction of Fibonacci means.

$$\begin{aligned}
\text{For}\quad i=1,\dots,n,\quad & w_i = L_{2i-1},\quad W_n = L_{2n}-2, && [7,\text{ p. }98]\\
& w_i = L_{2i},\quad W_n = L_{2n+1}-1, && [7,\text{ p. }98]\\
& w_i = iL_i,\quad W_n = nL_{n+2}-L_{n+3}+4, && [7,\text{ p. }98]\\
& w_i = L_i^2,\quad W_n = L_nL_{n+1}-2, && [7,\text{ p. }98]\\
& w_i = iL_i,\quad W_n = nL_{n+2}-L_{n+3}+4, && [7,\text{ p. }99]\\
\text{for } m\in\mathbb{N},\ i=0,1,\dots,n,\quad & w_i = L_{mi}L_{mn-mi},\quad W_n = 2^nL_{mn}+2L_m^n. && [7,\text{ p. }98]
\end{aligned}$$

Remark 2.6. *In a completely analogous way,* Lucas means *can also be defined using Lucas numbers: if $n \in \mathbb{N},\ u \in \mathbb{R}$ and if $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ stands for a strictly positive $n-$tuple*

$$L_n^{[r]}(\mathbf{w}; u) = \begin{cases} \left(\frac{\sum_{i=1}^n w_i L_i^{ru}}{W_n}\right)^{\frac{1}{r}}, & r \neq 0 \\ \prod_{i=1}^n L_i^{uw_i/W_n}, & r = 0 \\ \min\{L_1^u, L_n^u\}, & r = -\infty \\ \max\{L_1^u, L_n^u\}, & r = \infty. \end{cases} \tag{2.9}$$

Things become interesting when we combine mixed identities of these two numbers in Fibonacci means:

$$\text{for} \quad i = 0, 1, \ldots, n, \quad w_i = \binom{n}{i} F_i F_{n-i}, \quad W_n = (1/5)\left(2^n L_n - 2\right), \qquad [7, \text{ p. } 110]$$
$$w_i = \binom{n}{i} F_i L_{n-i}, \quad W_n = 2^n F_n. \qquad [7, \text{ p. } 110] \tag{2.10}$$

Corollary 2.7. *Let $u \in \mathbb{R},\ m = \min\{F_1^u, F_n^u\},\ M = \max\{F_1^u, F_n^u\},\ \delta = M/m$. Then for any $-\infty < r < s < \infty$*

$$2^{n\left(\frac{1}{s}-\frac{1}{r}\right)} \left(\sum_{i=0}^n \binom{n}{i} F_i^{su+1} L_{n-i}\right)^{1/s} \leq \Gamma_{r,s}(\delta) \cdot \left(\sum_{i=0}^n \binom{n}{i} F_i^{ru+1} L_{n-i}\right)^{1/r}, \tag{2.11}$$

where $\Gamma_{r,s}(\delta)$ is as in Theorem 2.4.

Proof. We use identity (2.10) in Theorem 2.4. □

Catalan and Narayana numbers. Catalan numbers are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \in \mathbb{N},\ C_0 = 1.$$

The following identity about Catalan numbers can be used in constructing examples of Fibonacci means:

$$\text{for} \quad i = 1, \ldots, n, \quad w_i = C_{i-1} C_{n-i}, \quad W_n = C_n. \qquad [7, \text{ p. } 232]$$

Narayana numbers are defined with

$$\begin{aligned} &N(0,0) = 1, \\ &N(n,0) = 0, \quad n \geq 1, \\ &N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}, \quad n \geq k \geq 1, \end{aligned}$$

and we can relate them to Catalan numbers (see [7]):

$$C_n = \sum_{i=1}^n N(n,i)$$

and again use this for Fibonacci means by choice:

$$\text{for} \quad i = 1, \ldots, n, \quad w_i = N(n,i), \quad W_n = C_n. \qquad [7, \text{ p. } 268]$$

Infinity case, generalized Fibonacci means.
It is natural to try to extend (1.6) to the case when $n = \infty$. The first condition is

$$W = \lim_{n\to\infty} W_n = \sum_{i=1}^{\infty} w_i < \infty$$

and the second condition is that the sequence $(x_i)_{i\in\mathbb{N}}$ is bounded from below and above.
Any bounded function f can serve the role of the power function $x \mapsto x^s$ in Fibonacci means. We can set

$$x_i = f(F_i),\ i \in \mathbb{N}$$

in extension of power means, using limit, and now

$$F^{[r]}(\mathbf{w}; f) = \begin{cases} \left(\frac{\sum_{i=1}^{\infty} w_i f(F_i)^r}{W}\right)^{\frac{1}{r}}, & r \neq 0 \\ \prod_{i=1}^{\infty} f(F_i)^{w_i/W}, & r = 0 \\ \inf\{f(F_i) : i \in \mathbb{N}\}, & r = -\infty \\ \sup\{f(F_i) : i \in \mathbb{N}\}, & r = \infty, \end{cases} \tag{2.12}$$

stands for *generalized Fibonacci means.*
For example, we can use for weights $(w_i)_{i\in\mathbb{N}}$ the following identities:

$$\text{for } \ i \in \mathbb{N},\ w_i = \frac{F_{i+1}}{2^i},\ \ W = 3, \qquad \text{(J. H. Butchart, 1968) [5]}$$
$$\text{for } \ i \in \mathbb{N},\ w_i = \frac{F_i}{3^{i+1}},\ \ W = \frac{1}{5}, \qquad \text{[7, p. 63]}$$
$$\text{for } \ i \in \mathbb{N},\ w_i = \frac{1}{F_i F_{i+2}},\ \ W = 1. \qquad \text{[7, p. 63]}$$

For functions, we can take, for example, $f = \arctan$, or $f = \tanh$.
Note here that the function arctan can be used in two ways: by extending the identity (2.8) to

$$\sum_{i=1}^{\infty} \arctan\left(\frac{1}{F_{2i+1}}\right) = \frac{\pi}{4}$$

i.e.

$$\text{for } \ i \in \mathbb{N},\ w_i = \arctan\left(\frac{1}{F_{2i+1}}\right),\ \ W = \frac{\pi}{4},$$

and putting $f = \arctan$ we get the following means

$$F^{[r]}(\mathbf{w}) = \begin{cases} \left(\frac{4}{\pi}\sum_{i=1}^{\infty} \arctan\left(\frac{1}{F_{2i+1}}\right) \arctan^r(F_i)\right)^{\frac{1}{r}}, & r \neq 0 \\ \prod_{i=1}^{\infty} \left[\arctan(F_i)\right]^{\frac{4}{\pi}\arctan\left(\frac{1}{F_{2i+1}}\right)}, & r = 0 \\ \frac{\pi}{4}, & r = -\infty \\ \frac{\pi}{2}, & r = \infty, \end{cases} \tag{2.13}$$

concluding

$$\frac{\pi}{4} < \frac{\pi}{4}\frac{1}{\sum_{i=1}^{\infty}\frac{\arctan(1/F_{2i+1})}{\arctan(F_i)}} < \prod_{i=1}^{\infty}[\arctan(F_i)]^{\frac{4}{\pi}\arctan\left(\frac{1}{F_{2i+1}}\right)} <$$
$$< \frac{4}{\pi}\sum_{i=1}^{\infty}\arctan\left(\frac{1}{F_{2i+1}}\right)\arctan(F_i) < \frac{\pi}{2},$$

after use of moment comparison.

References

[1] H. Alzer and F. Luca, *An inequality for the Fibonacci numbers*, Math. Bohem. 147 (2022), 587–590.

[2] H. Alzer, M. K. Kwong, *Extension of an Inequality for Fibonacci Numbers*, Integers 22(2022), Paper No. A85.

[3] A. Brousseau *Fibonacci-Lucas Infinite Series-Research Topic*, The Fibonacci Quarterly, Volume 7(2), (1968), 211–217.

[4] P.S. Bullen, D.S. Mitrinović, and P.M. Vasić, *Means and Their Inequalities, Mathematics and Its Applications,* D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, Tokyo, 1988.

[5] J. H. Butchart, *Problem B - 124* The Fibonacci Quarterly,6(4), (1968).

[6] H. T. Freitag, *Problem B - 286,* The Fibonacci Quarterly, 13(3),(1975)

[7] R. P. Grimaldi, *Fibonacci and Catalan Numbers,* John Wiley & Sons, Inc., Hoboken, NJ, 2012.

[8] T. Koshy, *New Fibonacci and Lucas Identities*, The Mathematical Gazette, Volume 82, November, 1998, 481–484.

[9] P.G. Popescu and J.L. Diaz-Barrero, *Certain inequalities for convex functions*, J. Inequal. Pure Appl. Math. 7(2) (2006), Article 41.

[10] J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Mathematics in science and engineering, **187** Academic Press, 1992.

[11] K. G. Recke, *Problem B - 153,* The Fibonacci Quarterly, 7(3) (1969).

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