FIBONACCI NUMBERS AND HÖLDER INEQUALITY

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ABSTRACT. The article establishes a chain of inequalities with power sums using Hölder's and Cauchy's inequalities and their conversions. Subsequently, applications are made to power sums whose terms are composed of Fibonacci numbers, for which the sum can be calculated with an appropriate choice of exponents.

1. INTRODUCTION

Fibonacci sequence. The Fibonacci sequence is defined recursively with

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-2} + F_{n-1}, \quad n = 2, 3 \ldots.$$

Fibonacci numbers can be calculated using the formula

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - (\phi - \sqrt{5})^n), \tag{1.1}$$

where $\phi = \frac{1+\sqrt{5}}{2}$, $n \in \mathbb{N}$.

In [14], the following inequality with Fibonacci numbers can be found.

Theorem 1. *Let n be a positive integer and ℓ an integer. Then,*

$$\left(F_1^\ell + F_2^\ell + \ldots + F_n^\ell\right)\left(\frac{1}{F_1^{\ell-4}} + \frac{1}{F_2^{\ell-4}} + \cdots + \frac{1}{F_n^{\ell-4}}\right) \geq F_n^2 F_{n+1}^2 \tag{1.2}$$

holds, where F_n is the n^{th} Fibonacci number.

The previous theorem is generalized in the paper [1].

Theorem 2. *Let $r, s \in \mathbb{R}$ with $r + s \geq 4$. Then, for $n \geq 1$,*

$$\sum_{k=1}^n F_k^r \sum_{k=1}^n F_k^s \geq (F_n F_{n+1})^2. \tag{1.3}$$

The sign of equality is valid in (1.3) *if and only if $n = 1, 2$ or $n \geq 3, r = s = 2$.*

Power sums, Hölder and Cauchy inequality. We present some results that will be needed in the exposition.

Let us denote, for $\alpha \in \mathbb{R}$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$

$$S_n^{[\alpha]}(\mathbf{x}) = \sum_{i=1}^n x_i^\alpha. \tag{1.4}$$

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Proposition 1. *If $\alpha > \beta > 0$ then*

$$\left(S_n^{[\alpha]}(\mathbf{x})\right)^{1/\alpha} \leq \left(S_n^{[\beta]}(\mathbf{x})\right)^{1/\beta}. \tag{1.5}$$

(See [9, p. 4].)

Theorem 3 (Hölder)**.** *Let $\mathbf{x},\ \mathbf{y} \in \mathbb{R}_+^n$.*

(i) *If $p > 1$ and $q = \frac{p}{p-1}$, then*

$$\sum_{i=1}^{n} x_i y_i \leq \left(S_n^{[p]}(\mathbf{x})\right)^{1/p} \left(S_n^{[q]}(\mathbf{y})\right)^{1/q}. \tag{1.6}$$

(ii) *If $0 < p < 1$ and $q = \frac{p}{p-1}$, then the reverse inequality holds in* (1.6)*.*

(See [9, p.26] and [15, p.113].)

Corollary 1 (Cauchy)**.** *Let $\mathbf{x},\ \mathbf{y} \in \mathbb{R}_+^n$.*

$$\sum_{i=1}^{n} x_i y_i \leq \left(S_n^{[2]}(\mathbf{x})\right)^{1/2} \left(S_n^{[2]}(\mathbf{y})\right)^{1/2}. \tag{1.7}$$

([9, p.16] and [15, p.131].)

Theorem 4 (Hölder's conversions)**.** *Let $\mathbf{x},\ \mathbf{y} \in \mathbb{R}_+^n,\ \frac{1}{p} + \frac{1}{q} = 1,\ 0 < m < M$, $m \leq x_i / y_i^{q/p} \leq M,\ i = 1, \ldots, n$.*

(i) *If $p > 1$, then*

$$(M-m)\sum_{k=1}^{n} x_k^p + (mM^p - Mm^p)\sum_{k=1}^{n} y_k^q \leq (M^p - m^p)\sum_{k=1}^{n} x_k y_k \tag{1.8}$$

and if $0 < p < 1$, then reversed inequality in (1.8) *is valid.*

(ii) *If $p > 1$, then*

$$\left(\sum_{k=1}^{n} x_k^p\right)^{1/p} \left(\sum_{k=1}^{n} y_k^q\right)^{l/q} \leq \lambda \sum_{k=1}^{n} x_k y_k \tag{1.9}$$

where

$$\lambda = |M^p - m^p|\, |p(M-m)|^{-1/p}\, |q\,(M^p - mM^p)|^{-1/q}.$$

If $0 < p < 1$, then reversed inequality in (1.9) *is valid.*

(See [11, p.64] and[15, p.114].)

Theorem 5 (Cauchy's conversions)**.** *Let $\mathbf{x},\ \mathbf{y} \in \mathbb{R}_+^n$ and*

$$0 < m_1 \leq x_i \leq M_1, \qquad 0 < m_2 \leq y_i < M_2, \quad i = 1, \ldots, n.$$

Then, the following transformations of Cauchy's inequality hold:

$$1 \leq \frac{\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} {y_k}^2\right)}{\left(\sum_{k=1}^{n} x_k y_k\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2, \tag{1.10}$$

$$\frac{\sum_{k=1}^{n} {x_k}^2}{\sum_{k=1}^{n} x_k y_k} - \frac{\sum_{k=1}^{n} x_k y_k}{\sum_{k=1}^{n} {y_k}^2} \leq \left(\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}} - \left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right)^2, \tag{1.11}$$

$$\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) - \left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq \frac{n^2}{4}\left(M_1M_2 - m_1m_2\right)^2, \tag{1.12}$$

$$\sum_{k=1}^{n} {y_k}^2 + \frac{m_2}{M_1}\frac{M_2}{m_1}\sum_{k=1}^{n} {x_k}^2 \leq \left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)\sum_{k=1}^{n} x_k y_k. \tag{1.13}$$

The above theorem can be found in [11, p. 61]. The inequality (1.10) was proven in [13], (1.11) in [17], (1.12) in [12], and (1.13) in [6].

2. Preliminary results

Theorem 6. *Let* $p > 1, q = \frac{p}{p-1},\ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$.

(i) *If* $x_i \geq 1,\ i = 1, \ldots, n,$ *and if* $u, v \in \mathbb{R},\ \alpha = \frac{u}{p} + \frac{v}{q}$ *and* $\alpha \geq \beta > 0$, *then*

$$\left(S_n^{[u]}(x)\right)^{1/p}\left(S_n^{[v]}(x)\right)^{1/q} \geq S_n^{[\alpha]}(x) \geq$$
$$\geq \frac{1}{n^{\frac{\alpha}{\beta}-1}}\left(S_n^{[\beta]}(x)\right)^{\alpha/\beta} \geq S_n^{[\beta]}(x) \geq \left(S_n^{[\alpha]}(x)\right)^{\beta/\alpha}. \tag{2.1}$$

(ii) *If* $u, v \in \mathbb{R}$, *such that* $\alpha = \frac{u}{p} + \frac{v}{q}$ *and* $0 \leq \alpha < \beta$, *then*

$$\left(S_n^{[u]}(x)\right)^{1/p}\left(S_n^{[v]}(x)\right)^{1/q} \geq S_n^{[\alpha]}(x) \geq \left(S_n^{[\beta]}(x)\right)^{\alpha/\beta}. \tag{2.2}$$

Proof. (i)

$$\left(\sum_{k=1}^{n} x_k^u\right)^{1/p}\left(\sum_{k=1}^{n} x_k^v\right)^{1/q} = \left(\sum_{k=1}^{n}\left((x_k^{u/p})^p\right)\right)^{1/p}\left(\sum_{k=1}^{n}\left((x_k^{v/q})^q\right)\right)^{1/q} \tag{2.3}$$

$$\geq \sum_{k=1}^{n} x_k^{u/p} x_k^{v/q} = \sum_{k=1}^{n} x_k^{\alpha} = \sum_{k=1}^{n} (x_k^{\beta})^{\alpha/\beta} \tag{2.4}$$

$$\geq n\left(\frac{1}{n}\sum_{k=1}^{n} x_k^{\beta}\right)^{\alpha/\beta} \tag{2.5}$$

$$\geq \sum_{k=1}^{n} x_k^{\beta} \tag{2.6}$$

$$\geq \left(\sum_{k=1}^{n} x_k^{\alpha}\right)^{\beta/\alpha}, \tag{2.7}$$

where in (2.4) we used Hölder's inequality with substitutions $x_i \to x_i^{u/p},\ y_i \to x_i^{v/q}$. Then, in (2.5), we used Jensen's inequality for the function $x \mapsto x^{\alpha/\beta},\ \alpha \geq \beta$, and in (2.6), we used the monotonicity of the exponential function $x \mapsto b^x,\ b =$

$\frac{1}{n}\sum_{k=1}^{n} x_k^{\beta} \geq 1$.
(ii) Similar as in (i)-part, we apply Hölder's inequality and then the moment inequality (1.5). □

Remark 1. *Observe that the condition* $x_i \geq 1, \ i = 1, \ldots, n$ *in the (i)-part of the Theorem can be weakened to the condition* $\frac{1}{n}\sum_{k=1}^{n} x_k^{\beta} \geq 1$, *as used in* (2.5).

The above proof can also be adapted for the following theorem, which uses the reverse Hölder inequality.

Theorem 7. *Let* $0 < p < 1$, $q = \frac{p}{p-1}$, $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$.

(i) *If* $u, v \in \mathbb{R}$, $\alpha = \frac{u}{p} + \frac{v}{q}$ *and* $\alpha \geq \beta > 0$, *then*

$$\left(S_n^{[u]}(x)\right)^{1/p}\left(S_n^{[v]}(x)\right)^{1/q} \leq S_n^{[\alpha]}(x) \leq \left(S_n^{[\beta]}(x)\right)^{\alpha/\beta}. \tag{2.8}$$

(ii) *Let* $x_i \geq 1, \ i = 1, \ldots, n$. *If* $u, v \in \mathbb{R}$, *such* $\alpha = \frac{u}{p} + \frac{v}{q}$ *and* $0 \leq \alpha < \beta$, *then*

$$\left(S_n^{[u]}(x)\right)^{1/p}\left(S_n^{[v]}(x)\right)^{1/q} \leq S_n^{[\alpha]}(x) \leq \frac{1}{n^{\frac{\alpha}{\beta}-1}}\left(S_n^{[\beta]}(x)\right)^{\alpha/\beta} \leq S_n^{[\beta]}(x) \leq \left(S_n^{[\alpha]}(x)\right)^{\beta/\alpha}. \tag{2.9}$$

The results from Theorems 6 and 7 can be further extended using the conversions from Theorem 4.

Theorem 8. *Let* $u, v \in \mathbb{R}$, *such that* $\alpha = \frac{u}{p} + \frac{v}{q}$. *If* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, $m = \min_i\{x_i^{\frac{u-v}{p}}\}$ *and* $M = \max_i\{x_i^{\frac{u-v}{p}}\}$.

(i) *If* $p > 1$ *then*

$$(M-m)S_n^{[u]}(x) + (mM^p - Mm^p)\, S_n^{[v]}(x) \leq (M^p - m^p)\sum_{k=1}^{n} S_n^{[\alpha]}(x) \tag{2.10}$$

and if $0 < p < 1$ *then reversed inequality in* (2.10) *is valid.*

(ii) *If* $p > 1$, *then*

$$\left(S_n^{[u]}(x)\right)^{1/p}\left(S_n^{[v]}(x)\right)^{l/q} \leq \lambda S_n^{[\alpha]}(x) \tag{2.11}$$

where

$$\lambda = |M^p - m^p|\,|p(M-m)|^{-1/p}\,|q\,(M^p - mM^p)|^{-1/q}.$$

If $0 < p < 1$ *then reversed inequality in* (2.11) *is valid.*

Proof. By substituting $x_i \to x_i^{u/p}$ and $y_i \to x_i^{v/q}$ into Theorem 4, we observe that the condition $m \leq x_i/y_i^{q/p} \leq M$ is fulfilled for $m = \min_i x_i^{\frac{u-v}{p}}$ and $M = \max_i x_i^{\frac{u-v}{p}}$, and that the inequalities (1.8) and (1.9) transform into (2.10) and (2.11). □

Similarly, from the Cauchy conversions in Theorem 5, we obtain the following theorem:

Theorem 9. *Let* $\mathbf{x} \in \mathbb{R}_+^n$, $u, v \in \mathbb{R}$, $\alpha = \frac{u}{2} + \frac{v}{2}$ *and*

$$m_1 = \min_i\{x_i^{u/2}\}, \ M_1 = \max_i\{x_i^{u/2}\} \quad \text{and} \quad m_2 = \min_i\{x_i^{v/2}\}, \ M_2 = \max_i\{x_i^{v/2}\}.$$

Then

$$1 \leq \frac{S_n^{[u]}(x)\, S_n^{[v]}(x)}{\left(S_n^{[\alpha]}(x)\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2, \tag{2.12}$$

$$\frac{S_n^{[u]}(x)}{S_n^{[\alpha]}(x)} - \frac{S_n^{[\alpha]}(x)}{S_n^{[v]}(x)} \leq \left(\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}} - \left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right)^2, \tag{2.13}$$

$$S_n^{[u]}(x)\, S_n^{[v]}(x) - \left(S_n^{[\alpha]}(x)\right)^2 \leq \frac{n^2}{4}\left(M_1M_2 - m_1m_2\right)^2 \tag{2.14}$$

and

$$S_n^{[v]}(x) + \frac{m_2}{M_1}\frac{M_2}{m_1} S_n^{[u]}(x) \leq \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right) S_n^{[\alpha]}(x). \tag{2.15}$$

3. Applications

In this section, we provide applications by estimating the sums of powers of sequences whose elements include Fibonacci numbers. The inequalities become interesting when we have exponents for which the sum can be calculated explicitly.

In the papers [1], [2], and [14] the identity

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \tag{3.1}$$

is used. Using our notation, $x_i = F_i,\ i = 1, \ldots, n$ (3.1) can be rephrased as

$$S_n^{[2]}(\mathbf{x}) = F_n F_{n+1}. \tag{3.2}$$

If we relate (3.2) with Theorems 6 and 7 under $\beta = 2$ and observing that condition $x_i \geq 1$ is satisfied, we get the following Theorem.

Theorem 10. *(i) Let $p > 1, q = \frac{p}{p-1}$ and let $u, v \in \mathbb{R}$ such that $\alpha = \frac{u}{p} + \frac{v}{q} \geq 2$. Then*

$$\left(\sum_{i=1}^{n} F_i^u\right)^{1/p} \left(\sum_{i=1}^{n} F_i^v\right)^{1/q} \geq \sum_{i=1}^{n} F_i^\alpha \geq$$
$$\geq \frac{1}{n^{\frac{\alpha}{2}-1}} \left(F_nF_{n+1}\right)^{\alpha/2} \geq F_nF_{n+1} \geq \left(\sum_{i=1}^{n} F_i^\alpha\right)^{2/\alpha}. \tag{3.3}$$

(ii) Let $p > 1, q = \frac{p}{p-1}$ and let $u, v \in \mathbb{R}$ such that $\alpha = \frac{u}{p} + \frac{v}{q} \leq 2$. Then

$$\left(\sum_{i=1}^{n} F_i^u\right)^{1/p} \left(\sum_{i=1}^{n} F_i^v\right)^{1/q} \geq \sum_{i=1}^{n} F_i^\alpha \geq \left(F_nF_{n+1}\right)^{\alpha/2}. \tag{3.4}$$

(iii) Let $0 < p < 1,\ q = \frac{p}{p-1}$ and let $u, v \in \mathbb{R}$ such that $\alpha = \frac{u}{p} + \frac{v}{q} \geq 2$. Then

$$\left(\sum_{i=1}^{n} F_i^u\right)^{1/p} \left(\sum_{i=1}^{n} F_i^v\right)^{1/q} \leq \sum_{i=1}^{n} F_i^\alpha \leq \left(F_nF_{n+1}\right)^{\alpha/2}.$$

(iv) Let $0 < p < 1,\ q = \frac{p}{p-1}$ and let $u, v \in \mathbb{R}$ such that $\alpha = \frac{u}{p} + \frac{v}{q} \leq 2$. Then

$$\left(\sum_{i=1}^{n} F_i^u\right)^{1/p}\left(\sum_{i=1}^{n} F_i^v\right)^{1/q} \le \sum_{i=1}^{n} F_i^\alpha \le (F_nF_{n+1})^{\alpha/2} \le$$
$$\le \frac{1}{n^{\frac{\alpha}{2}-1}}(F_nF_{n+1})^{\alpha/2} \le F_nF_{n+1} \le \left(\sum_{i=1}^{n} F_i^\alpha\right)^{\alpha/2}.$$

Remark 2.

(i) *Note that if we set* $p = 2$ *(Cauchy case) in the* (*i*)*-part of Theorem 10, then the chain of inequalities in* (3.3) *provides a refinement of* (1.3)*.*

(ii) *Observe that the analysis of equality in* (3.3) *(or* (1.3)*) is now simpler using the central terms* $\sum_{i=1}^n F_i^\alpha$ *and* $\frac{1}{n^{\frac{\alpha}{2}-1}}(F_nF_{n+1})^{\alpha/2}$.

The previous results can be extended using Hölder's transformations from Theorem 8, with $x_i = F_i$ and $S_n^{[2]}(\mathbf{x}) = F_nF_{n+1}$. Observe that due to the monotonicity of the Fibonacci sequence, we have

$$m = \min_i\{x_i^{\frac{u-v}{p}}\} = \min\{F_1^{\frac{u-v}{p}}, F_n^{\frac{u-v}{p}}\} \text{ and } M = \max_i\{x_i^{\frac{u-v}{p}}\} = \max\{F_1^{\frac{u-v}{p}}, F_n^{\frac{u-v}{p}}\}.$$

Theorem 11. *Let* $p > 1, q = \frac{p}{p-1}$, $u, v \in \mathbb{R}$ *such that* $\frac{u}{p} + \frac{v}{q} = 2$ *and let* $m = \min\{F_1^{\frac{u-v}{p}}, F_n^{\frac{u-v}{p}}\}$ *and* $M = \max\{F_1^{\frac{u-v}{p}}, F_n^{\frac{u-v}{p}}\}$.

(i) *If* $p > 1$ *then*

$$(M-m)\sum_{i=1}^{n} F_i^u + (mM^p - Mm^p)\sum_{i=1}^{n} F_i^v \le (M^p - m^p)\,F_nF_{n+1} \tag{3.5}$$

and if $0 < p < 1$ *then reversed inequality in* (2.10) *is valid.*

(ii) *If* $p > 1$*, then*

$$\left(\sum_{i=1}^{n} F_i^u\right)^{1/p}\left(\sum_{i=1}^{n} F_i^v\right)^{l/q} \le \lambda F_nF_{n+1}. \tag{3.6}$$

where

$$\lambda = |M^p - m^p|\,|p(M-m)|^{-1/p}\,|q\,(M^p - mM^p)|^{-1/q}.$$

If $0 < p < 1$ *then reversed inequality in* (3.6) *is valid.*

We will apply Cauchy's transformations from Theorem 9 to the following identity

$$x_i = F_iF_{i+1},\ i = 1,\dots,n;\quad S_n^{[1]}(\mathbf{x}) = F_{n+1}^2 - \frac{1+(-1)^n}{2} \tag{3.7}$$

(see [8]).

Here, again due to the monotonicity of the sequence x_i, we have that for any $u, v \in \mathbb{R}$

$$m_1 = \min_i\{x_i^{u/2}\} = \min\{(F_1F_2)^{u/2}, (F_nF_{n+1})^{u/2}\},$$
$$M_1 = \max_i\{x_i^{u/2}\} = \max\{(F_1F_2)^{u/2}, (F_nF_{n+1})^{u/2}\}$$

and similarly

$$m_2 = \min\{(F_1F_2)^{v/2}, (F_nF_{n+1})^{v/2}\},\ M_2 = \max\{(F_1F_2)^{v/2}, (F_nF_{n+1})^{v/2}\}.$$

We now use Theorem 9 with $\alpha = 1$ to utilize the identity from (3.7).

Theorem 12. *Let* $u, v \in \mathbb{R},\ u+v=2$ *and*

$$m_1 = \min\{(F_1F_2)^{u/2}, (F_nF_{n+1})^{u/2}\},\ M_1 = \max\{(F_1F_2)^{u/2}, (F_nF_{n+1})^{u/2}\}$$

$$m_2 = \min\{(F_1F_2)^{v/2}, (F_nF_{n+1})^{v/2}\},\ M_2 = \max\{(F_1F_2)^{v/2}, (F_nF_{n+1})^{v/2}\}.$$

Then

$$1 \le \frac{\sum_{i=1}^n (F_iF_{i+1})^u \sum_{i=1}^n (F_iF_{i+1})^v}{F_nF_{n+1} - \frac{1+(-1)^n}{2}} \le \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2, \tag{3.8}$$

$$\frac{\sum_{i=1}^n (F_iF_{i+1})^u}{F_nF_{n+1} - \frac{1+(-1)^n}{2}} - \frac{F_nF_{n+1} - \frac{1+(-1)^n}{2}}{\sum_{i=1}^n (F_iF_{i+1})^v} \le \left(\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}} - \left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right)^2, \tag{3.9}$$

$$\sum_{i=1}^n (F_iF_{i+1})^u \sum_{i=1}^n (F_iF_{i+1})^v - \left(F_nF_{n+1} - \frac{1+(-1)^n}{2}\right)^2 \le \frac{n^2}{4}\left(M_1M_2 - m_1m_2\right)^2, \tag{3.10}$$

$$\sum_{i=1}^n (F_iF_{i+1})^v + \frac{m_2}{M_1}\frac{M_2}{m_1}\sum_{i=1}^n (F_iF_{i+1})^u \le \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\left(F_nF_{n+1} - \frac{1+(-1)^n}{2}\right). \tag{3.11}$$

4. Concluding remarks

We have demonstrated how, by using Theorems 6, 7, 8, and 9, as well as the identities (3.7) and (3.2) related to Fibonacci numbers, a series of inequalities can be derived, as shown in Theorems 10, 11, and 12. In the following lines, we present additional identities that can be utilized in a similar manner.

For $i = 1, \ldots, n,\ x_i = F_i,\ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_{n+2} - 2,$ [8, p. 11]

$x_i = F_{2i-1},\ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_{2n},$ [8, p. 11]

$x_i = F_{2i},\ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_{2n+1} - 1,$ [8, p. 11]

$x_i = iF_i,\ \beta = 1\ \ S_n^{[1]}(\mathbf{x}) = F_{n+2} - F_{n+3} + 2,$ [8, p. 11]

$x_i = F_iF_{3i},\ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_nF_{n+1}F_{2n+1},$ [16]

$x_i = F_{4i-2},\ \ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_{2n}^2,$ [8, p. 61]

$x_i = \binom{n}{i}F_i,\ \ \beta = 1,\ S_n^{[1]}(\mathbf{x}) = F_{2n},$ [8, p. 61]

for $i = 1, \ldots, 2n+1,\ x_i = F_i\sqrt{\binom{2n+1}{i}},\ \beta = 2,\ S_{2n+1}^{[2]}(\mathbf{x}) = 5^nF_{2n+1},$ [8, p. 56]

Particularly interesting are those identities involving Fibonacci numbers for which the sum $S_n^{[\alpha]}(\mathbf{x}) = \sum_{i=1}^n x_i^\alpha$ can be calculated for several different choices of $\alpha \in \mathbb{R}$. For example, with the choice $\alpha = -1,\ x_i = F_iF_{i+2}$, we have (see [8])

$$S_n^{[-1]}(\mathbf{x}) = 1 - \frac{1}{F_{n+1}F_{n+2}}. \tag{4.1}$$

Furthermore, by continuing the calculations, we obtain

$$\begin{aligned} S_n^{[1]}(\mathbf{x}) &= \sum_{i=1}^n F_iF_{i+2} = \sum_{i=1}^n F_i(F_i+F_{i+1}) = \sum_{i=1}^n (F_i^2 + F_iF_{i+1}) = \\ &= \sum_{i=1}^n F_i^2 + \sum_{i=1}^n F_iF_{i+1} = F_nF_{n+1} + F_{n+1}^2 - \frac{1+(-1)^n}{2} \\ &= F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2}. \end{aligned} \tag{4.2}$$

These two facts are used in the following theorem.

Theorem 13. *(i) Let $p>1$ and let $\beta \le 1$. Then*

$$\left(\sum_{i=1}^n (F_iF_{i+2})^{2p-1}\right)^{1/p} \left(1 - \frac{1}{F_{n+1}F_{n+2}}\right)^{\frac{p-1}{p}} \ge F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2} \ge$$
$$\ge \frac{1}{n^{\frac{1}{\beta}-1}} \left(\sum_{i=1}^n (F_iF_{i+2})^{\beta}\right)^{1/\beta} \ge \sum_{i=1}^n (F_iF_{i+2})^{\beta} \ge \left(F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2}\right)^{\beta}.$$

(ii) Let $p>1$ and let $\beta > 1$. Then

$$\left(\sum_{i=1}^n (F_iF_{i+2})^{2p-1}\right)^{1/p} \left(1 - \frac{1}{F_{n+1}F_{n+2}}\right)^{\frac{p-1}{p}} \ge F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2} \ge$$
$$\ge \left(\sum_{i=1}^n (F_iF_{i+2})^{\beta}\right)^{1/\beta}.$$

(iii) Let $0<p<1$ and let $\beta > 1$. Then

$$\left(\sum_{i=1}^n (F_iF_{i+2})^{2p-1}\right)^{1/p} \left(1 - \frac{1}{F_{n+1}F_{n+2}}\right)^{\frac{p-1}{p}} \le F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2} \le$$
$$\le \frac{1}{n^{\frac{1}{\beta}-1}} \left(\sum_{i=1}^n (F_iF_{i+2})^{\beta}\right)^{1/\beta} \ge \sum_{i=1}^n (F_iF_{i+2})^{\beta} \le \left(F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2}\right)^{\beta}.$$

(iv) Let $0<p<1$ and let $\beta \le 1$. Then

$$\left(\sum_{i=1}^n (F_iF_{i+2})^{2p-1}\right)^{1/p} \left(1 - \frac{1}{F_{n+1}F_{n+2}}\right)^{\frac{p-1}{p}} \le F_{n+1}F_{n+2} - \frac{1+(-1)^n}{2} \le$$
$$\le \left(\sum_{i=1}^n (F_iF_{i+2})^{\beta}\right)^{1/\beta}.$$

Proof. First, observe that the condition $x_i = F_iF_{i+2} \ge 1,\ i=1,\dots,n$ is satisfied in Theorems 6 and 7. In the expression $\alpha = \frac{u}{p} + \frac{v}{q}$ from the mentioned theorems, we choose $\alpha = 1$ and $v = -1$, and we utilize the identities (4.1) and (4.2). □

Cauchy, Fibonacci, Lucas, and Vandermonde. Here, we simply mention that some interesting inequalities can be obtained through the direct application of Cauchy's inequalities.

(i) If we set $x_i = i,\ y_i = F_i,\ i = 1, \ldots, n$ then $\sum_{i=1}^{n} x_i y_i = nF_{n+2} - F_{n+3} + 2$ (see [8, p. 11]). By application of Cauchy inequality (1.7) we get

$$(nF_{n+2} - F_{n+3} + 2)^2 \leq \frac{(2n+1)\, n\, (n+1)}{6} F_n F_{n+1} \tag{4.3}$$

(ii) Let us take $x_i = \binom{n}{i},\ y_i = F_i,\ i = 0, 1, \ldots, n$. Then from [8, p. 61] we know $\sum_{i=0}^{n} x_i y_i = F_{2n}$. Now we use Vandermonde identity

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n},$$

(3.1) and Cauchy inequality to get

$$F_{2n}^2 \leq \binom{2n}{n} F_n F_{n+1}. \tag{4.4}$$

(iii) The above model can also be applied to Lucas numbers (and similar sequences)

$$L_n = F_{n+1} + F_{n-1}, \quad n \geq 1,$$

and $L_0 := 2$. From [8, p. 111] we have

$$\sum_{i=0}^{n} \binom{n}{i} F_i F_{n-i} = (1/5)\left(2^n L_n - 2\right) \tag{4.5}$$

and from [8, p. 56] we have

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F_i^2 = 5^n F_{2n+1}.$$

We set $x_i = F_i \sqrt{\binom{2n+1}{i}},\ y_i = F_{2n-i} \sqrt{\binom{2n+1}{2n-i}}, i = 0, 1, \ldots 2n+1$. Using (4.5) for $n \to 2n+1$, symmetry of binomial coefficients and Cauchy inequality we get

$$2^{2n+1} L_{2n+1} \leq 5^{n+1} F_{2n+1} + 2.$$

Thus, we have proven the following theorem.

Theorem 14. *For any $n \in \mathbb{N}$, the following holds*

$$(nF_{n+2} - F_{n+3} + 2)^2 \leq \frac{(2n+1)\, n\, (n+1)}{6} F_n F_{n+1}, \tag{4.6}$$

$$F_{2n}^2 \leq \binom{2n}{n} F_n F_{n+1}, \tag{4.7}$$

$$2^{2n+1} L_{2n+1} \leq 5^{n+1} F_{2n+1} + 2. \tag{4.8}$$

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