ELLIPTIC CURVES WITH TORSION GROUPS $\mathbb{Z}/8\mathbb{Z}$ AND $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$

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Abstract. In this paper, we present details of seven elliptic curves over $\mathbb{Q}(u)$ with rank 2 and torsion group $\mathbb{Z}/8\mathbb{Z}$ and five curves over $\mathbb{Q}(u)$ with rank 2 and torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. We also exhibit some particular examples of curves with high rank over $\mathbb{Q}$ by specialization of the parameter. We present several sets of infinitely many elliptic curves in both torsion groups and rank at least 3 parametrized by elliptic curves having positive rank. In some of these sets we have performed calculations about the distribution of the root number. This has relation with recent heuristics concerning the rank bound for elliptic curves by Park, Poonen, Voight and Wood.

1. Introduction

We are interested in curves having torsion group $\mathbb{Z}/8\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. We describe a general model for curves with these torsions, which jointly with the curves having torsion $\mathbb{Z}/7\mathbb{Z}$ are the three cases in which the general model is a $K3$ surface. Then we try to impose the existence of new points.

1.1. New points from quadratic conditions.

We explain how we get new points in our models.

1) All the curves in the paper can be written as $y^2 = x^3 + Ax^2 + Bx$. This is due to the fact that all of them have at least one torsion point of order two.

2) In these cases it is well known that the x-coordinates of rational points should be either divisors of B or rational squares times divisors of B.

3) We know some of the divisors of B, namely the polynomial factors of B.

4) We construct the list of all such known divisors of B and we look for those that become a new point after parametrizing a conic.

If d is a divisor of B, we try to force d to be the x-coordinate of a new point, so we have to impose that $d^3 + Ad^2 + Bd = d^2(d + A + B/d)$ is a square. So for every known divisor of B we consider the equation

$$d + A + \frac{B}{d} = \text{Square}.$$

In some cases this equation is equivalent to solving a conic and in this way we get a new point in the curve in case of solvability.

We try the same with a divisor of B, say d, times a rational square, say $\frac{U^2}{V^2}$, and we get the following equations

$$d^3\frac{U^6}{V^6} + Ad^2\frac{U^4}{V^4} + Bd\frac{U^2}{V^2} = \frac{d^2U^2}{V^6}\left(dU^4 + AU^2V^2 + V^4\frac{B}{d}\right) = \text{Square}.$$

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These equations describe the homogeneous spaces corresponding to the pair (U, V). In some cases this equation is equivalent to a quadratic condition and in case of solvability we get a new point on the curve, which in case of independence raise the rank by one.

1.2. **Torsion group $\mathbb{Z}/8\mathbb{Z}$.** For torsion $\mathbb{Z}/8\mathbb{Z}$, we list the main data for eighteen curves with rank 1. Curves having rank 1 over $\mathbb{Q}(u)$ were found by Kulesz and Lecacheux and Woo, see [Ku], [Le1], [Le2] and [W].

Kulesz used a quadratic section while Lecacheux used fibrations of the corresponding surfaces, as the ones given explicitly in Beauville [Be], in Bertin and Lecacheux [BL] or Livné and Yui [LY], in order to find elliptic curves with positive rank over $\mathbb{Q}(u)$ for several torsion groups.

Woo in his Doctoral Dissertation [W] studied in a systematic way quadratic sections of $K3$ surfaces and in this way he found several quadratic sections leading to rank 1 curves over $\mathbb{Q}(u)$ for torsion groups $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ in which we are interested in this paper. These results can be found in [W, Chapter 3].

In 2013, Dujella and Peral (see [DP1] and [DP2]) described ten of these curves, including those previously discovered by Kulesz and Lecacheux and Woo. The main details of these ten curves are given first. In 2014, MacLeod (see [ML]) listed twelve examples. Six of these twelve are new, and we list them in second place. Finally, we discovered two new examples wich are given in third place. We list all such curves in order to complete the presentation. That they are non-isomorphic is deduced by comparing their j-invariants.

In some cases, we are able to repeat the procedure of quadratic sections, and so we reach curves with rank 2, as mentioned before. Details will be the subject of another section. As already said, the first two examples for such curves with rank 2 over $\mathbb{Q}(u)$ were found by Dujella-Peral. Details were presented first on ArXiv [DP1] and later in [DP2]. In 2014, MacLeod found other two curves with rank 2, see [ML]. In this paper, we construct three new curves with rank 2 over $\mathbb{Q}(u)$ and several examples of new sets of infinitely many elliptic curves with such torsion group and rank at least 3 parametrized by elliptic curves having positive rank. Again an argument with their j-invariant shows that the seven curves are different.

We also exhibit particular examples of curves with high rank by specialization of the parameter. In particular, for the torsion group $\mathbb{Z}/8\mathbb{Z}$, the rank record for elliptic curves over $\mathbb{Q}$ at the moment is 6. The first example is due to Elkies (2006), the second was found by Dujella-MacLeod-Peral (2013) and the third and fourth by Voznyy (2021). See [D] for details of these curves were also the details of more than 50 curves with rank 5 are given.

1.3. **Torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.** Elliptic curves with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and rank at least 1 over $\mathbb{Q}(u)$ have been constructed by several authors, see [Ca], [Ku], [Le1], [Du1] and [Ra], and as mentioned before by Woo in [W]. For example the transformation leading the Woo model (written in terms of t) to our model is $t \to -\frac{2(v-1)}{v+1}$.

Here we describe five elliptic curves with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and rank 2 over $\mathbb{Q}(u)$.

The first four examples with rank 2 over $\mathbb{Q}(u)$ and torsion $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ were found by Dujella-Peral in 2013. Three of them were found using Diophantine triples, and the fourth by starting with a model due to Hadano. The last example is new. We have checked that the five curves are non-isomorphic to each other by comparing their j-invariants. It should be observed that the seven curves presented by MacLeod in 2014 are all isomorphic to one of the first three curves.

Let us mention at this point that in [DP3] it was shown that the elliptic curves over $\mathbb{Q}(t)$ induced by Diophantine triples can have as torsion group any of the non cyclic groups in Mazur's theorem, and several results with highest known ranks for such curves were obtained. In [DJS], it was proved that there are infinitely many Diophantine triples over quadratic fields which induce elliptic curves with torsion groups $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

We also exhibit, in the present paper, some particular examples of curves with high rank over $\mathbb{Q}$ by specialization of the parameter. In the case of torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, the rank record for elliptic curves over $\mathbb{Q}$ at the moment is 6. The first example was found by Elkies in 2006, the second was found by Dujella-Peral-Tadić in 2015 and the third and fourth by Dujella-Peral in 2020. See [D] for details of these curves where also the details of more than 60 curves with rank 5 are given.

We include several sets of infinitely many elliptic curves in both torsion groups having rank at least 3 parametrized by elliptic curves having positive rank. In some of these sets, we have performed calculations concerning the distribution of the root number. This has relation with recent heuristics about the rank bound for elliptic curves by Park, Poonen, Voight and Wood [PPVW]. In their paper, the authors describe a heuristic argument according to which the rank for elliptic curves over $\mathbb{Q}$ would be bounded. For example, they assert that only a finite number of curves will have rank greater than 21. And for the torsion groups considered here, only a finite number of curves will have rank greater than 3.

This is in contrast with an old conjecture predicting the existence of elliptic curves of arbitrarily high rank over $\mathbb{Q}$ even for each torsion group in the Mazur theorem. Now observe that if both the parity conjecture and the predictions of the heuristic were true, then the distribution of root number over these infinite families with rank 3 would not be evenly distributed. We dedicate a section for the calculation related to these questions.

2. General model for torsion groups $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

The Tate normal form for an elliptic curve is given by

$$E(b,c): \quad y^2 + (1-c)xy - by = x^3 - bx^2$$

(see [Kn]). It is nonsingular if and only if $b \neq 0$.

Using the addition law for $P = (0,0) \in E(b,c)$ and taking $d = b/c$ (assuming that $c \neq 0$), we have, in terms of c and d:

$$\begin{aligned} P &= (0,0), & -P &= (0, c\,d) \\ 2P &= (c\,d, 0), & -2P &= (0,b) \\ 3P &= (c, c\,(d-1)), & -3P &= (c, c^2) \\ 4P &= \big(d(d-1), d^2(c-d+1)\big), & -4P &= \big(d(d-1), d(d-1)^2\big). \end{aligned}$$

From here we get that P is a torsion point of order 6 for $d = 1+c$ or $b = c + c^2$, P is a torsion point of order 7 for $c = d(d-1)$ and $b = d^2(d-1)$ and P is a torsion point of order 8 for

$$b = (2d-1)(d-1),$$
$$c = \frac{(2d-1)(d-1)}{d}$$

with d rational, see [Kn].

For these values we can write the general curve with torsion $\mathbb{Z}/8\mathbb{Z}$ in the form $y^2 = x^3 + A_8(v)x^2 + B_8(v)x$, where

$$A_8(v) = 1 - 8v + 16v^2 - 16v^3 + 8v^4,$$
$$B_8(v) = 16(v-1)^4v^4$$

(we replaced here d by v to adjust notation with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ case since we will similarly treat these two cases).

The j-invariant for this model is

$$j_8(v) = \frac{\left(16v^8 - 64v^7 + 224v^6 - 448v^5 + 480v^4 - 288v^3 + 96v^2 - 16v + 1\right)^3}{(v-1)^8v^8(2v-1)^4\left(8v^2 - 8v + 1\right)}.$$

This invariant remains the same with the substitution $v \to 1 - v$.

Finally for torsion $\mathbb{Z}/6\mathbb{Z}$ the general curve is $y^2 = x^3 + A_6(d)x^2 + B_6(d)x$, where

$$A_6(d) = 1 + 6d - 3d^2,$$
$$B_6(d) = -16d^3.$$

Now the discriminant of $X^3 + A_6(d)X^2 + B_6(d)X$ is $256d^6(d+1)^3(9d+1)$ so we get complete factorization, and hence torsion $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, if this is a square. By taking $(d+1)(9d+1) = (3d+v)^2$, we get that the general curve with this torsion group can be written as: $y^2 = x^3 + A_{26}(v)x^2 + B_{26}(v)x$ where

$$A_{26}(v) = 37 - 84v + 102v^2 - 36v^3 - 3v^4,$$
$$B_{26}(v) = 32(v-1)^3(v+1)^3(3v-5).$$

The j-invariant for this model is

$$j_{2\times6}(v) = \frac{\left(3v^2 - 6v + 7\right)^3\left(3v^6 - 18v^5 + 345v^4 - 1260v^3 + 1605v^2 - 738v + 127\right)^3}{4(v-3)^6(v-1)^6(v+1)^6(3v-5)^2(3v-1)^2}.$$

In this case the transformations leading to unchanged j-invariant are the following:

$$\{v \to 2 - v\}, \left\{v \to \frac{v-7}{3v-5}\right\}, \left\{v \to \frac{v+5}{3v-1}\right\}, \left\{v \to \frac{5v-7}{3v-1}\right\}, \left\{v \to \frac{5v-3}{3v-5}\right\}.$$

As said before in both cases $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, this form of the curve, i.e., $y^2 = x^3 + Ax^2 + Bx$, is a convenient way to search for candidates for new rational points. In fact, their x-coordinates should be either divisors of B or rational squares times divisors of B. For example, if we define

$$el_8(x) = x^3 + A_8(v)x^2 + B_8(v)x,$$

we look for values of x such that the corresponding expression can be transformed into a square by solving a quadratic equation in v. Observe that $el_8(4v^4) = 16v^8(2v-1)^2(4v^2 - 4v + 5)$, so we get a new point by solving

$$4v^2 - 4v + 5 = \text{Square}.$$

This is the first example in our list of rank 1 curves with such torsion.

3. Rank 1 curves for torsion group $\mathbb{Z}/8\mathbb{Z}$.

We list eighteen points together with the condition and the specialization of v, which leads to a new point on the curve and so, eventually, to rank 1 curves over $\mathbb{Q}(w)$. Observe that the parametrization it is not unique.

We split the list into three parts, having ten, six and two entries, respectively. First we present the ten conditions that were in the papers of Dujella and Peral already quoted. As said, all of them were included in Dujella-Peral [DP1] and

[DP2]. Some typos there are now corrected. The curve corresponding to place 4 were already in Lecacheux, and the curve in place 5 is both in Lecacheux and Kulesz.

In the second part of the list, we present the six new curves discovered by MacLeod in [ML]. MacLeod reported twelve curves of rank 1 in his paper, of which six are equivalent to some in the preceding ten and the other six were new. These curves form the second part of the list. Finally, the last two entries correspond to new curves and are the third part of the list.

3.1. **First part of the list.**

$$x_1 = 4v^4, \qquad c_1 = 5-4v+4v^2, \qquad v_1 = \frac{5-w^2}{4(w+1)},$$

$$x_2 = -(-1+v)v, \qquad c_2 = 1+v-v^2, \qquad v_2 = \frac{(w-2)w}{w^2+1},$$

$$x_3 = -4v^3(-2+3v), \qquad c_3 = -(2+v)(-2+3v), \qquad v_3 = -\frac{2(w-1)(w+1)}{w^2+3},$$

$$x_4 = 16(-1+v)^2v^2(1-2v+2v^2), \qquad c_4 = (1-2v+2v^2), \qquad v_4 = \frac{(w-2)w}{w^2-2},$$

$$x_5 = -2v^2(-1+2v^2), \qquad c_5 = -(-1+2v^2), \qquad v_5 = -\frac{2w}{w^2+2},$$

$$x_6 = -(-1+v)^2(1-6v+4v^2), \qquad c_6 = -1+6v-4v^2, \qquad v_6 = \frac{w^2-2w+2}{w^2+4},$$

$$x_7 = -\frac{4(v-1)^4(6v-1)}{2v-3}, \qquad c_7 = -(-3+2v)(-1+6v), \qquad v_7 = \frac{3w^2+1}{2\left(w^2+3\right)},$$

$$x_8 = -\frac{4(v-1)^4(4v-1)}{4v-3}, \qquad c_8 = -(-3+4v)(-1+4v), \qquad v_8 = \frac{w^2+3}{4\left(w^2+1\right)},$$

$$x_9 = -\frac{(v-1)^4(8v-5)(18v-5)}{4(3v-2)^2}, \qquad c_9 = -(-5+8v)(-5+18v), \qquad v_9 = \frac{5\left(w^2+1\right)}{2\left(4w^2+9\right)}$$

$$x_{10} = \frac{1}{8}\left(-4v^2+4v+1\right), \qquad c_{10} = -2(-1-4v+4v^2), \qquad v_{10} = \frac{w^2-4w+2}{2\left(w^2+2\right)}.$$

3.2. **Second part of the list.**

$$x_{11} = -\frac{(v-1)^2(2v-5)^2\left(36v^2-70v+25\right)}{(6v-7)^2}, \qquad c_{11} = -25+70v-36v^2, \qquad v_{11} = \frac{w^2-6w+34}{w^2+36},$$

$$x_{12} = -\frac{4(v-1)^3v(2v+1)^2}{(2v-3)^2}, \qquad c_{12} = -16(-9-28v+28v^2), \qquad v_{12} = -\frac{2(3w-14)}{w^2+28},$$

$$x_{13} = -\frac{4(v-1)^3v(10v-1)}{10v-9}, \qquad c_{13} = (-9+10v)(-1+10v), \qquad v_{13} = \frac{(3w-1)(3w+1)}{10(w-1)(w+1)},$$

$$x_{14} = -\frac{27}{2}(v-1)v, \qquad c_{14} = -6(-3+v)(2+v), \qquad v_{14} = -\frac{2(w-3)(w+3)}{w^2+6},$$

$$x_{15} = -\frac{1}{64}\left(16v^2-16v+1\right)^2, \qquad c_{15} = (7-128v+128v^2), \qquad v_{15} = -\frac{w^2+80w+1152}{8\left(w^2-128\right)},$$

$$x_{16} = \frac{(v-1)^2(4v-1)^2(10v-1)}{8(3v-1)}, \qquad c_{16} = 2(-1+3v)(-1+10v), \qquad v_{16} = \frac{2w^2-1}{2\left(3w^2-5\right)}.$$

3.3. **Third part of the list.**

$$x_{17} = -\frac{(v-1)^2(2v+1)^2(8v+1)}{8(v-3)}, \qquad c_{17} = -2(-3+v)(1+8v), \qquad v_{17} = \frac{6w^2-1}{2\left(w^2+4\right)},$$

$$x_{18} = \frac{4(4v-3)^2(10v-9)\left(18v^2-26v+9\right)^2}{(6v-5)^2(18v-13)}, \quad c_{18} = (-9+10v)(-13+18v), \quad v_{18} = \frac{9w^2-13}{2\left(5w^2-9\right)}.$$

All of them have rank at least 1, and all of them are non-isomorphic to each other. That the rank of these three curves is at least 1 over $\mathbb{Q}(w)$ can be proved using a specialization argument since the specialization map is a homomorphism, see [S, Theorem 11.4]. That all of them are non-isomorphic to each other can be proved by comparing their j-invariants.

4. Rank 2 families for the torsion group $\mathbb{Z}/8\mathbb{Z}$

In this section and in the next one we present curves with torsion group $\mathbb{Z}/8\mathbb{Z}$ and rank 2 over $Q(u)$. The search for these curves is made again by looking for solvable quadratic conditions leading to new points. Most of these conditions can be obtained also by comparing the j-invariants of the different curves with rank 1, for example from $v_3(u) = v_{14}(s)$ we deduce w_3. But we have followed mainly the first alternative in the presentation because in this way we get directly the x-coordinates of the new point.

4.1. **First Dujella and Peral case.** We present here some details for the family in which we have found three subfamilies with torsion $\mathbb{Z}/8\mathbb{Z}$ and generic rank at least 2. It corresponds to the third entry in the table of rank 1 families above. By inserting in the general family $y^2 = x^3 + A_8(v)x^2 + B_8(v)x$ the value $v = v_3(w)$ and clearing denominators we get the rank 1 family given by $y^2 = x^3+a_3(w)x^2+b_3(w)x$, where

$$a_3(w) = -31 - 148w^2 + 214w^4 - 116w^6 + 337w^8,$$
$$b_3(w) = 256(-1+w)^4(1+w)^4(1+3w^2)^4.$$

The X-coordinate of a point of infinite order is

$$X = -256(-1+w)^3w^2(1+w)^3.$$

By searching on several homogeneous spaces of this curve, we have found three conditions that lead to new points.

We list the values X_1, X_2 and X_3 jointly with the specialization of the parameter that converts them into a point of the specialized curve

$$X_1 = \frac{(-1+w)^2(1+w)^2(5+7w^2)^2(11+25w^2)}{16}, \qquad w_1 = \frac{11-u^2}{10u},$$
$$X_2 = \frac{(-1+w)^2(1+w)^2(1+11w^2)^2(7+29w^2)}{16w^2}, \qquad w_2 = \frac{29-12u+u^2}{-29+u^2},$$
$$X_3 = -27(w-1)(w+1)\left(w^2+3\right)^2\left(3w^2+1\right), \qquad w_3 = \frac{u^2-12u+15}{u^2-15}.$$

In this way we get three curves of rank 2 over $\mathbb{Q}(u)$. The two curves corresponding to w_1 and w_2 were reported in [DP1] and [DP2]. The curve corresponding to w_3 is new.

Once we insert w_1 into the coefficients a_3, b_3, we get as new coefficients aa_1, bb_1 given by

$$\begin{aligned} aa_1 &= 337u^{16} - 41256u^{14} + 4047356u^{12} - 288332632u^{10} + 2363813190u^8 \\ &\quad - 34888248472u^6 + 59257339196u^4 - 73087520616u^2 + 72238942897, \\ bb_1 &= 256\,(363+34u^2+3u^4)^4\,(11+u)^4\,(-11+u)^4\,(-1+u)^4\,(1+u)^4. \end{aligned}$$

The X-coordinates of two independent infinite order points are

$$-16(u-11)^3(u-1)(u+1)^3(u+11)\left(3u^4+34u^2+363\right)^2,$$
$$\frac{(-11+u)^2(-1+u)^2(1+u)^2(11+u)^2(11+u^2)^2(847+346u^2+7u^4)^2}{64u^2}.$$

4.2. **Second Dujella and Peral case.** By inserting w_2 into the coefficients a_3, b_3, we get as new coefficients aa_2, bb_2 given by

$$\begin{aligned} aa_2 &= 500246412961 - 2069985157080u + 3162080774436u^2 - 2895517882032u^3 + \\ &\quad 1873181389706u^4 - 906769167048u^5 + 333391978480u^6 - 93284915496u^7 + \\ &\quad 19860033555u^8 - 3216721224u^9 + 396423280u^{10} - 37179432u^{11} + \\ &\quad 2648426u^{12} - 141168u^{13} + 5316u^{14} - 120u^{15} + u^{16}, \\ bb_2 &= 256(-6+u)^4u^4(-29+6u)^4(841-522u+137u^2-18u^3+u^4)^4. \end{aligned}$$

The X-coordinates of two independent infinite order points are

$$16(u-6)u(6u-29)^3\left(u^4-18u^3+137u^2-522u+841\right)^2,$$
$$\frac{(-6+u)^2u^2(-29+6u)^2(87-29u+3u^2)^2(2523-1914u+541u^2-66u^3+3u^4)^2}{4(29-12u+u^2)^2}.$$

4.3. **The first new curve.** Now we insert w_3 into the coefficients a_3, b_3. We get as new coefficients aa_3, bb_3 given by

$$\begin{aligned}aa_3 = {} & 2562890625 - 20503125000u + 58638937500u^2 - 98524350000u^3 + \\ & 112751696250u^4 - 92004903000u^5 + 54062154000u^6 - 22880209320u^7 + \\ & 6966724707u^8 - 1525347288u^9 + 240276240u^{10} - 27260712u^{11} + \\ & 2227194u^{12} - 129744u^{13} + 5148u^{14} - 120u^{15} + u^{16}, \\ bb_3 = {} & 20736(-6+u)^4u^4(-5+2u)^4(75-15u+u^2)^4(3-3u+u^2)^4.\end{aligned}$$

The X-coordinates of two independent infinite order points are

$$1728(-6+u)^3u^3(-5+2u)^3(15-12u+u^2)^2,$$
$$81(-6+u)u(-5+2u)(75-15u+u^2)(3-3u+u^2)(225-90u+21u^2-6u^3+u^4)^2.$$

4.4. **The first MacLeod curve with rank 2.** We give here some details for the curve in which MacLeod discovered a curve having rank 2. It corresponds to the entry v_{12} in the table for rank 1 curves above. By inserting $v = v_{12}(w)$ in the general family $y^2 = x^3 + A_8(v)x^2 + B_8(v)x$, we get the rank 1 curve given by $y^2 = x^3 + a_{12}(w)x^2 + b_{12}(w)x$ where

$$\begin{aligned}a_{12}(w) = {} & 614656 - 1053696w + 363776w^2 - 59136w^3 - 7328w^4 + 2112w^5 \\ & + 464w^6 + 48w^7 + w^8, \\ b_{12}(w) = {} & 256w^4(6+w)^4(-14+3w)^4.\end{aligned}$$

The X-coordinate of the point of infinite order is

$$X = -\frac{8w^3(w+6)^3(3w-14)\left(w^2-12w+84\right)^2}{\left(3w^2+12w+28\right)^2}.$$

Now MacLeod imposed $-((256w^3(6+w)^2(-14+3w)^3)/(14+w)^2)$ as the X-coordinate of a new point. This is the same as to spezialize $w_4 = -\frac{(u-28)(u+28)}{2(u-63)}$. Using this value into the coefficients a_{12}, b_{12}, we get as new coefficients aa_4, bb_4 given by

$$\begin{aligned}aa_4 = {} & 1058387660788345388204032 - 141209336315730168643584u + \\ & 7118408590330053918720u^2 + 46091099527055278080u^3 - \\ & 2052153461221797068 8u^4 + 473831305485189120u^5 + 19585996741025792u^6 - \\ & 545185218600960u^7 - 18026420955648u^8 + 234415749120u^9 + 22250170880u^{10} - \\ & 242597376u^{11} - 14269120u^{12} + 276096u^{13} + 1632u^{14} - 96u^{15} + u^{16}, \\ bb_4 = {} & 4096(-63+u)^4(-28+u)^4(-14+u)^4(2+u)^4(28+u)^4(42+u)^4(-98+3u)^4.\end{aligned}$$

The X-coordinates of two independent infinite order points are

$$16(u-63)(u-28)^3(u-14)^3(u+2)^3(u+28)^3(u+42)(3u-98)\times$$
$$\frac{\left(u^4+24u^3-2744u^2-61152u+3133648\right)^2}{\left(3u^4-24u^3-3080u^2+4704u+1103088\right)^2},$$
$$-\frac{1024(u-63)^2(u-28)^3(u-14)^2(u+2)^2(u+28)^3(u+42)^3(3u-98)^3}{\left(u^2-28u+980\right)^2}.$$

4.5. **The second MacLeod curve with rank 2.** We present here some details for the curve in which MacLeod [ML] discovered a second curve having rank 2. It corresponds to the entry v_{13} in the table for rank 1 curves above. By inserting $v = v_{13}(w)$ in the general family $y^2 = x^3 + A_8(v)x^2 + B_8(v)x$, we get the rank 1 curve given by $y^2 = x^3 + a_{13}(w)x^2 + b_{13}(w)x$ where

$$a_{13}(w) = 2(431 + 3524w^2 - 3814w^4 + 3524w^6 + 431w^8)$$
$$b_{13}(w) = (-3 + w)^4(3 + w)^4(-1 + 3w)^4(1 + 3w)^4.$$

The X-coordinate of the point of infinite order is

$$X = (-3 + w)^3 w^2 (3 + w)^3 (-1 + 3w)(1 + 3w).$$

Now MacLeod [ML] imposed

$$\frac{10(w-3)^2 w^2 (w+3)^2 \left(13w^2+3\right)^2}{17w^2+7}$$

as the X-coordinate of a new point. This is the same as to spezialize $w_5 = -\frac{3u^2-80u+510}{u^2-170}$. Using this value into the coefficients a_{13}, b_{13}, we get as new coefficients aa_5, bb_5 given by

$$aa_5 = 4u^{16} - 768u^{15} + 68736u^{14} - 3816768u^{13} + 147831608u^{12} - 4261407840u^{11} + 95281085176u^{10} - 1698380209632u^9 + 24531870965502u^8 - 288724635637440u^7 + 2753623361586400u^6 - 20936296717920000u^5 + 123470437317680000u^4 - 541926476217600000u^3 + 1659119942784000000u^2 - 3151401008640000000u + 2790302976400000000,$$

$$bb_5 = u^4(3u-40)^4(4u-51)^4\left(u^2-24u+136\right)^4\left(2u^2-60u+425\right)^4.$$

The X-coordinates of two independent infinite order points are

$$-u(3u-40)(4u-51)^4\left(2u^2-60u+425\right)\left(u^3-44u^2+660u-3400\right)^2,$$
$$-u(3u-40)(4u-51)^2\left(2u^2-60u+425\right)\left(35u^3-1236u^2+14620u-57800\right)^2.$$

4.6. **The second new curve with rank 2.** We give here some details for the curve of rank 1 in which we have found another new curve with rank 2. It corresponds to the entry v_{17} in the table for rank 1 curves above. By inserting $v = v_{17}(w)$ in the general family $y^2 = x^3 + A_8(v)x^2 + B_8(v)x$, we get the rank 1 curve given by $y^2 = x^3 + a_{17}(w)x^2 + b_{17}(w)x$ where

$$a_{17}(w) = 2(1169 - 3956w^2 + 3704w^4 - 2216w^6 + 674w^8)$$
$$b_{17}(w) = (-3 + 2w)^4(3 + 2w)^4(-1 + 6w^2)^4.$$

The X-coordinate of the point of infinite order is

$$X = \frac{1}{4}w^2(2w-3)^2(2w+3)^2\left(7w^2+3\right)^2.$$

Observe that in order to have

$$X = -(27/2)(-3 + 2w)(3 + 2w)(4 + w^2)^2(-1 + 6w^2)$$

as a new point, we have to solve $30(3+2w^2)$ equal to a rational square. It is enough to specialize to

$$w_6 = \frac{3\left(u^2-20u+60\right)}{2\left(u^2-60\right)}.$$

Now we insert w_6 in the coefficients a_{17}, b_{17}, and we get as new coefficients aa_6, bb_6 given by

$$
\begin{aligned}
aa_6 = {} & 625u^{16} - 180000u^{15} + 17872800u^{14} - 1010171520u^{13} + \\
& 37753002432u^{12} - 973296787968u^{11} + 17592030254592u^{10} - \\
& 225415897049088u^9 + 2063161668920832u^8 - 13524953822945280u^7 + \\
& 63331308916531200u^6 - 210232106201088000u^5 + 489278911518720000u^4 - \\
& 785509373952000000u^3 + 833873356800000000u^2 \\
& - 503884800000000000u + 104976000000000000, \\
bb_6 = {} & 6879707136(u-10)^4(u-6)^4u^4\left(u^2-36u+300\right)^4\left(5u^2-36u+60\right)^4.
\end{aligned}
$$

The X-coordinates of two independent infinite order points are

$$
\begin{aligned}
& 104976(u-10)^2(u-6)^2u^2\left(u^2-20u+60\right)^2 \times \\
& \frac{\left(5u^4-168u^3+2088u^2-10080u+18000\right)^2}{\left(u^2-60\right)^2}, \\
& 1944(u-10)(u-6)u\left(u^2-36u+300\right)\left(5u^2-36u+60\right) \times \\
& \left(5u^4-72u^3+552u^2-4320u+18000\right)^2.
\end{aligned}
$$

4.7. **The third new curve with rank 2.** We describe here some details for the curve of rank 1 in which we have found another new curve with rank 2. It corresponds to the entry v_{18} in the table for rank 1 curves above. By inserting $v = v_{18}(w)$ in the general family $y^2 = x^3 + A_8(v)x^2 + B_8(v)x$, we get the rank 1 curve given by $y^2 = x^3 + a_{18}(w)x^2 + b_{18}(w)x$ where

$$
\begin{aligned}
a_{18}(w) = {} & 2(-3713 + 5492w^2 - 1462w^4 - 1004w^6 + 431w^8) \\
b_{18}(w) = {} & ((-5 + w^2)^4(-13 + 9w^2)^4.
\end{aligned}
$$

The X-coordinate of the point of infinite order is

$$
X = \frac{\left(3w^2+1\right)^2\left(9w^4+70w^2-63\right)^2}{w^2\left(w^2+3\right)^2}.
$$

If we impose

$$
X = (4(1+3w^2)^2(-13+9w^2)^2)/(-3+7w^2)
$$

as a new point, we have to solve $(-3 + 7w^2)$ equal to a rational square. This is done with

$$
w_7 = \frac{u^2-6u+21}{u^2-14u+21}
$$

Now we use w_7 in the coefficients a_{18}, b_{18} and we get as new coefficients aa_7, bb_7 given by

$$
\begin{aligned}
aa_7 = {} & -2u^{16} + 384u^{15} - 30128u^{14} + 1278592u^{13} - 32804472u^{12} + 545481088u^{11} - \\
& 6133914960u^{10} + 47788256896u^9 - 261061974220u^8 + 1003553394816u^7 - \\
& 2705056497360u^6 + 5051700355968u^5 - 6379846519032u^4 + \\
& 5221898865792u^3 - 2583961693488u^2 + 691617999744u - 75645718722, \\
bb_7 = {} & \left(u^2-56u+147\right)^4\left(u^2-8u+3\right)^4\left(u^4-32u^3+278u^2-672u+441\right)^4.
\end{aligned}
$$

The X-coordinates of two independent infinite order points are

$$
\begin{aligned}
& \left(u^2-8u+3\right)^2\left(u^4-32u^3+278u^2-672u+441\right)\left(2u^5-55u^4+508u^3-1834u^2+3234u-3087\right)^2, \\
& \frac{u^2\left(u^2-56u+147\right)^2\left(u^2-8u+3\right)^4\left(u^4-32u^3+278u^2-672u+441\right)^3}{(u^5-22u^4+262u^3-1524u^2+3465u-2646)^2}.
\end{aligned}
$$

4.8. **Rank 2 results.**

Theorem 1. *The seven curves corresponding to the specializations w_i, $i = 1, ..., 7$ have rank 2 over $\mathbb{Q}(u)$, and the points listed in each case, jointly with the torsion points, are generators for the full Mordell-Weil group.*

First we have proved that the curves have rank at least 2 using the specialization theorem [S, Theorem 11.4]. Then we have used the Gusić-Tadić algorithm [GT, Theorem 1.3] to find an injective specialization, and mwrank and magma to compute the rank and the Mordell-Weil group for the specialized curves. The specializations listed below prove that the rank is exactly 2 and that the points listed for each curve generate, jointly with the torsion points, the full Mordel-Weil group. In some cases, the Gusić-Tadić algorithm indicated that the originally found points generated a subgroup of order 2, so by "halving" corresponding points, we found the generators of the full group.

w_i	u value
1	22
2	19
3	11
4	17
5	3
6	-48
7	10

5. Infinite families of rank 3 for the torsion group $\mathbb{Z}/8\mathbb{Z}$

5.1. **First example.** It can be proved that there exist infinitely many elliptic curves with torsion group $\mathbb{Z}/8\mathbb{Z}$ parametrized by the points on an elliptic curve with positive rank. For example, it is enough to see that the equation $w_1(r) = w_2(s)$, i.e.,

$$\frac{11 - r^2}{10r} = \frac{29 - 12s + s^2}{-29 + s^2}$$

has infinitely many solutions. This is the same as to solve

$$r^2s^2 - 29r^2 + 10rs^2 - 120rs + 290r - 11s^2 + 319 = 0$$

in rationals, so the discriminant $\Delta = 3509 + 62r^2 + 29r^4$ has to be a square.

Observe that this is the same as imposing

$$\frac{(u-11)^2(u-1)^2(u+1)^2(u+11)^2\left(11u^4 - 142u^2 + 1331\right)^2\left(29u^4 + 62u^2 + 3509\right)}{16\left(u^2-11\right)^2}$$

as a new point on the curve of rank 2 corresponding to w_1.

But $t^2 = 3509 + 62r^2 + 29r^4$ has a solution, for example $(r, t) = (1, 60)$, hence it is equivalent to the cubic $Y^2 = X^3 - 463X^2 + 45936X$ whose rank is 2 as proved with mwrank [Cr]. This, jointly with the independence of the corresponding points, implies the existence of infinitely many solutions parametrized by the points of the elliptic curve, see [Le1] or [Ra] for this kind of argument.

5.2. **Other examples.** For the rank 2 curve corresponding to w_1, imposing

$$X = -27(u-11)(u-1)(u+1)(u+11)\left(u^4+278u^2+121\right)^2\left(3u^4+34u^2+363\right)$$

as a new point it is equivalent to solve

$$15(121+118u^2+u^4)=t^2$$

This equation has for example the solution $(u,t)(=1,60)$. So it is equivalent to the elliptic curve $Y^2 = X^3+1770X^2-108900X-192753000$ whose rank is 1.

For the rank 2 curve corresponding to w_2, imposing

$$X = 27(u-6)u(6u-29)\left(u^4-18u^3+137u^2-522u+841\right)\left(u^4-6u^3+7u^2-174u+841\right)^2$$

as a new point is equivalent to solve

$$3(2523-870u+151u^2-30u^3+3u^4)=t^2,$$

having the solution $(u,t)=(0,87)$, so the quartic can be shown equivalent to the elliptic curve $Y^2 = X^3+453X^2-37584X-817452$ whose rank is 2.

6. Examples of curves with high rank

The highest known rank of an elliptic curve over $\mathbb{Q}$ with torsion group $\mathbb{Z}/8\mathbb{Z}$ is 6. The first was discovered by Elkies in 2006, the second was found by Dujella MacLeod and Peral in 2013, and the third and fourth by Voznyy in 2021.

The second curve with rank 6 corresponds to $w=-\frac{261}{70}$ in the curve number 12. The third curve with rank 6 corresponds to $w=\frac{1327}{989}$ in the curve number 13. See [D] for the details of these curves.

The following list includes examples of rank 5 curves found in the rank 1 curves. First column indicates the number of the curve, and the second the value(s) of the parameter producing a rank 5 curve. The details on the curves, including the authors and years of discoveries, can be found on the web page [D].

Curve number	w values
1	$\frac{72}{19}, \frac{101}{145},$
2	$\frac{317}{10}, \frac{235}{46}, \frac{309}{130},$
3	$\frac{73}{83}, \frac{37}{157}, \frac{131}{419}, \frac{699}{1226}, \frac{166}{121},$
4	$\frac{245}{12}, \frac{95}{396}, \frac{-87}{28},$
6	$\frac{100}{29}, \frac{-28}{79}, \frac{304}{55},$
7	$\frac{79}{431},$
9	$\frac{287}{109}, \frac{65}{71},$
10	$\frac{21}{95}, \frac{195}{154},$
11	$\frac{94}{31}, \frac{103}{136}, \frac{508}{201},$
12	$\frac{3}{22}, \frac{117}{40}, \frac{48}{209}, \frac{24}{43}, \frac{-153}{4}, \frac{193}{2}, \frac{533}{126}, \frac{1440}{319}, \frac{-7446}{2773}, \frac{2365}{426},$
13	$\frac{501}{2407}, \frac{77}{188}, \frac{427}{1341},$
14	$\frac{838}{331}, \frac{484}{283}, \frac{1538}{631}, \frac{382}{121}, \frac{367}{94}, \frac{1226}{477},$
16	$\frac{657}{262}, \frac{283}{82}, \frac{4969}{2796}, \frac{2742}{839}, \frac{906}{719}, \frac{762}{521},$
17	$\frac{1557}{1538}, \frac{27}{382}, \frac{489}{670}, \frac{811}{1351}, \frac{198}{97}$
18	$\frac{89}{51}, \frac{1099}{1371}.$

7. Rank 1 curves for torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

We have found over forty curves having rank 1 for this torsion group, but we list only those in which we have found a further quadratic sections producing the five non-isomorphic rank 2 curves.

We present in the next section five non-isomorphic curves of rank 2, the four already known and another one that is new. The first four curves of rank 2 for this torsion group were discovered using Diophantine triples or a model due to Hadano [DP1], [DP2]. The fifth one has been found by using two consecutive quadratic sections. As said before, the seven cases presented by MacLeod [ML], are all isomorphic to one of the first three curves mentioned.

In this presentation, we get all these curves by imposing pair of consecutive quadratic sections. The data for the rank 1 curves in which we found rank 2 curves are in the next list. In the first rank 1 curve, we have found two rank 2 curves.

$$
\begin{aligned}
x_1 &= 16(-2+v)(1+v)^2, & c_1 &= 3(-2+v)v, & v_1 &= -\frac{6}{w^2-3},\\
x_2 &= \frac{64(v-1)^2(v+1)^3}{(v+5)^2}, & c_2 &= -6(-7+v)(3+v), & v_2 &= \frac{3\left(14w^2-1\right)}{6w^2+1},\\
x_3 &= (1+v)^2(7-4v+v^2)^2, & c_3 &= 6-2v+v^2, & v_3 &= \frac{2w^2-4w-1}{2w-3},\\
x_4 &= \frac{64}{3}(v-1)(v+1)(3v-1)^2, & c_4 &= 6(-1+2v)(-1+7v), & v_4 &= \frac{6w^2-1}{12w^2-7}.
\end{aligned}
$$

8. Rank 2 curves for torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

8.1. **First Dujella-Peral curve with rank 2.** As said before, this curve was first discovered using Diophantine triples [DP1], [DP2]. Here we give the details using a pair of quadratic sections. The curve of rank 1 corresponding to v_1 in the previous table is the following:

$$
\begin{aligned}
a_1 &= -4779 - 4644w^2 + 1134w^4 + 60w^6 + 37w^8,\\
b_1 &= 32(-3+w)^3(3+w)^3(-3+w^2)(3+w^2)^3(3+5w^2).
\end{aligned}
$$

A point of infinite order has the following X-coordinate

$$X = -32(-3+w)^2w^2(3+w)^2(-3+w^2).$$

Now we impose $-(-3+w)^2(3+w)^2(-3+w^2)(9+7w^2)$ as a new point, this is the same as specializing to $w_1 = \frac{3\left(u^2-8u+14\right)}{u^2-14}$. From here, we get the rank 2 curve whose main data are:

$$
\begin{aligned}
aa_1 = {} & 1475789056 - 6324810240u + 12303261824u^2 - 14934296832u^3 + 12836014912u^4 - 8279778528u^5 +\\
& 4113507272u^6 - 1590783936u^7 + 480725533u^8 - 113627424u^9 + 20987282u^{10} - 3017412u^{11} + 334132u^{12} -\\
& 27768u^{13} + 1634u^{14} - 60u^{15} + u^{16},\\
bb_1 = {} & -27(-4+u)^3u^3(-7+2u)^3(196-336u+152u^2-24u^3+u^4)(196-168u+62u^2-12u^3+u^4)^3\\
& (392-420u+169u^2-30u^3+2u^4).
\end{aligned}
$$

The X-coordinates of two independent infinite order points are

$$
\begin{aligned}
& -27(-4+u)^2u^2(-7+2u)^2(14-8u+u^2)^2(196-336u+152u^2-24u^3+u^4),\\
& -\frac{27}{4}(u-4)^2u^2(2u-7)^2\left(u^2-7u+14\right)^2\left(u^4-24u^3+152u^2-336u+196\right).
\end{aligned}
$$

8.2. **Second Dujella-Peral curve with rank 2.** Now we impose

$$(-3+w)(3+w)(-3+w^2)(9+7w^2)^2$$

in the rank 1 curve for v_1. This is the same as specializing to $w_2 = \frac{u^2-8u+6}{u^2-6}$. We get the rank 2 curve whose coefficients are:

$$
\begin{aligned}
aa_2 = {} & -3359232 + 2239488u + 6905088u^2 - 11695104u^3 + 6925824u^4 - 2494368u^5 + 3007512u^6 -\\
& 3509088u^7 + 2015437u^8 - 584848u^9 + 83542u^{10} - 11548u^{11} + 5344u^{12} - 1504u^{13} + 148u^{14} + 8u^{15} - 2u^{16},\\
bb_2 = {} & (-3+u)^3(-2+u)^3(1+u)^3(6+u)^3(36-60u+43u^2-10u^3+u^4)(36-24u+10u^2-4u^3+u^4)^3\\
& 36+48u-56u^2+8u^3+u^4).
\end{aligned}
$$

The X-coordinates of two independent infinite order points are

$$(-3+u)^2(-2+u)^2(1+u)^2(6+u)^2(6-8u+u^2)^2(36+48u-56u^2+8u^3+u^4),$$
$$\frac{1}{4}(u-3)(u-2)(u+1)(u+6)\left(u^4+8u^3-56u^2+48u+36\right)\left(2u^4-14u^3+53u^2-84u+72\right)^2.$$

8.3. **Third Dujella-Peral curve with rank 2.** We use v_2 and we get the rank 1 curve whose coefficients are:

$$a_2 = 121-2136w^2-5184w^4+273024w^6-1223424w^8,$$
$$b_2 = 128(-1+3w)^3(1+3w)^3(1+6w^2)(-1+24w^2)^3(-7+48w^2).$$

A point of infinite order has the following X-coordinate

$$X = \frac{128(3w-1)^2(3w+1)^2\left(6w^2+1\right)\left(24w^2-1\right)^3}{\left(36w^2+1\right)^2}.$$

Now we impose $4(-1+3w)^2(1+3w)^2(1+36w^2)(-7+48w^2)$ as X-coordinate for a new point. This is the same as using $w_3 = \frac{u^2-30u+180}{3(u^2-180)}$ in the preceding rank 1 curve. We get the rank 2 curve whose coefficients are:

$$aa_3 = 1101996057600000000-587731230720000000u-901187887104000000u^2+226276523827200000u^3-$$
$$1225425058007040000u^4+335908714991616000u^5-57791877967872000u^6+6886398457405440u^7-$$
$$597067735693824u^8+38257769207808u^9-1783699937213-26496u^{14}-96u^{15}+u^{16},$$
$$bb_3 = 5971968(-15+u)^3(-12+u)^3u^3(32400-17280u+2232u^2-96u^3+u^4)^3$$
$$(32400-4320u+288u^2-24u^3+u^4)(32400+34560u-5544u^2+192u^3+u^4).$$

The X-coordinates of two independent infinite order points are

$$\frac{18432(u-15)^2(u-12)^2u^2\left(u^4-96u^3+2232u^2-17280u+32400\right)^3\left(u^4-24u^3+288u^2-4320u+32400\right)}{\left(u^2-24u+180\right)^4},$$
$$-15552(u-15)^2(u-12)^2u^2\left(u^2-24u+180\right)^2\left(u^4+192u^3-5544u^2+34560u+32400\right).$$

8.4. **Fourth Dujella-Peral curve with rank 2.** We use v_3 and we get the rank 1 curve whose coefficients are:

$$a_3 = 96-480w+1584w^2-3084w^3+3001w^4-1440w^5+306w^6-12w^7-3w^8,$$
$$b_3 = 16(-3+w)(-2+w)^3(1+w)^3(-3+2w)(-2+3w)(1-3w+w^2)^3.$$

A point P of infinite order has the following X-coordinate

$$X = \frac{4(w-2)^2(w+1)^2\left(w^4-8w^3+24w^2-29w+13\right)^2}{(2w-3)^2}.$$

Now we impose $(-2+w)^3(1+w)(-3+2w)(-2+3w)(1-3w+w^2)$ as X-coordinate for a new point. This is the same as using $w_4 = -\frac{4u+9}{u^2-3}$ in the preceding rank 1 curve. We get the rank 2 curve whose coefficients are:

$$aa_4 = -314928-7978176u-47134224u^2-141974208u^3-263196864u^4-321113808u^5-259493652u^6-$$
$$128609568u^7-23353995u^8+16908960u^9+16006092u^{10}+6735888u^{11}+1706128u^{12}+271104u^{13}+$$
$$27360u^{14}+1920u^{15}+96u^{16},$$
$$bb_4 = 16(u-6)^3u(u+2)^3(3u+4)\left(u^2-3\right)\left(u^2+3u+1\right)^3\left(u^2+9u+9\right)^3\left(2u^2+4u+3\right)^3$$
$$\left(2u^2+12u+21\right)\left(3u^2+8u+9\right).$$

The X-coordinates of two independent infinite order points are

$$-4(u-6)u(u+2)(3u+4)\left(u^2+3u+1\right)\left(u^2+9u+9\right)\left(2u^4+8u^3+22u^2+48u+45\right)^2,$$
$$(6-u)(2+u)(1+3u+u^2)(9+9u+u^2)(3+4u+2u^2)^3(21+12u+2u^2)(9+8u+3u^2).$$

Note that the point P is in $2E(\mathbb{Q})$, so we replaced it by a point Q such that $2Q=P$ (mod $E(\mathbb{Q})_{\text{tors}}$) in order to obtain generators of the Mordell-Weil group.

8.5. **New curve with rank** 2. We use v_4 and we get the rank 1 curve whose coefficients are:

$$a_4 = 4048-22512w^2+49248w^4-50652w^6+20493w^8,$$
$$b_4 = 432(-1+w)^3(1+w)^3(-2+3w)^3(2+3w)^3(-7+12w^2)(-16+21w^2).$$

A point of infinite order has the following X-coordinate

$$X=-64(-1+w)(1+w)(-2+3w)(2+3w)(2+3w^2)^2.$$

Now we impose $-\frac{243}{4}(w-1)^2(w+1)^2\left(12w^2-7\right)\left(21w^2-16\right)$ as X-coordinate for a new point. This is the same as to use $w_5=\frac{4\left(u^2+1\right)}{5(u^2-1)}$ in the preceding rank 1 curve. We get the rank 2 curve whose coefficients are:

$$aa_5=-675347-8801576u^2+443877484u^4-944081432u^6+22507829710u^8-944081432u^{10}+443877484u^{12}-8801576u^{14}-675347u^{16},$$
$$bb_5=6912(-3+u)^3(3+u)^3(-1+3u)^3(1+3u)^3(11+u^2)^3(1-5u+u^2)(1+5u+u^2)\times(1+11u^2)^3(17+734u^2+17u^4).$$

The X-coordinates of two independent infinite order points are

$$192(u-3)^2(u+3)(3u+1)\left(u^2-5u+1\right)\left(11u^2+1\right)^2\left(u^3+28u^2+11u+8\right)^2,$$
$$243(-3+u)^2(3+u)^2(-1+3u)^2(1+3u)^2(1-5u+u^2)(1+5u+u^2)(17+734u^2+17u^4).$$

8.6. **Rank** 2 **results.**

Theorem 2. *The five curves corresponding to the specializations w_i, $i=1,\dots,5$ have rank* 2 *over $\mathbb{Q}(u)$, and the points listed in each case, jointly with the torsion points, are generators for the full Mordell-Weil group.*

As for the torsion group $\mathbb{Z}/8\mathbb{Z}$, we use the Gusić-Tadić algorithm again to find injective specializations and mwrank and magma to compute the rank and the generators for the specialized curves. However, since for the curves with the torsion group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$, the cubic polynomial factorizes into linear factors, here we use [GT, Theorem 1.1]. The specializations listed below prove that the rank is exactly 2 and that the points listed for each curve generate, jointly with the torsion points, the full Mordell-Weil group.

w_i	u value
1	15
2	17
3	22
4	19
5	20

9. Infinite families of rank 3 for the torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$

Imposing

$$-\frac{27}{4}(u-4)u(2u-7)\left(u^2-7u+14\right)^4\left(u^4-24u^3+152u^2-336u+196\right)$$

as the X-coordinate of a a new point in the first rank 2 curve is equivalent to solve

$$784-756u+293u^2-54u^3+4u^4=t^2.$$

This quartic has a rational point $(u,t)=(0,28)$. The condition is equivalent to an elliptic curve with rank 1.

In the same curve, we have that imposing

$$(-4+u)u(-7+2u)(196-336u+152u^2-24u^3+u^4)(196-168u+62u^2-12u^3+u^4)(392-420u+169u^2-30u^3+2u^4)$$

as the X-coordinate of a new point is the same as solving $196-420u+197u^2-30u^3+u^4=t^2$, which has a rational solution $(u,t)=(0,14)$ and can be seen to be equivalent to an elliptic curve with rank 1.

In the same curve, imposing

$$\frac{27}{4}(u-4)u(2u-7)\left(u^2-7u+14\right)^2\left(u^4-12u^3+62u^2-168u+196\right)^2$$

as the X-coordinate of a new point is the same that solving

$$784-924u+383u^2-66u^3+4u^4=t^2,$$

which has a rational solution $(u,t)=(0,28)$, and can be seen to be equivalent to an elliptic curve with rank 2.

In the third curve, imposing

$$-62208(u-15)^2(u-12)^2u^2\left(u^4-96u^3+2232u^2-17280u+32400\right)^2$$

as the X-coordinate of a new point is the same as solving

$$32400+60480u-9432u^2+336u^3+u^4=t^2$$

which has a rational solution $(u,t)=(0,180)$ and can be seen to be equivalent to an elliptic curve with rank 2.

10. Examples of curves with high rank

The highest known rank of an elliptic curve over $\mathbb{Q}$ with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ is 6. The first was discovered by Elkies in 2006, the second was found by Dujella, Peral and Tadić in 2015 and the third and fourth by Dujella and Peral in 2020. The Elkies curve with rank 6 corresponds to $u=-\frac{5}{6}$ in the rank 2 curve number 1 and to $u=\frac{3}{4}$ in the rank 2 curve number 3.

The following list includes examples of rank 5 curves found in the rank 2 curves. First column indicates the number of the curve, and the second the value(s) of the parameter producing a rank 5 curve. See [D] for the details of these curves.

Curve number	v values
1	$-\frac{5}{2}$,
2	$7, -66, \frac{21}{17}, \frac{14}{9}, \frac{65}{27}$,
4	$\frac{2}{5}, \frac{35}{4}, -\frac{1}{10}, -\frac{9}{62}$,
5	$\frac{13}{7}, \frac{77}{6}$.

11. On the rank: conjectures and heuristics.

There is an old conjecture predicting the existence of elliptic curves of arbitrarily high rank over $\mathbb{Q}$. This has also been conjectured for each torsion group in the Mazur theorem.

But recently, some heuristic predicts the existence of a universal bound for the rank of the elliptic curves over $\mathbb{Q}$. See [PPVW] for details. In that paper, the authors state heuristic bounds for each torsion group. In the case of torsion groups $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$, their heuristic claims that 3 is this bound, meaning that only a finite number of elliptic curves with this torsion group would have rank over $\mathbb{Q}$ greater or equal to 4.

As we said, we have several families of elliptic curves with torsion groups $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and rank at least 3 over $\mathbb{Q}$ parametrized by elliptic curves of positive rank.

Now observe that if we assume both the parity conjecture and the heuristic bound, then as a consequence, only a finite number of these curves can have root number equal to 1, and so the root number would not be evenly distributed over these families. The numerical data that we have collected suggests that this might not be the case.

In our first example, see Section 5.1, the elliptic curves with torsion group $\mathbb{Z}/8\mathbb{Z}$ and rank at least 3 over $\mathbb{Q}$ are parametrized by rational points on the curve

$$C : r^2s^2 - 29r^2 + 10rs^2 - 120rs + 290r - 11s^2 + 319 = 0.$$

This is a genus one curve (since it has two singular points with multiplicity 2, $[1 : 0 : 0]$ and $[0 : 1 : 0]$, in its projective closure) with a rational point $[-1, 0]$, so it is birationally equivalent to the elliptic curve in Weierstrass form

$$E : Y^2 + XY + Y = X^3 + X^2 - 1595X - 4768$$

by birational map f, which maps $[-1, 0]$ to the point at infinity $\mathcal{O} \in E(\overline{\mathbb{Q}})$. This map is unique up to the composition with $[-1]$. We choose one such f. Elliptic curve E has rank 2 and a rational 2-torsion. Denote the generators of the free part of the Mordell-Weil group by $P_1 = (-57/4, 1043/8)$ and $P_2 = (42, -89)$ while the generators of the torsion are $T_1 = (-3, 1)$ and $T_2 = (-39, 19)$. The root numbers of elliptic curves corresponding (via f) to the points $nP_1 + mP_2$ for small $n, m \in \mathbb{Z}$ are presented in Figure 1 (for the other choice of f, the figure would be centrally symmetric with respect to the origin). As we will show below, the points which differ by the point of order two correspond to the isomorphic elliptic curves, so they are not included in the figure. There are 38 curves with the root number 1, and 46 curves with the root number -1, which suggests that the root numbers are evenly distributed in the family.

The bottleneck of the root number computation is the factorization of the discriminant, which we need to determine the primes of bad reduction. One can speed up the computation by factoring the discriminant of the rank two elliptic curve over $\mathbb{Q}(w)$ from Section 4 (before specializing w) and using special number field sieve algorithm for factorization, but already for small values of (n, m) one ends up factoring numbers with hundreds of digits. We are grateful to the members of Mersenne Forum (https://mersenneforum.org/) for their help with factorization that permitted us to compute the parity for the much larger number of curves than we initially thought it is possible (especially in our third example).

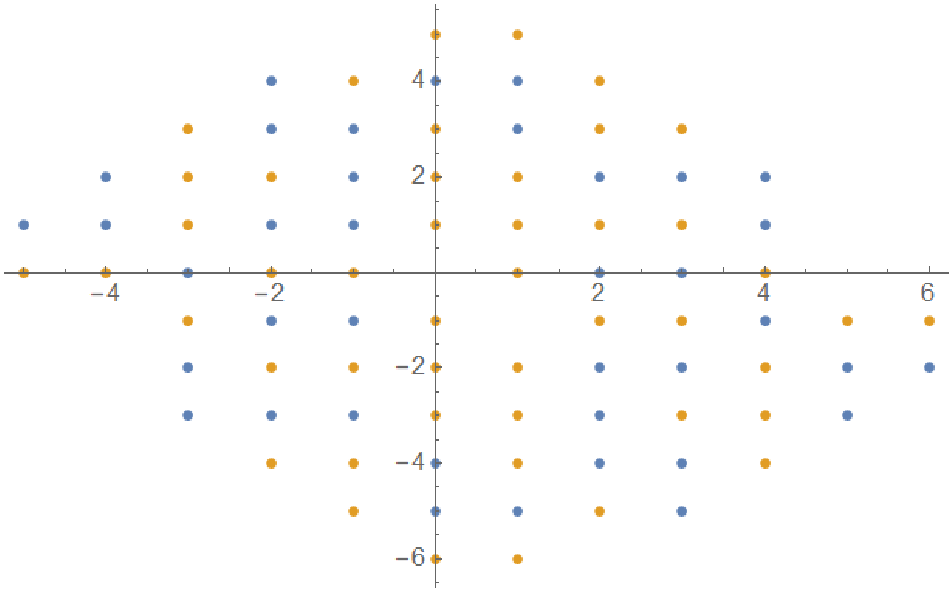

Figure 1. The blue (orange) point with coordinates (n, m) represents the elliptic curve with root number one (minus one) that corresponds to the point $nP_1 + mP_2$ from the first example.

Moreover, note that the figure has a symmetry $(n, m) \leftrightarrow (-n+1, -m-1)$. This symmetry comes from the properties of two involutions τ_1 and τ_2 on C, which are birational automorphisms on C that act on generic point (r, s) as $\tau_1(r, s) = (r, \tilde{s})$ and $\tau_2(r, s) = (\tilde{r}, s)$ where $\tilde{s}$ (respectively $\tilde{r}$) is the second solution of the quadratic equation in s (respectively r) defined by C, i.e. $\tilde{s} = \frac{120r}{r^2+10r-11} - s$. Since these functions extend to the regular involutions on the blowup of C (which we identify with E - the points in the blowup above the singular point $[0:1:0]$ are both rational and correspond to the points $P_1 - P_2$ and T_2 in $E(\mathbb{Q})$), we can identify them with involutions on E, which in general are of the form $T \mapsto -T + P$ for any point $P \in E(\overline{\mathbb{Q}})$ or of the form $T \mapsto T + P$ for P of order 2. Since both τ_1 and τ_2 have fixed points, they are of the form $T \mapsto -T + S_1$ and $T \mapsto -T + S_2$ for some points S_1 and S_2, respectively. One can check that, with our choice of generators, $S_1 = P_1 - P_2 + T_1 + T_2$ and $S_2 = P_1 - P_2$, thus the composition $\tau_1 \circ \tau_2$ is given by $T \mapsto T_1 + T_2$. Since C is equivalent to the condition $w_1(r) = w_2(s)$ (see Section 4), it follows that the points in the same orbit under the action of group generated by τ_1 and τ_2 correspond to the isomorphic elliptic curve (with $\mathbb{Z}/8\mathbb{Z}$ torsion and rank at least 3). In particular, it means that the points T and $T + T_1 + T_2$ correspond to the same elliptic curve, and likewise, points T and $-T + P_1 - P_2$, which explains observed symmetry.

It remains to prove that points T, $T+T_1$ and $T+T_2$ correspond to the same curve. We first note that elliptic curves from Section 4 corresponding to parameters $w_1(r)$ and $w_1(-r)=-w_1(r)$ are the same. Moreover, we have another pair of involutions on C mapping $(r,s)\mapsto(-r,s')$ (since discriminant of the defining equation of C with respect to s is even function in r). More precisely, define $\psi_1(r,s)=(-r,\frac{s(r^2+10r-11)-120r}{r^2-10r-11})$ and $\psi_2(r,s)=\tau_1(\psi_1(r,s))$. Same as before, we can identify ψ_1 and ψ_2 with involutions on E, and since $\psi_1\circ\psi_2=\tau_1$ it follows that one of them is given by $T\mapsto T+R$ for some R of order two. Since ψ_1 and ψ_2 are different from $\tau_1\circ\tau_2$ it follows that $R\neq T_1+T_2$, thus since T and $T+R$ correspond to the same curve, the claim follows.

In our second example, see Section 5.2, the elliptic curves with torsion group $\mathbb{Z}/8\mathbb{Z}$ and rank at least 3 over $\mathbb{Q}$ are parametrized by rational points on elliptic curve $Y^2=X^3-105987X+11743634$ of rank 2 and with rational 2-torsion subgroup. Denote the generators of the free part of the Mordell-Weil group by $Q_1=(-77,-4410)$ and $Q_2=(805,21168)$. The root numbers of elliptic curves corresponding to the points nQ_1+mQ_2 for small $n,m\in\mathbb{Z}$ are presented in Figure 2 (as in the previous example, the points which differ by the point of order two correspond to the isomorphic elliptic curves, so they are not included in the figure). There are 52 curves with the root number 1, and 52 curves with the root number -1, which suggests that the root numbers are evenly distributed also in this family. Note that in this case, the figure has a symmetry $(n,m)\leftrightarrow(-n-1,-m)$.

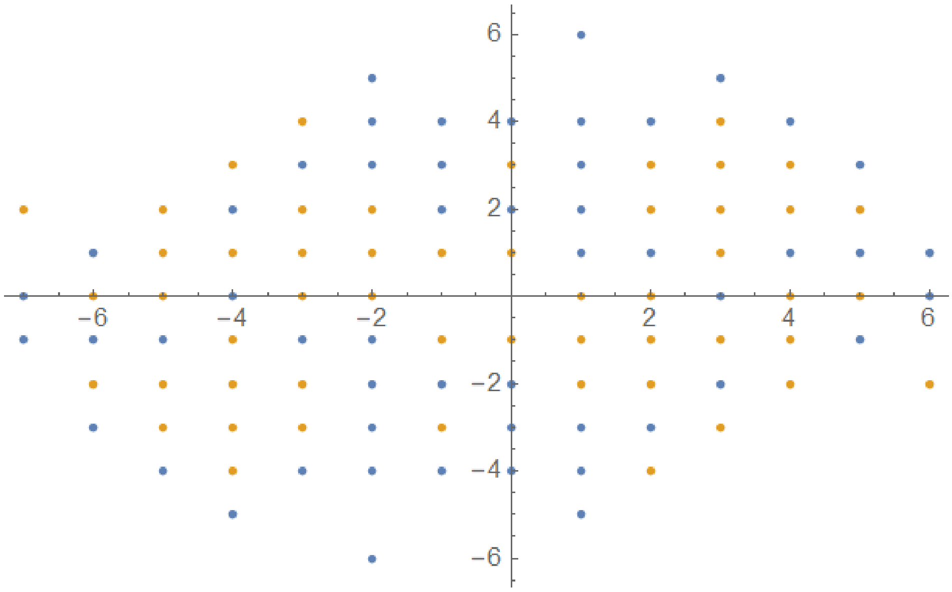

FIGURE 2. The blue (orange) point with coordinates (n,m) represents the elliptic curve with root number one (minus one) that corresponds to the point nQ_1+mQ_2 from the first example.

Two parameterizations of rank three elliptic curves with torsion group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$ by rank two elliptic curves from Section 9 are contained in the intersection of the rank two families from Sections 8.1 and 8.3. More precisely, these families are described by curves $D_1: r^2s-24rs^2+168rs-336r+360s^2-2700s+5040=0$ and $D_2: r^2s^2-\frac{15}{2}r^2s+14r^2-12rs^2+84rs-168r+90s=0$ where r and s are parameters of the families from Sections 8.1 and 8.3. Since these two equations have the same discriminant with respect to r and s, we conclude that these (genus one) curves are not only birationally equivalent but also that the set of s (and r) coordinates of the rational points on both of these curves agree. Hence, these two parameterizations are equal and can be described as follows.

The elliptic curves with torsion group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and rank at least 3 over $\mathbb{Q}$ are parameterized by rational points on elliptic curve $y^2 = x^3 - x^2 - 456x + 3456$ of rank 2 and with rational 2-torsion subgroup. Denote the generators of the free part of the Mordell-Weil group by $R_1 = (20, -44)$ and $R_2 = (4/9, -1540/27)$. The root numbers of elliptic curves corresponding to the points $nR_1 + mR_2$ for small $n, m \in \mathbb{Z}$ are presented in Figure 3 (as in the previous examples, the points which differ by the point of order two correspond to the isomorphic elliptic curves, so they are not included in the figure). There are 194 curves with the root number 1, and 168 curves with the root number -1, which suggests that the root numbers are evenly distributed also in this family. Note that in this case, the figure has a symmetry $(n, m) \leftrightarrow (-n+1, -m+1)$ which can be explained using involutions on curve D_1 (or D_2) as in the first example.

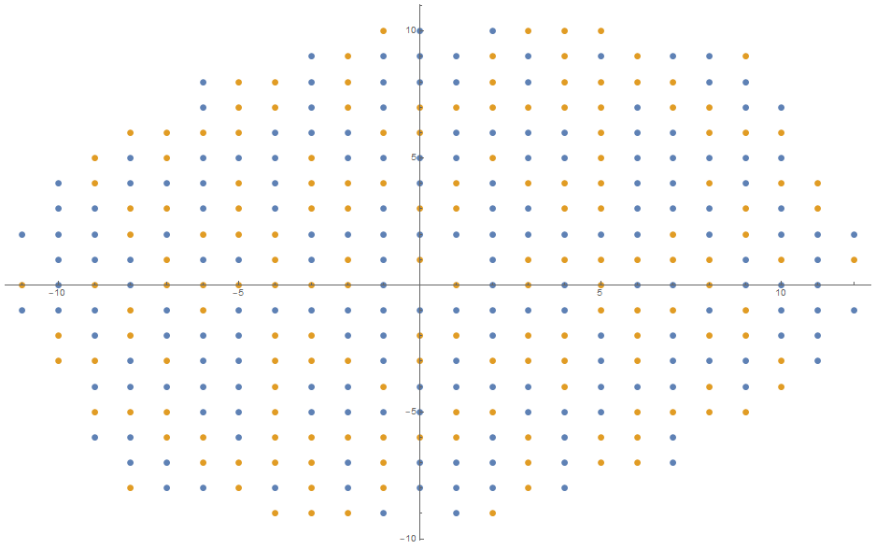

FIGURE 3. The blue (orange) point with coordinates (n, m) represents the elliptic curve with root number one (minus one) that corresponds to the point $nR_1 + mR_2$.

Final remark. Although our calculations of root numbers are limited to a relatively small number of cases, it seems that they indicate that the heuristic in [PPVW] needs some adjustments, at least in the case of curves with torsion groups $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

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