The extendibility of the parametric $D(-1)$-triples

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(joint work with Mirela Jukić Bokun)

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[1] M. Jukić Bokun, I. Soldo, *The extendibility of the parametric $D(-1)$-triples*, Publ. Math. Debrecen, to appear.

Let N be a positive integer such that $4N^2+1=q^j$, q is a prime and j a positive integer. It follows that q has to be odd. Moreover, $j=1$; A002496.

The extendibility of $D(-1)$-triples $\{1, 4N^2+1, 1-N\}$ in the ring $\mathbb{Z}[\sqrt{-N}]$.

If $N=1,2$, the extendibility of $D(-1)$-pairs $\{1,5\}$ and $\{1,17\}$ has already been considered and established the non-extendibility.

If $N>2$ is a square, say $N=x^2, x\in\mathbb{Z}$, then

$$4N^2+1=4x^4+1=(2x^2-2x+1)(2x^2+2x+1)$$

is not a prime. In that case, it is easy to check that the set $\{1, 4N^2+1, 1-N, 1+N\}$ is a $D(-1)$-quadruple in the ring $\mathbb{Z}[\sqrt{-N}]$, so in the ring of the Gaussian integers as well.



Main result:

Theorem 1 [Jukić Bokun, Soldo, 2025]

Let $N > 2$ be a non-square integer such that $4N^2 + 1 = q$, where q is a prime. There does not exist a $D(-1)$-quadruple of the form $\{1, 4N^2 + 1, 1 - N, d\}$ in the ring $\mathbb{Z}[\sqrt{-N}]$.

Auxiliary results:

Theorem 2 [Fujita, Soldo, 2022, 2023]

Let i be a non-negative integer, j, k positive integers and p, q different odd primes with $p^i < q^j$. If $\{p^i, q^j, c\}$ is a $D(-1)$-triple in the ring $\mathbb{Z}[\sqrt{-k}]$, then $c \in \mathbb{Z}$.



Theorem 3 [Nagell]

If the (u, v) is the fundamental solution of the equation

$$u^2 - Dv^2 = -N,$$

and if (x_1, y_1) is the fundamental solution of the corresponding Pell's equation, the following inequalities holds

$$0 < v \leq \frac{y_1}{\sqrt{2(x_1 - 1)}}\sqrt{N},$$
$$0 \leq u \leq \sqrt{\frac{1}{2}(x_1 - 1)N}.$$



The assumption that the set $\{1, 4N^2+1, 1-N, d\}$ is a $D(-1)$-quadruple in the ring $\mathbb{Z}[\sqrt{-N}]$ and Theorem 2 imply that $d \in \mathbb{Z}$.

1. Case $d < 0$:

Consider the system of equations

$$\begin{aligned} d-1 &= -Nx^2 \\ (4N^2+1)d-1 &= -Ny^2 \\ (1-N)d-1 &= z^2 \end{aligned} \tag{1}$$

in integers x, y, z.

If we eliminate d from (1), we obtain the system of pellian equations

$$(4N^2+1)x^2 - y^2 = 4N, \tag{2}$$
$$z^2 - N(N-1)x^2 = -N. \tag{3}$$

Consider the equation

$$z^2 - N(N-1)x^2 = \ 1. \tag{4}$$

Since $\sqrt{N(N-1)} = [N-1, \overline{2, 2(N-1)}]$, $(z_0, x_0) = (2N-1, 2)$ is the fundamental solution of (4). Then Nagell's theorem implies the following estimate for the fundamental solution x^* of (3):

$$0 \leq x^* \leq \sqrt{\frac{N}{N-1}} < \sqrt{2}.$$

Obviously, $x^* = 1$ and $z^{*2} = N(N-2)$, which yields $N = 0, 2$. This contradicts to the initial assumption $N > 2$.



2. Case $d > 0$:

Similarly, eliminating d from

$$\begin{aligned} d - 1 &= x^2 \\ (4N^2 + 1)d - 1 &= y^2 \\ (1 - N)d - 1 &= -Nz^2 \end{aligned} \tag{5}$$

we obtain the system of the pellian equations



$$y^2 - (4N^2+1)x^2 = 4N^2, \tag{6}$$
$$Nz^2 - (N-1)x^2 = N, \tag{7}$$

in integers x, y, z.

Since $\sqrt{4N^2+1} = [2N, \overline{4N}]$ and $\sqrt{N(N-1)} = [N-1, \overline{2, 2(N-1)}]$, then

$$(y, x) = (8N^2+1, 4N),$$
$$(z, x) = (2N-1, 2),$$

are the fundamental solutions of the Pell equations $y^2 - (4N^2+1)x^2 = 1$ and $z^2 - N(N-1)x^2 = 1$, respectively.



There exist non-negative integers m, n, and the fundamental solutions (y_0, x_0), (z_1, x_1) of (6) and (7), respectively, such that the equalities

$$y+x\sqrt{4N^2+1}=(y_0+x_0\sqrt{4N^2+1})(8N^2+1+4N\sqrt{4N^2+1})^m, \tag{8}$$

$$z\sqrt{N}+x\sqrt{N-1}=(z_1\sqrt{N}+x_1\sqrt{N-1})(2N-1+2\sqrt{N(N-1)})^n \tag{9}$$

with

$$|y_0| \leq 2N(2N+1) < 6N^2, \quad 0 \leq x_0 \leq \frac{4N^2}{\sqrt{4N^2+1}} < 2N, \tag{10}$$

$$|z_1| \leq \sqrt{N} < N, \quad 0 \leq x_1 \leq \sqrt{N} \tag{11}$$

hold.



From (8) we conclude that $x = v_m$, where

$$v_0 = x_0,\ v_1 = (8N^2+1)x_0 + 4Ny_0,\ v_{m+2} = 2(8N^2+1)v_{m+1} - v_m, \tag{12}$$

and from (9) we may write $x = w_n$, where

$$w_0 = x_1,\ w_1 = (2N-1)x_1 + 2Nz_1,\ w_{n+2} = 2(2N-1)w_{n+1} - w_n. \tag{13}$$

Thus we reformulated the system of equations (6) and (7) to Diophantine equations of the form $v_m = w_n$.
Next, we have to find the relation between the indices m and n.

From the recursive sequences (12) and (13) it follows by induction that

$$\begin{aligned} v_m &\equiv x_0 \pmod{2N}, \\ w_n &\equiv (-1)^n x_1 \pmod{2N}, \end{aligned} \tag{14}$$

for all $m, n \geq 0$. This implies $(y_0, x_0) = (\pm 2N, 0), (z_1, x_1) = (\pm 1, 0)$ and

$$y + x\sqrt{4N^2+1} = \pm\, 2N(8N^2 + 1 + 4N\sqrt{4N^2+1})^m, \tag{15}$$

$$z\sqrt{N} + x\sqrt{N-1} = \pm\, \sqrt{N}(2N - 1 + 2\sqrt{N(N-1)})^n. \tag{16}$$



We consider $v_m, w_n > 0$:

$$v_m = \frac{N}{\sqrt{4N^2+1}}\left((8N^2+1+4N\sqrt{4N^2+1})^m - (8N^2+1-4N\sqrt{4N^2+1})^m\right) \quad (17)$$

$$w_n = \frac{\sqrt{N}}{2\sqrt{N-1}}\left((2N-1+2\sqrt{N(N-1)})^n - (2N-1-2\sqrt{N(N-1)})^n\right), \quad (18)$$

i.e.,

$$v_0 = 0,\ v_1 = 8N^2,\ v_{m+2} = 2(8N^2+1)v_{m+1} - v_m, \quad (19)$$

$$w_0 = 0,\ w_1 = 2N,\ w_{n+2} = 2(2N-1)w_{n+1} - w_n. \quad (20)$$

If $m = 0$ or $n = 0$, we get $d = 1$, which cannot be.



It can be proved by induction that for $m, n \geq 1$ the following inequalities hold:

$$v_m \geq 8N^2(16N^2)^{m-1}, \tag{21}$$
$$w_n \leq 2N(4N-2)^{n-1}, \tag{22}$$

which imply

$$2 \leq 2m < n. \tag{23}$$



Further we will consider the equation $x = v_m = w_n$, with the condition $2 \leq 2m < n$. We will transform it into an inequality for linear form in logarithms of the following algebraic numbers with the corresponding minimal polynomials with integer coefficients:

$$\alpha_1 = \frac{\sqrt{4N^2+1}}{2\sqrt{N(N-1)}}, \qquad P_{\alpha_1}(x) = 4N(N-1)x^2 - (4N^2+1),$$

$$\alpha_2 = 8N^2 + 1 + 4N\sqrt{4N^2+1}, \qquad P_{\alpha_2}(x) = x^2 - (16N^2+2)x + 1,$$

$$\alpha_3 = 2N - 1 + 2\sqrt{N(N-1)}, \qquad P_{\alpha_3}(x) = x^2 - (4N-2)x + 1,$$

Taking

$$P = \frac{N}{\sqrt{4N^2+1}}(8N^2+1+4N\sqrt{4N^2+1})^m,$$
$$Q = \frac{\sqrt{N}}{2\sqrt{N-1}}(2N-1+2\sqrt{N(N-1)})^n,$$

we have that

$$\Lambda := \log\frac{Q}{P} = \log\alpha_1 - m\log\alpha_2 + n\log\alpha_3.$$



Proposition 1

If $v_m = w_n$ and $m \geq 1$, then

$$0 < \log \Lambda < \log 2 - 2n \log \alpha_3 = \log 2\alpha_3^{-2n}. \tag{24}$$

Lemma 1 [Laurent, 2008]

Let $\gamma_1 > 1$ and $\gamma_2 > 1$ be two real multiplicatively independent algebraic numbers, b_1, b_2 positive integers and

$$\Lambda = b_2 \log \gamma_2 - b_1 \log \gamma_1.$$

Let $D := [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}]$,

$$h_i \geq h'(\gamma_i) = \max\left\{h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D}\right\}, \quad i = 1, 2,$$

where $h(\gamma_i)$ is the absolute logarithmic height of γ_i, and

$$b' \geq \frac{b_1}{Dh_2} + \frac{b_2}{Dh_1}.$$

Then

$$\log|\Lambda| \geq -17.9 \cdot D^4 \left(\max\left\{\log b' + 0.38, \frac{30}{D}, 1\right\}\right)^2 h_1 h_2.$$



In order to do that,

$$\Lambda = \log \alpha_1 - m \log \alpha_2 + n \log \alpha_3$$

is transformed to

$$\Lambda = \log \gamma_1 - m \log \gamma_2,$$

where $\gamma_1 = \frac{\alpha_2}{\alpha_3^2}$ and $\gamma_2 = \alpha_3^{n-2m}\alpha_1$.

In our case we have $D = 4$, $b_1 = m$, $b_2 = 1$.



Moreover, we have

$$h(\alpha_1) = \frac{1}{2}\log(4N^2+1), \quad h(\alpha_2) = \frac{1}{2}\log\alpha_2, \quad h(\alpha_3) = \frac{1}{2}\log\alpha_1,$$

We also get

$$h_1 = 2.241\log N \quad h_2 = (0.561(n-2m)+1.061)\log N,$$
$$b' > \frac{0.661N(n-2m)}{4(0.561(n-2m)+1.061)}.$$

By applying Laurent's Theorem we get

$$\log\Lambda \geq -10269.16\left(\log b' + 0.38\right)^2(0.561(n-2m)+1.061)\log^2 N.$$



By combining that with the bound

$$\log \Lambda < \log 2 - 2n \log \alpha_3 < \log 2 - 2n \log N,$$

we obtain

$$b' - 0.01 < 642 \left(\log b' + 0.38 \right)^2,$$

which implies $b' < 89041$.

Lemma 2

If $v_m = w_n$, $m \geq 1$, then $N < 8.8 \cdot 10^5$.

Now we have to find an upper bound for n.

Theorem 4 [Matveev, 2000]

Let $\alpha_1, \alpha_2, \alpha_3$ be three distinct non-zero algebraic numbers, let $\log \alpha_1, \log \alpha_2$ and $\log \alpha_3$ be $\mathbb{Q}$-linearly independent logarithms of these algebraic numbers and let b_1, b_2, b_3 be rational integers with $b_1 \neq 0$. Put

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3.$$

Let $D = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}]$ and $\chi = [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}]$. Let A_1, A_2, A_3 be positive real numbers, which satisfy

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|\}, \quad 1 \leq j \leq 3,$$

where h is the absolute logarithmic height.

Assume that

$$B \geq \max\{|b_j|A_j/A_1 : 1 \leq j \leq 3\}.$$

Also define

$$C_1 = \frac{5 \cdot 16^5}{6\chi} e^3(7+2\chi)\left(\frac{3e}{2}\right)^{\chi}\left(26.25 + \log(D^2\log(eD))\right).$$

Then

$$\log|\Lambda| > -C_1 D^2 A_1 A_2 A_3 \log\left(1.5eDB\log(eD)\right).$$



In our case we have $D = 4, \chi = 1, b_1 = 1, b_2 = -m, b_3 = n$. Moreover we have

$$h(\alpha_1) = \frac{1}{2}\log(4N^2+1), \quad h(\alpha_2) = \frac{1}{2}\log\alpha_2 < \frac{1}{2}\log(16N^2+2), \quad h(\alpha_3) = \frac{1}{2}\log\alpha_3,$$

and we can choose

$$\begin{aligned} A_1 &= 4h(\alpha_1) = 2\log(4N^2+1), \\ A_2 &= 2\log(16N^2+2), \\ A_3 &= 4h(\alpha_3) = 2\log\alpha_3. \end{aligned}$$

Since $B \geq \max\{1, 1.5m, 0.7n\}$, $2m < n$ implies that we can take $B = 0.75n$. Thus, by applying Matveev's Theorem we get:

$$\log\Lambda > -24.64336 \cdot 10^{11} \cdot \log(4N^2+1) \cdot \log(16N^2+2) \cdot \log\alpha_3 \cdot \log(29.1898n).$$



By combining that with the bound

$$\log \Lambda < \log 2 - 2n \log \alpha_3,$$

we obtain

$$\begin{aligned} 2n \log \alpha_3 <& 2.464336 \cdot 10^{12} \cdot \log(4N^2+1) \cdot \log(16N^2+2) \\ & \times \log \alpha_3 \cdot \log(29.1898N) + \log 2 \end{aligned}$$

and

$$\frac{n}{\log(29.1898n)} < 1.23217 \cdot 10^{12} \cdot \log(4N^2+1) \cdot \log(16N^2+2). \tag{25}$$

For non-square integer $N < 8.8 \cdot 10^5$ from (25) we obtain $n < 4.5 \cdot 10^{16}$.

To complete the proof of the main result we use the reduction method of Dujella and Pethő in order to find a lower upper bound for n.

Lemma 3 [Dujella, Pethő, 1998]

Let $\kappa \neq 0$ and μ be real numbers. Suppose that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let $\varepsilon = ||\mu q|| - M \cdot ||\kappa q||$, where $|| \cdot ||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < n\kappa - m + \mu < A \cdot B^{-n}, \tag{26}$$

in integers m and n with

$$\frac{\log \frac{Aq}{\varepsilon}}{\log B} \leq n \leq M.$$



Dividing the inequality

$$\Lambda = \log \alpha_1 - m \log \alpha_2 + n \log \alpha_3 < 2\alpha_3^{-2n}.$$

by $\log \alpha_2$ we obtain the inequality

$$0 < n\kappa - m + \mu < A \cdot B^{-n},$$

where

$$\kappa = \frac{\log \alpha_3}{\log \alpha_2}, \quad \mu = \frac{\log \alpha_1}{\log \alpha_2}, \quad A = \frac{2}{\log \alpha_2}, \quad B = \alpha_3^2.$$



We apply Lemma 3 for $M = 4.5 \cdot 10^{16}$, and by using the computer algebra system PARI/GP we do the reduction method. After two steps of the reduction method we obtain $n \leq 2$ and we arrive to contradiction since we have that $2 \leq 2m < n$.
That implies that the system with positive d from the beginning has no the non-trivial solutions.



Happy birthbay to Andrej, Borka, and Ivica!

Thank you for your attention!