Schneider's p-adic continued fractions of rational numbers

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Happy milestone birthdays, Borka, Duje, and Ivica!

Continued fractions of p-adic numbers

Two main types of continued fractions in $\mathbb{Q}_p$:

Schneider (1968) and

Ruban (1970) modified by Browkin (1978).

Schneider's continued fraction

$\xi \in \mathbb{Q}_p, \quad |\xi|_p = 1$

$$\xi = b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{\ddots}}} = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, p^{a_3} \ldots]$$

a_j positive integers

$b_j \in \{1, 2, \ldots, p-1\}$

Algorithm:

b_0 such that $|\xi - b_0|_p < 1$

if $b_0 = \xi$, stop,

$p^{a_1} = |\xi - b_0|_p^{-1}$

$\xi_1 = p^{a_1}/(\xi - b_0)$, ...

Example

$$p = 7$$

$$\frac{716}{79} = 1 + \cfrac{7^2}{5 + \cfrac{7}{3 + \cfrac{7}{2}}} = [1, 7^2 : 5, 7 : 3, 7 : 2]$$

$$\frac{715}{79} = [4, 7 : 2, 7 : 4, 7 : 4, 7^2 : 3, 7 : 3, 7 : 5, 7 : \underbrace{6, 7 : 6, 7 : 6, 7 : 6, \ldots}_{6,\,7 \text{ repeats}}]$$

$$\xi = b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{\ddots}}}, \qquad \frac{P_n}{Q_n} = b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{\ddots + \cfrac{p^{a_n}}{b_n}}}}$$

$$\begin{pmatrix} P_0 & P_{-1} \\ Q_0 & Q_{-1} \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{matrix} P_n = b_n P_{n-1} + p^{a_n} P_{n-2} \\ Q_n = b_n Q_{n-1} + p^{a_n} Q_{n-2} \end{matrix}, \text{ for } n \geqslant 1$$

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} b_n & 1 \\ p^{a_n} & 0 \end{pmatrix} = \prod_{i=0}^{n} \begin{pmatrix} b_i & 1 \\ p^{a_i} & 0 \end{pmatrix}$$

Taking determinant

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n+1} \prod_{i=1}^{n} p^{a_i} \;\Rightarrow\; \left| \frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} \right|_p = p^{-\sum_{i=1}^{n} a_i}.$$

Thus (P_n/Q_n) is Cauchy, $\xi = \lim_{n\to\infty} P_n/Q_n$ and

$$|Q_n \xi - P_n|_p = |\xi - P_n/Q_n|_p = p^{-\sum_{i=1}^{n+1} a_i}.$$

Finiteness of continued fraction expansion of rationals

If the continued fraction expansion is finite, then the number is rational.

A rational number has a finite or ultimately periodic continued fraction expansion (Bundschuh, 1977) ending in

$$p-1+\cfrac{p}{p-1+\cfrac{p}{p-1+\cfrac{p}{\ddots}}}=-1$$

e.g. $1+\frac{p}{1+\frac{p^2}{-1}}=\frac{p^2-p-1}{p^2-1}>0$

No characterization of rational numbers with infinite Schneider's p-adic continued fraction expansion.
Hirsh and Washington (2011) handle some special cases.

For a prime p and a real number $x > 1$, denote by

$$N_p(x) = \{u/v \,:\, u, v \in \mathbb{Z}, 1 \leqslant u \leqslant x,\ 1 \leqslant v \leqslant x,\ \gcd(u, v) = \gcd(uv, p) = 1\}.$$

Since the number of coprime pairs (u, v) with $1 \leqslant u \leqslant x$ and $1 \leqslant v \leqslant x$ is $\frac{6}{\pi^2}x^2 + O(x \log x)$, by excluding pairs of the form (pm, n) or (n, pm), where $\gcd(n, p) = 1$ and $pm,\ n \leqslant x$, we can crudely bound

$$\operatorname{card} N_p(x) > \Big(\frac{6}{\pi^2} - \frac{1}{200} - 2\,\frac{1}{p}\,\frac{p-1}{p}\Big)x^2 > \frac{1}{10}\,x^2, \qquad (1)$$

for x large enough.
Because of symmetry taking (u, v) to (v, u), we have

$$\operatorname{card}\big(N_p(x) \cap (1, +\infty)\big) = \frac{1}{2}(\operatorname{card} N_p(x) - 1) \qquad (2)$$

Upper bound on the number of fractions with terminating expansion

Our result confirming a conjecture by Hirsh and Washington, for which they gave numerical support and heuristic arguments, that rational numbers with nonterminating expansion are more common than those with terminating expansion.

Theorem 1

For a prime p and a positive number x, the number of elements in $N_p(x)$ with terminating p-adic continued fraction expansion is $\ll x^{2(1-p^{-8})}$. The number of elements in $N_p(x) \cap (1, +\infty)$ with nonterminating expansion is $\gg x^2$.

Note that throughout, the constants implicit in Vinogradov $\gg$, $\ll$, and Landau $O(\cdot)$ notation depend at most on p.

Proof (sketch)

Let $x = p^D$ for some $D > 0$ large enough. Consider any fraction from $N_p(p^D)$ with terminating p-adic continued fraction expansion

$$\frac{P_n}{Q_n} = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots, p^{a_n} : b_n],$$

We can assume that P_n is large enough in terms of p.
Since $P_n \geqslant P_{n-1}$ and $Q_n \geqslant Q_{n-1}$ for $n \geqslant 0$ and the inequalities are strict for $n \geqslant 2$, we have from the basic recurrences that $P_n \geqslant (p^{a_n}+1)P_{n-2}$ and $Q_n \geqslant (p^{a_n}+1)Q_{n-2}$. It follows that

$$\begin{aligned} P_nQ_n &\geqslant P_nQ_{n-1} \geqslant (p^{a_n}+1)(p^{a_{n-1}}+1)P_{n-2}Q_{n-3} \geqslant \ldots \geqslant \\ &\geqslant (p^{a_n}+1)(p^{a_{n-1}}+1)\cdots(p^{a_2}+1)P_1Q_0 \\ &\geqslant (p^{a_n}+1)(p^{a_{n-1}}+1)\cdots(p^{a_2}+1)(p^{a_1}+1)P_{-1}Q_0 \end{aligned}$$

for n odd and

$$\begin{aligned} P_nQ_n &\geqslant P_{n-1}Q_n \geqslant (p^{a_n}+1)(p^{a_{n-1}}+1)P_{n-3}Q_{n-2} \geqslant \ldots \geqslant \\ &\geqslant (p^{a_n}+1)(p^{a_{n-1}}+1)\cdots(p^{a_1}+1)P_{-1}Q_0 \end{aligned}$$

for n even, so the same lower bound holds for P_nQ_n for all $n \geqslant 1$.

Since $P_n/Q_n \in N_p(p^D)$, this implies

$$p^{2D} \geqslant (p^{a_1}+1)(p^{a_2}+1)\cdots(p^{a_n}+1), \quad \text{so}$$

$$2D \geqslant \sum_{i=1}^{n} \log_p(p^{a_i}+1) = \sum_{i=1}^{n} \left(a_i + \log_p\left(1+\frac{1}{p^{a_i}}\right)\right)$$

$$> \sum_{i=1}^{n} \left(a_i + \frac{1}{p^{a_i+1}}\right),$$

where the last inequality follows from Bernoulli inequality. Multiplying by p^2, we obtain

$$2Dp^2 > \sum_{i=1}^{n} \left(p^2 a_i + \frac{1}{p^{a_i-1}}\right) \geqslant \sum_{i=1}^{n} (p^2 a_i + \varepsilon_i), \tag{3}$$

where $\varepsilon_i = 1$ for $a_i = 1$ and $\varepsilon_i = 0$ for $a_i > 1$.
Therefore, the number of pairs of sequences $\left((a_i)_{1\leqslant i\leqslant n}, (b_i)_{0\leqslant i\leqslant n}\right)$ such that the value of the continued fraction $[b_0, p^{a_1} : b_1, \ldots, p^{a_n} : b_n]$ is in $N_p(p^D)$ is less than the number of solutions of inequality (3) (for $(a_i)_i$) multiplied by $(p-1)^{n+1}$ (for $(b_i)_i$).

Now, employing generating functions, we see that the number of fractions in $N_p(p^D)$ with terminating expansion is smaller than the coefficient of $z^{\lfloor 2Dp^2 \rfloor}$ in the power series

$$(1+z+z^2+\cdots)\cdot \tag{4}$$

$$\sum_{n=1}^{\infty}(p-1)\big((p-1)z^{p^2+1}+(p-1)z^{2p^2}+(p-1)z^{3p^2}+\cdots\big)^n$$

$$= \frac{1}{1-z}\,\frac{(p-1)^2\big(z^{p^2+1}+\frac{z^{2p^2}}{1-z^{p^2}}\big)}{1-(p-1)\big(z^{p^2+1}+\frac{z^{2p^2}}{1-z^{p^2}}\big)}$$

$$= \frac{(p-1)^2(z^{p^2+1}+z^{2p^2}-z^{2p^2+1})}{(1-z)\big(1-z^{p^2}-(p-1)(z^{p^2+1}+z^{2p^2}-z^{2p^2+1})\big)}. \tag{5}$$

From this, we obtain that the number of such fractions is

$$\ll r^{(2D-1)p^2-1}, \tag{6}$$

where r is an upper bound on the (standard) absolute value of all roots in complex numbers of the polynomial reciprocal to the polynomial in the denominator of the rational function (5).

Thus, we have to find an upper bound for the roots of the polynomial

$$\begin{aligned} R(z) &= z^{2p^2+1} - z^{p^2+1} - (p-1)z^{p^2} - (p-1)z + (p-1) \\ &= (z^{p^2+1} - p + 1)(z^{p^2} - 1) - (p-1)z \\ &= \big(z^{2p^2} - z^{p^2} - (p-1)\big)z - (p-1)(z^{p^2} - 1). \end{aligned}$$

We show that we can take

$$r = p^{\frac{1}{p^2}} - \frac{1}{p^8},$$

Now, (6) and some auxiliary bounds show that the number of elements in $N_p(x)$ with terminating p-adic continued fractions is

$$\ll \left(r^{p^2}\right)^{2D} \frac{1}{r^{p^2+1}} \ll \left(\frac{p}{1+p^{-7}}\right)^{2D} \ll \left(p^{1-p^{-8}}\right)^{2D} = x^{2(1-p^{-8})}.$$

The last claim in the theorem follows by comparing the previous inequality with (1) and (2).

While the conjecture and the discussion in the paper by Hirsh and Washington are made for fractions u/v with bounded $|u| + |v|$ and our results are given for u/v with bound on $\max\{|u|, |v|\}$ (see the definition of $N_p(x)$), this poses no difficulty since

$$\max\{|u|, |v|\} \leqslant |u| + |v| \leqslant 2\max\{|u|, |v|\}$$

so the asymptotic bounds in our results are still valid in the their setting.

Lower bound on the number of fractions with terminating expansion

Although rational numbers with nonterminating p-adic continued fractions are more common than those with terminating expansions, the next theorem shows that the latter are still fairly common.

Theorem 2

For a prime p and a positive number x, the number of elements in $N_p(x)$ with terminating p-adic continued fraction expansion is $\gg x^{1+\varepsilon}$, where ε is a small positive number depending only on p. For $p > 7$, even if we restrict ourselves to elements in $N_p(x)$ with terminating expansion whose every partial numerator is equal to p, the same bound still holds.

Finite expansion with fixed initial part

Corollary 3

Let p be any prime number. For any $k \geqslant 0$, any sequences $(a_i)_{1 \leqslant i \leqslant k}$ of positive integers and $(b_i)_{0 \leqslant i \leqslant k}$ of elements in $\{1, \ldots, p-1\}$, and any positive number C, there are infinitely many positive integers u such that there are at least C reduced positive rational numbers u/v with terminating p-adic continued fraction expansion, all of them having initial part

$$[b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots, p^{a_k} : b_k, \ldots].$$

Using the simple identity

$$b + \frac{p^a}{\frac{u}{v}} = \frac{bu + p^a v}{u},$$

we see that a result analogous to the previous corollary holds if we look at denominators instead of numerators.

Various results on expansion of rational numbers

For a positive integer n, we slightly modify and generalize the notation by Hirsh and Washington and denote by $S_p(n)$ the set of all positive rationals of the form u/v, where $u+v=n$ and $\gcd(uv,p)=\gcd(u,v)=1$. Let $T_p(n)$ be the set of elements in $S_p(n)$ with terminating p-adic continued fraction expansion.

Proposition 4

If p divides n, then $\operatorname{card} T_p(pn) = \operatorname{card} T_p(n)$.

$S_p(n)$ the set of all positive rationals of the form u/v, where $u+v=n$ and $\gcd(uv,p)=\gcd(u,v)=1$.

$T_p(n)$ the set of elements in $S_p(n)$ with terminating p-adic continued fraction expansion.

Proposition 5

For a positive integer n, the set $T_p(p^n)$ contains only the rational number u_n/u_{n-1}, where

$$u_k = \frac{1}{p+1}\left(p^{k+1}+(-1)^k\right), \quad k \geqslant 0.$$

$S_p(n)$ the set of all positive rationals of the form u/v, where $u+v=n$ and $\gcd(uv,p)=\gcd(u,v)=1$.

$T_p(n)$ the set of elements in $S_p(n)$ with terminating p-adic continued fraction expansion.

Proposition 6

For every prime p, $\limsup_{n\to+\infty} \operatorname{card} T_p(n) = +\infty$ holds.
For primes $p>2$,

$$\limsup_{\substack{n\to+\infty \\ p\nmid n}} \operatorname{card} T_p(n) = +\infty,$$

where $\limsup$ *is taken over positive integers not divisible by p.*

$S_p(n)$ the set of all positive rationals of the form u/v, where $u+v=n$ and $\gcd(uv,p)=\gcd(u,v)=1$.

$T_p(n)$ the set of elements in $S_p(n)$ with terminating p-adic continued fraction expansion.

Proposition 7

For any prime p, there exists a positive integer n divisible by p such that $T_p(n)$ is an empty set while $S_p(n)$ is not empty.

Note that we are not interested in the cases like $p=3$ and $n=10$ where already $S_p(n)$ is empty which tells us nothing about continued fraction expansions.

$S_p(n)$ the set of all positive rationals of the form u/v, where $u+v=n$ and $\gcd(uv,p)=\gcd(u,v)=1$.

$T_p(n)$ the set of elements in $S_p(n)$ with terminating p-adic continued fraction expansion.

Corollary 8

For any prime p, there are infinitely many positive integers n such that $S_p(n)$ is nonempty while $T_p(n)$ is empty.

Proof. The statement follows immediately from Propositions 4 and 7. The only detail left to mention is that from $u/v \in S_p(n)$, we get $p-1+\frac{p}{u/v} \in S_p(pn)$, so $S_p(n)$ being nonempty implies that $S_p(pn)$ is nonempty as well.

Theorem 9

For every prime p and every positive integer $n \notin \{1,2,3,4,6\}$ such that $(p,n) \notin E$, there is a rational number $\frac{u}{v} > 1$ such that $\frac{u}{v} \in S_p(n)$ and the p-adic continued fraction expansion of $\frac{u}{v}$ is nonterminating. Here E is the set containing all pairs $(2, 2k+1)$, $k \geqslant 2$ and the following elements

$$(2,10),$$
$$(3,5),\ (3,7),\ (3,10),\ (3,11),\ (3,13),\ (3,16),\ (3,28),$$
$$(5,8),\ (5,9),$$
$$(7,10),\ (7,12),\ (7,15).$$

For $(p,n) \in E$, such a rational number does not exist.

Exceptions, for example

$$S_3(28) \cap (1,+\infty) = \{\tfrac{23}{5}, \tfrac{17}{11}\} = T_3(28),$$
$$S_7(15) \cap (1,+\infty) = \{\tfrac{13}{2}, \tfrac{11}{4}\} = T_7(15).$$

We prove the theorem by considering the following six cases:

1. $p = 2$,
2. $p > 3$, n odd,
3. $p > 3$, n even,
4. $p > 3$, $n \in \{2p - 2, 2p - 1, 2p + 1\}$,
5. $p = 3$, n odd,
6. $p = 3$, n even.

Theorem 10

For an odd prime p, there exist positive numbers c_1 and c_2 depending only on p such that for all integers $n \geqslant c_1$, we have

$$\operatorname{card}\left(S_p(n) \setminus T_p(n)\right) > c_2 \frac{n}{\log n}. \tag{7}$$

For $p = 2$, the same statement holds if we further require that such n be even.
For $p > 3$, the claim remains true if instead of the inequality (7), we set

$$\operatorname{card}\Big(\left(S_p(n) \setminus T_p(n)\right) \cap (1, +\infty)\Big) > c_2 \frac{n}{\log n}. \tag{8}$$

Theorem 11

Let $\frac{u}{v} \notin \{-1, 1, 2, \ldots, p-1\}$ *be a rational number such that*

$$\gcd(u, v) = \gcd(uv, p) = 1 \quad \textit{and} \quad |u| < p, \ |v| < p.$$

Then the p*-adic continued fraction expansion of* $\frac{u}{v}$ *is infinite*

$$\frac{u}{v} = [b_0, p : b_1, p : b_2, \ldots, p : b_k, p : -1],$$

where we noted the first appearance of -1 *as a complete quotient. Moreover, for* $0 \leqslant i \leqslant k$*, we have* $k \leqslant i + b_i$ *and thus* $k \leqslant p - 1$*.*

The bound $k \leqslant p-1$ on the length of the preperiod is better than $k \leqslant 2p-5$ which can be obtained by inspecting the proof of Bundschuh's already mentioned result.

At first glance, it is worse than $k = \mathcal{O}\big((\log(p-1))^2\big)$ following from (P 2023, Theorem 1) but the implied constant there depends on p and making it explicit in this case, gives a worse bound on k.

However, numerical experiments suggest that a much smaller upper bound on the length of preperiod should hold for rationals in Theorem 11.

Sylvester's sequence

Recall that a variant of Sylvester's sequence (OEIS A000058 and A129871) is defined by $s_0 = 1$, $s_1 = 2$, and

$$s_n = s_{n-1}^2 - s_{n-1} + 1 \text{ for } n > 1.$$

Thus, the first few elements in the sequence are

$$s_0 = 1,\ s_1 = 2,\ s_2 = 3,\ s_3 = 7, s_4 = 43.$$

It is easily shown that $s_n = 1 + \prod_{k=0}^{n-1} s_k$ holds for $n \geqslant 1$.

Theorem 12

If, for some $n \geqslant 1$, we have $p = s_{n+1}$, then

$$\left[p-1, \left(p : \tfrac{p-1}{s_i}, p : p - s_i^2\right)_{i=1,\ldots,n-1}, p : \tfrac{p-1}{s_n}, p : p - s_n + 1, p : -1\right]$$

is the p-adic continued fraction expansion of $\frac{1}{p-1}$. Here, for continued fraction as before,

$$\begin{aligned} a_i &= 1 && \text{for} \quad 1 \leqslant i \leqslant 2n+1, \\ b_{2i-1} &= \frac{p-1}{s_i} && \text{for} \quad 1 \leqslant i \leqslant n, \\ b_{2i} &= p - s_i^2 && \text{for} \quad 0 \leqslant i \leqslant n-1, \\ b_{2n} &= p - s_n + 1. \end{aligned} \tag{9}$$

The largest currently known prime in the Sylvester sequence is $s_6 = 3263443$ for which we obtain

$$\frac{1}{p-1} = \Big[p-1, p : \tfrac{p-1}{2}, p : p-4, p : \tfrac{p-1}{3}, p : p-9,$$
$$p : \tfrac{p-1}{7}, p : p-49, p : \tfrac{p-1}{43}, p : p-1849,$$
$$p : \tfrac{p-1}{1807}, p : p-1806, p : -1\Big].$$

It is conjectured that there are no larger primes in this sequence, but already the status of s_{33} is currently unknown.