Tetragonal intermediate modular curves

Second part is joint work with Maarten Derickx

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Modular curves

Modular curves $X(\Gamma)$ are a type of algebraic curves which can be constructed as quotients of the upper half plane $\mathcal{H}^*$ with $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. Examples of modular curves are $X(N)$, $X_0(N)$, $X_1(N)$ which correspond to congruence subgroups

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{N},\ b, c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{N},\ c \equiv 0 \pmod{N} \right\}.$$

Noncuspidal points on modular curves represent isomorphism classes of elliptic curves with some additional property.

For every group $\Delta \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$, there exists a modular curve $X_\Delta(N)$ defined over $\mathbb{Q}$. It corresponds to the congruence subgroup

$$\Gamma_\Delta(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : (a \bmod N) \in \Delta,\ c \equiv 0 \pmod{N} \right\}.$$

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Since $-I$ acts trivially on the upper half plane $\mathcal{H}^*$, the curves $X_\Delta(N)$ and $X_{\pm\Delta}(N)$ are isomorphic. Therefore, we can always assume that $-1 \in \Delta$. The intermediate modular curve $X_\Delta(N)$ lies between $X_1(N)$ and $X_0(N)$.

$$\begin{aligned} \Delta = (\mathbb{Z}/N\mathbb{Z})^\times &\implies X_\Delta(N) = X_0(N) \\ \Delta = \{\pm 1\} &\implies X_\Delta(N) = X_{\pm 1}(N) \cong X_1(N) \end{aligned}$$

if $\{\pm 1\} \subseteq \Delta_1 \subseteq \Delta_2 \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$, then we have natural projections

$$X_1(N) \to X_{\Delta_1}(N) \to X_{\Delta_2}(N) \to X_0(N).$$

Definition

Let C/k be a curve. The k-gonality of C $\mathrm{gon}_k C$ is the minimal degree of a nonconstant morphism $g : C \to \mathbb{P}^1$ defined over k.

$$
\begin{aligned}
X_0(N): \ & \mathrm{gon}_{\mathbb{C}} = 2 \text{ (Ogg - 1974)} \\
& \mathrm{gon}_{\mathbb{C}} = 3, \mathrm{gon}_{\mathbb{Q}} = 3 \text{ (Hasegawa, Shimura - 1999)} \\
& \mathrm{gon}_{\mathbb{C}} = 4 \text{ (Jeon, Park - 2005)} \\
& \mathrm{gon}_{\mathbb{Q}} = 4, 5, 6, \ N \leq 144 \text{ (Najman, O. - 2023)} \\
X_1(N): \ & \mathrm{gon}_{\mathbb{C}} = 2 \text{ (Kenku, Momose - 1988)} \\
& \mathrm{gon}_{\mathbb{C}} = 3, \mathrm{gon}_{\mathbb{Q}} = 3 \text{ (Jeon, Kim, Schweizer - 2004)} \\
& \mathrm{gon}_{\mathbb{C}} = 4, \mathrm{gon}_{\mathbb{Q}} = 4 \text{ (Jeon, Kim, Park - 2006)} \\
& \mathrm{gon}_{\mathbb{Q}} = 5, 6, 7, 8, \ N \leq 40 \text{ (Derickx, van Hoeij - 2014)} \\
X_\Delta(N): \ & \mathrm{gon}_{\mathbb{C}} = 2 \text{ (Ishii, Momose - 1991, with an error)} \\
& \mathrm{gon}_{\mathbb{C}} = 3 \text{ (Jeon, Kim - 2007, fixed this error)} \\
& \mathrm{gon}_{\mathbb{Q}} = 3 \text{ (Derickx, Najman - 2024)}
\end{aligned}
$$

Theorem (O.)

Determined all curves $X_\Delta(N)$ *with* $\mathrm{gon}_{\mathbb{Q}} = 4, 5$ *and all curves* $X_\Delta(N)$ *with* $\mathrm{gon}_{\mathbb{C}} = 4$.

Theorem (O.)

Determined $\mathrm{gon}_{\mathbb{Q}}(X_\Delta(N))$ *for all* $N \leq 40$ *and* $\{\pm 1\} \subsetneq \Delta \subsetneq (\mathbb{Z}/N\mathbb{Z})^\times$.

Canonical models

Let C/k be a curve of genus g. Take a canonical divisor K. We know that $\ell(K) = g$. Choose a basis $f_1, \ldots, f_g$ of $L(K)$. The canonical map of C is

$$(f_1, \ldots, f_g) : C \to \mathbb{P}^{g-1}.$$

For non-hyperelliptic curves of genus $g \geq 3$, this map is an embedding, and its image is called the canonical model of C.

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$\mathcal{L}_i = \{P \in \mathbb{Q}[x_1, \ldots, x_g] \text{ homogenous}, \deg P = i, P(f_1, \ldots, f_g) = 0\}$

$\implies$ $\mathbb{Q}$-vector space, $\dim \mathcal{L}_2 = \dfrac{(g-2)(g-3)}{2}$

If $g = 3$, then $\dim \mathcal{L}_2 = \dim \mathcal{L}_3 = 0, \dim \mathcal{L}_4 = 1$.

If $g > 3$, then the basis of $\mathcal{L}_2 \bigoplus \mathcal{L}_3$ provides a canonical model.

For a curve C of genus $g \geq 4$ that is neither hyperelliptic, trigonal, nor a smooth plane quintic, its canonical model consists of $\frac{(g-2)(g-3)}{2}$ quadrics.

We are only interested in curves of genus $g \geq 5$ (none of them are trigonal) and there are no curves $X_\Delta(N)$ that are smooth plane quintics.

MD Sage function vanishing_quadratic_forms() finds these quadrics.

Due to the natural projections

$$X_1(N) \to X_{\Delta_1}(N) \to X_{\Delta_2}(N) \to X_0(N),$$

we conclude that

$$\begin{aligned}
\mathrm{gon}_{\mathbb{Q}}(X_1(N)) \geq \mathrm{gon}_{\mathbb{Q}}(X_{\Delta_1}(N)) \geq \mathrm{gon}_{\mathbb{Q}}(X_{\Delta_2}(N)) \geq \mathrm{gon}_{\mathbb{Q}}(X_0(N)),\\
\mathrm{gon}_{\mathbb{C}}(X_1(N)) \geq \mathrm{gon}_{\mathbb{C}}(X_{\Delta_1}(N)) \geq \mathrm{gon}_{\mathbb{C}}(X_{\Delta_2}(N)) \geq \mathrm{gon}_{\mathbb{C}}(X_0(N)).
\end{aligned}$$

Theorem

The curve $X_0(N)$ has $\mathbb{C}$-gonality at most 4 if and only if

$$\begin{aligned}
N \in \{&1-75, 77-81, 83, 85, 87-89, 91, 92, 94-96, 98, 100, 101,\\
&103, 104, 107, 109, 111, 119, 121, 125, 131, 142, 143, 167, 191\}.
\end{aligned}$$

Theorem

The curve $X_0(N)$ has $\mathbb{Q}$-gonality equal to 5 if and only if $N = 109$.

We can also eliminate those levels N for which the $\mathbb{Q}$-gonality of the curve $X_1(N)$ is at most 3, namely

$$N \in \{1 - 16, 18, 20\}.$$

Therefore, there are only finitely many intermediate modular curves $X_\Delta(N)$ we need to deal with. Moreover, from the group structure of the group $(\mathbb{Z}/N\mathbb{Z})^\times$, we can easily see that for

$$N \in \{22, 23, 46, 47, 59, 83, 94, 107, 167\},$$

there are actually no intermediate modular curves $X_\Delta(N)$ because in these cases $(\mathbb{Z}/N\mathbb{Z})^\times \cong \mathbb{Z}/2p\mathbb{Z}$ for some prime p.

Methods used

Proposition

Let X be a curve of genus g over a field k.

i. *If $g \geq 2$, then $\mathrm{gon}_k(X) \leq 2g - 2$.*
ii. *If $g \geq 2$ and $X(k) \neq \emptyset$, then $\mathrm{gon}_k(X) \leq g$.*
iii. *If k is algebraically closed, then $\mathrm{gon}_k(X) \leq \frac{g+3}{2}$.*
iv. *If $\pi : X \to Y$ is a dominant k-rational map, then $\mathrm{gon}_k(Y) \leq \mathrm{gon}_k(X) \leq \deg \pi \cdot \mathrm{gon}_k(Y)$.*

- $\mathbb{F}_p$-gonality - $\mathrm{gon}_{\mathbb{Q}} \geq \mathrm{gon}_{\mathbb{F}_p}$, Magma computations (counting $\mathbb{F}_{p^2}$ points, Riemann-Roch dimensions)
- Castelnuovo-Severi inequality
- Abramovich bound - $[\mathrm{SL}_2(\mathbb{Z}) : \pm\Gamma] \leq \lambda \cdot \mathrm{gon}_{\mathbb{C}}(X(\Gamma))$, Kim and Sarnak improved the constant to $\lambda = \frac{12000}{119}$
- Betti numbers $\beta_{i,j}$

Degree d points on $X_\Delta(N)$

Theorem (Faltings' Theorem)

Let K be a number field and let C be a non-singular curve defined over K of genus $g \geq 2$. Then the set $C(K)$ is finite.

Theorem (Harris, Silverman)

Let K be a number field, and let C/K be a curve of genus $g \geq 2$. Assume that C is neither hyperelliptic nor bielliptic. Then the set of quadratic points on C/K is finite.

Theorem (Frey)

Let C be a curve defined over a number field K. If C has infinitely many points of degree $\leq d$ over K, then $\mathrm{gon}_K C \leq 2d$.

$X_1(M,N)$: $d = 1$ (Mazur - 1977)

$d = 2$ (Kamienny, Kenku, Momose - 1992)

$d = 3$ (Jeon, Kim, Schweizer - 2004)

$d = 4$ (Jeon, Kim, Park - 2006)

$d = 5, 6$ (Derickx, Sutherland - 2017)

$d = 7, 8$ for $X_1(N)$ (Derickx, van Hoeij - 2014)

$X_0(N)$: $d = 1$ (Mazur, Kenku - 1982)

$d = 2$ (Bars - 1999)

$d = 3$ (Jeon - 2021)

$d = 4$ (Hwang, Jeon - 2023, Derickx, O. - 2024)

$d = 5$ partially (Derickx, Hwang, Jeon, O. - 2025)

$X_\Delta(N)$: $d = 2$ (Jeon, Kim, Schweizer - 2020)

$d = 3$ (Dalal - 2023)

$d = 4$ (Derickx, O. - 2025)

Definition

Let C be a curve defined over a number field K. The *density degree set* $\delta(C/K)$ is the set of integers d for which the collection of closed points of degree d on C is infinite.

Theorem (Kadets, Vogt)

Suppose C/K *is a curve of genus* g *such that* $\min(\delta(C/K)) = d$. *Let* $m := \lceil d/2 \rceil - 1$ *and* $\epsilon := 3d - 1 - 6m < 6$. *If* $g > \max\left(\frac{d(d-1)}{2} + 1, 3m(m-1) + m\epsilon\right)$, *then there exists a morphism of curves* $\phi : C \to Y$ *of degree at least* 2 *such that* $d = \min(\delta(Y/K)) \cdot \deg\phi$.

Corollary

Suppose $C/\mathbb{Q}$ *is a curve of genus* $g \geq 8$ *such that* $\min(\delta(C/\mathbb{Q})) = 4$. *Then there exists a morphism of degree* 4 *from* C *to* $\mathbb{P}^1$ *or an elliptic curve* $E/\mathbb{Q}$ *with positive* $\mathbb{Q}$*-rank.*

We have two cases regarding the conductor of E: Cond $E = N$ and Cond E $< N$. We first consider the case Cond $E = N$.

Proposition (Derickx, O.)

Let $E/\mathbb{Q}$ be a strong Weil curve of conductor N and let us take $\{\pm 1\} \subseteq \Delta \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$. Suppose that d is the minimal degree of a parametrization $f : X_0(N) \to E$. If $E'/\mathbb{Q}$ is a strong Weil curve for $X_\Delta(N)$ in the isogeny class of E, then the minimal degree of a parametrization $f' : X_\Delta(N) \to E'$ equals

$$\frac{d \cdot \#((\mathbb{Z}/N\mathbb{Z})^\times/\Delta)}{\#(E \cap \Sigma(\Delta))}.$$

Here $E \cap \Sigma(\Delta)$ is the kernel of the map

$$E \to J_0(N) \to J_\Delta(N)$$

induced by the morphism $X_\Delta(N) \to X_0(N) \xrightarrow{f} E$.

Proposition (Derickx, O.)

Let $E/\mathbb{Q}$ be a strong Weil curve of conductor N and odd analytic rank. Then the kernel of the composition map

$$E \to J_0(N) \to J_\Delta(N)$$

is contained in $E[2]$ for all groups $\{\pm 1\} \subseteq \Delta \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$. Furthermore, it is trivial for $N \leq 800$.

We also computed the Shimura subgroups $\Sigma(N) \cap E$ of all elliptic curves of even analytic rank and conductor $N \leq 800$. They were always cyclic and of size ≤ 3 for $N \geq 18$.
So, it makes sense to ask what is the possible size of $\Sigma(N) \cap E$ and is it bounded by 3 when $N \geq 18$. We can show that $\#(\Sigma(N) \cap E) \leq 16$.

In the case Cond $E = M < N$ we have the following diagram. Here $E/\mathbb{Q}$ is a strong Weil curve for $X_0(M)$ and $\phi : E \to E'$ is rational isogeny of minimal degree.

$$X_\Delta(N) \xrightarrow[\text{projection map}]{u_\Delta} X_0(N) \xrightarrow[\text{degeneracy map}]{\iota_{d,N,M}} X_0(M) \xrightarrow{f} E \xrightarrow{\psi} E'. \quad (1)$$

Proposition (Derickx, O.)

Suppose that we have a setup as in (1) and that the map $E^n \to J_\Delta(N)$ has trivial kernel. Then the quadratic form

$$\deg \psi \cdot \#((\mathbb{Z}/N\mathbb{Z})^\times/\Delta) \cdot (\text{quadratic form for } X_0(N) \to E)$$

represents the degrees of all rational morphisms $h : X_\Delta(N) \to E'$.

Thank you for your attention.