Twists arising from torsion points

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Twists of a curve

Definition

Let C/K be a smooth projective curve. A *twist* of C/K is a smooth curve C'/K that is isomorphic to C over $\overline{K}$. We treat two twists as equivalent if they are isomorphic over K. The set of twists of C/K, modulo K-isomorphism, is denoted by $\mathrm{Twist}(C/K)$.

Theorem

There is a one-to-one correspondence between elements in $\mathrm{Twist}(C/K)$ *and the elements of the cohomology set* $H^1\left(\mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K}), \mathrm{Isom}(C)\right)$ *where* $\mathrm{Isom}(C)$ *is the group of all* $\overline{K}$*-isomorphisms from* C *to itself).*

Motivation

Çiperiani and Ozman in [cO15] have studied the following twist of an elliptic curve E/K. For any quadratic extension $F = K(\sqrt{d})$ and any point $S \in E(K)$ they considered the cocycle $\xi_{F,S} \in H^1\left(\mathrm{Gal}(\overline{K}/K), \mathrm{Aut}(E)\right)$ defined by

$$\xi_{F,S}(\sigma) = \begin{cases} (-1, S), & \text{if } \sigma(\sqrt{d}) = -\sqrt{d}, \\ (1, O), & \text{otherwise} \end{cases}$$

where $(\pm 1, S)$ sends a point $X \in E$ to the point $\pm X + S$.

Motivation

In their work, Çiperiani and Ozman manage to find necessary and sufficient conditions for a point P of an elliptic curve to lie in the image of the global trace map. After they applied this conditions to determine whether the previous twist has a rational points or not.

Motivated by their research, we asked ourselves what can we say about the twist obtained by changing $(-1, S)$ to $(1, S)$?

Problem

Let p be a prime number, $K = \mathbb{Q}(\zeta_p)$ be the p-th cyclotomic field, d a p power-free integer and $L = K(\sqrt[p]{d})$.
Let E/K be an elliptic curve with full p-torsion and $S, T \in E(K)[p]$ the generators for $E(K)[p]$.
We define a cocycle $\xi_d : \mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K}) \to E$ with

$$\xi_d(\sigma) = \begin{cases} O, & \text{if } \sigma(\sqrt[p]{d}) = \sqrt[p]{d}, \\ kS, & \text{if } \sigma(\sqrt[p]{d}) = \zeta_p^k \sqrt[p]{d}. \end{cases}$$

Denote by H_S^d the twist of E corresponding to the cocycle ξ_d.

Problem

We are interested in the following **questions**:

- Can we give a model for the twist H^d_S ?
- What can we say about the locally solvability of H^d_S?

Case $p = 2$ - setting

We have the following setting:

- $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$
- $T \in E(\mathbb{Q})[2]$ a rational 2-torsion point
- d a square-free integer
- H_T^d the twist of E that corresponds to the cocycle

$$\zeta(\sigma) = \begin{cases} O, & \text{if } \sigma(\sqrt{d}) = \sqrt{d}, \\ T, & \text{if } \sigma(\sqrt{d}) = -\sqrt{d}. \end{cases}$$

Case $p = 2$ - model

By changing the coordinates we may assume that $T = (0, 0)$ and $E : y^2 = x^3 + ax^2 + bx$.
In this case the model for the twist H_T^d is known and can be found in [Sil09] (Chapter X.3, Remark 3.7)

$$H_T^d : dy^2 = (a^2 - 4b)x^4 - 2adx^2 + d^2.$$

Case $p = 2$ - ELS

Surprisingly the local solvability of the twist H^d_T was not studied before. By playing a bit around we got the next result.

Proposition

Let $E/\mathbb{Q}$ be an elliptic curve and $T \in E(\mathbb{Q})[2]$ a rational 2-torsion point on E. For a square-free integer d, let H^d_T be a twist of E defined as above. If the curve H^d_T is ELS, then all prime factors of d divide $a^2 - 4b$. Specially, H^d_T is ELS only for finitely many square-free integers d.

Case $p = 2$ - ELS

Proof.

Let d be a square-free integer and p a prime factor of d such that $p \nmid (a^2 - 4b)$. Suppose that H_T^d is ELS and let $(x, y) \in E(\mathbb{Q}_p)$.
Let us write $x = p^\alpha x_1$ and $y = p^\beta y_1$ for some integers α, β and x_1, $y_1 \in \mathbb{Z}_p$.
Now we have that

$$v_p(dy^2) = v_p(dp^{2\beta} y_1) = 2\beta + 1,$$

$$v_p((a^2 - 4b)x^4 - 2adx^2 + d^2) = v_p((a^2 - 4b)p^{4\alpha} x_1^4 - 2adp^{2\alpha} x_1^2 + d^2)$$

$$= \begin{cases} 2, & \text{if } \alpha \geq 1, \\ 0, & \text{if } \alpha = 0, \\ 4\alpha, & \text{if } \alpha < 0. \end{cases}$$

Since $v_p(dy^2)$ is always odd and $v_p((a^2 - 4b)x^4 - 2adx^2 + d^2)$ is always even we get a contradiction. Thus, H_T^d is not ELS for such d. □

Case $p = 3$ - setting

We have the following setting:

- $K = \mathbb{Q}(\zeta_3)$ is the 3rd cyclotomic field
- d a cube-free integer and $L = K(\sqrt[3]{d})$
- E/K an elliptic curve over K with full 3-torsion
- $S, T \in E(K)[3]$ the generators for $E(K)[3]$
- H_S^d the twist of E corresponding to the cocycle

$$\xi_d(\sigma) = \begin{cases} O, & \text{if } \sigma(\sqrt[3]{d}) = \sqrt[3]{d}, \\ S, & \text{if } \sigma(\sqrt[3]{d}) = \zeta_3\sqrt[3]{d}, \\ 2S, & \text{if } \sigma(\sqrt{d}) = \zeta_3^2\sqrt[3]{d}. \end{cases}$$

Case $p = 3$ - model

By changing the coordinates and using Sutherland's equation for $X_1(3,3)$ we may assume that

$$E : y^2 + ((2+\zeta_3)a + 1 - \zeta_3)xy + ((1+\zeta_3)a^2 - \zeta_3 a)y = x^3,$$

$S = (0,0)$ and $T = (-a, a)$ for some $a \in K$.
In order to obtain an equation for the twist H_S^d, we define functions $z, w \in K(H_S^d)$ as functions with the following divisors:

$$\operatorname{div}(z) = (T) + (T+S) + (T+2S) - (O) - (S) - (2S),$$
$$\operatorname{div}(w) = (2T) + (2T+S) + (2T+2S) - (O) - (S) - (2S).$$

Case $p = 3$ - model

After some calculation we obtain explicit formulas (that are too long to fit onto the slide) for the functions z and w via x and y and that the following holds.

Proposition

With the above notation we have that functions z and w generate the function filed $K(H_S^d)$ and satisfy the equation

$$H_S^d \ : \ dz^3 + 3d\alpha zw + d^2w^3 + \beta = 0$$

where $\alpha = a^3 + 2\zeta_3^2 a^2 + 2\zeta_3 a + 1$ *and*
$\beta = -a^9 + 3(3+\zeta_3)a^8 - 3(\zeta_3+11)a^7 + (63\zeta_3+64)a^6 - 3(35\zeta_3+23)a^5 + 3(35\zeta_3+12)a^4 + (1-63\zeta_3)a^3 + 3(7\zeta_3-4)a^2 + 3(2-\zeta_3)a - 1.$

Case $p = 3$ - ELS

Using the previous model for the twist H_S^d we get a similar result as in case $p = 2$.

Proposition

With the above notation the following holds. If the curve H_S^d is ELS, then for all prime factors p of d we have that $p^2 \mid N(\beta)$ where N is the norm in the filed K. Specially, H_S^d is ELS for only finitely many cube-free integers d.

General case - setting

The setting is a direct generalisation of the case $p = 3$:

- $K = \mathbb{Q}(\zeta_p)$ is the p-th cyclotomic field
- d a p power-free integer and $L = K(\sqrt[p]{d})$
- E/K an elliptic curve over K with full p-torsion
- $S, T \in E(K)[p]$ the generators for $E(K)[p]$
- H_S^d the twist of E corresponding to the cocycle

$$\xi_d(\sigma) = \begin{cases} O, & \text{if } \sigma(\sqrt[p]{d}) = \sqrt[p]{d}, \\ kS, & \text{if } \sigma(\sqrt[p]{d}) = \zeta_p^k \sqrt[p]{d}. \end{cases}$$

General case - model

Motivated by the example from the previous case we define functions $z, w \in K(H_S^d)$ as functions with the following divisors:

$$\operatorname{div}(z) = (T) + \ldots + (T + (p-1)S) - (O) - (S) - \ldots - ((p-1)S),$$
$$\operatorname{div}(w) = (2T) + \ldots + (2T + (p-1)S) - (O) - (S) - \ldots - ((p-1)S).$$

As expected, this functions turn out to be good candidates for obtaining the model of H_S^d. After some computation (and various tricks) we manage to proof the following.

Proposition

With the notation as above we have that the set $\{z^p, z^{p-2}w, \ldots, zw^{\frac{p-1}{2}}, w^p, 1\}$ *of functions in* $K(H_S^d)$ *is* K*-linearly dependent, i.e. there exist constants* $\alpha_1, \alpha_2, \ldots, \alpha_{\frac{p+1}{2}}, \beta, \gamma \in K$ *not all zero such that*

$$\alpha_1 z^p + \alpha_2 z^{p-2} w + \ldots + \alpha_{\frac{p+1}{2}} z w^{\frac{p-1}{2}} + \beta w^p + \gamma = 0.$$

General case - model

The last relation connecting the functions z and w from the previous proposition gives us a model of a curve C. With some additional work and analysis we were able to verify that this curve C is indeed the twist corresponding to the cocycle ξ_d.

Theorem

The functions z, $w \in K(H_S^d)$ generate the function field $K(H_S^d)$ and give a model of the twist

$$H_S^d \ : \ \alpha_1 z^p + \alpha_2 z^{p-2} w + \ldots + \alpha_{\frac{p+1}{2}} z w^{\frac{p-1}{2}} + \beta w^p + \gamma = 0.$$

General case - ELS

In the general case we also expect that H_S^d will be ELS only for finitely many integers d. To show this, we will need the next two Lemmas. The first Lemma is a small modification of Lemma 2.1 from Çiperiani and Ozman's paper [cO15].

Lemma

Let $v \in M_K$ be a prime and ξ_d^v the restriction of the cocylce ξ_d on $\mathrm{Gal}(\overline{K_v}/K_v)$. With the notation as above we have that $H_S^d(K_v) \neq \emptyset$ if and only if the $\{\xi_d^v\} = 0$ in $H^1(K_v, E)$.

General case - ELS

The next Lemma is due to Cassels [Cas65] (Lemma 4.1).

Lemma (Cassels)

Let E be an elliptic curve over the number field K, $v \in M_K$ a prime, p a rational prime and δ the connecting homomorphisam in the Kummer sequence

$$0 \longrightarrow E(K_v)/pE[K_v] \xrightarrow{\delta} H^1(K_v, E[p]) \longrightarrow H^1(K_v, E)[p] \longrightarrow 0.$$

If v is a prime of good reduction and $v \nmid p \cdot \infty$ them

$$H^1_f(K_v, E[p]) := \operatorname{Im} \delta = H^1(K_v^{ur}/K_v, E[p]).$$

General case - ELS

By combining the previous two Lemmas we get the desired result.

Theorem

Let $v \in M_K$ be a prime of good reduction such that $v \mid d$ and $v \nmid p \cdot \infty$. Than $\{\xi_d^v\} \neq 0$, i.e. $H_S^d(K_v) = \emptyset$. Specially, H_S^d is ELS only for finitely many p power-free integers d.

Future work

The above study has left some interesting question that one can try to answer in a future project:

- Can we classify all the integers d for which the twist H_S^d is ELS in the case $p = 2$, $p = 3$ or even in the general case?
- Can we say something more about the model

$$H_S^d \ : \ \alpha_1 z^p + \alpha_2 z^{p-2} w + \ldots + \alpha_{\frac{p+1}{2}} z w^{\frac{p-1}{2}} + \beta w^p + \gamma = 0?$$

What are the singular points of the above model? Is there some connection of the coefficients $\alpha_1, \ldots, \alpha_{\frac{p+1}{2}}, \beta$ and γ and the starting elliptic curve E, or perhaps even a formula on how to compute them?

References


J. W. S. Cassels.
Arithmetic on curves of genus 1. VIII. On conjectures of Birch and Swinnerton-Dyer.
J. Reine Angew. Math., 217:180–199, 1965.


Mirela Çiperiani and Ekin Ozman.
Local to global trace questions and twists of genus one curves.
Proc. Amer. Math. Soc., 143(9):3815–3826, 2015.


Joseph H Silverman.
The arithmetic of elliptic curves, volume 106.
Springer, 2009.

Hvala na pažnji! :)