Extensions of $D(-1)$-pairs in imaginary quadratic fields

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(joint work with Ivan Soldo)

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Happy birthday to Ivica Gusić, Borka Jadrijević and Andrej Dujella!

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$D(n)$-m-tuples



Definition 1.

A set of m non-zero elements $\{a_1, a_2, \ldots, a_m\}$ of commutative ring R such that $a_i a_j + n$ is a perfect square in R for all $1 \leq i < j \leq m$ is called a Diophantine m-tuple with the property $D(n)$ (or just a $D(n)$-m-tuple).

An overview of the most important historical and recent results and problems related to Diophantine m-tuples and their various generalizations:

A. Dujella, *Diophantine m-tuples and elliptic curves*, Springer Cham, Cham, 2024.

A. Dujella, *Diophantine m-tuples page*,
http://web.math.pmf.unizg.hr/∼duje/dtuples.html.

The most studied case is that one with $n = 1$.

- He, Togbé, and Ziegler (2019): There is no Diophantine $D(1)$ quintuple in integers.

$D(-1)$-m-tuples



- Bonciocat, Cipu, Mignotte (2022): There is no Diophantine $D(-1)$-quadruple in integers.
- Gupta (2021): For $m > 36$ there is no Diophantine $D(-1)$-m-tuple in the ring of integers of an imaginary quadratic number field.
- Cipu, Fujita (2024): For $m > 24$ there is no Diophantine $D(-1)$-m-tuple in the ring of integers of an imaginary quadratic number field.
- $\mathbb{Z}[\sqrt{-k}]$, with k positive integer: Dujella, Franušić, Soldo, Abu Muriefah, Al-Rashed, Bayad, Filipin, Togbe, A. Dossavi-Yovo, Fujita, Cipu

Y. Fujita, I. Soldo, 2022, 2023, 2025

- proved a general result on the nonexistence of a $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$, with integer $k \geq 2$ and integers a, b, c, d such that $a < b \leq 8a - 3$ and negative integers c and d
- applied the obtained results to the $D(-1)$-pair $\{p^i, q^j\}$ with an arbitrary different primes p, q and positive integers i, j
- consider the extendibility of $D(-1)$-pairs $\{a, p^j\}$ and $\{a, 2p^j\}$, where a, j are positive integers and p is an odd prime, to quadruples in the ring $\mathbb{Z}[\sqrt{-k}]$, with an integer $k \geq 2$



M. J. B., I. Soldo, *Extensions of D(-1)-pairs in some imaginary quadratic fields*, New York Journal of Mathematics **30** (2024), 745 – 755.

- consider the extendibility of $D(-1)$-pairs $\{a, b\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$, where a, b, k are positive integers
- prove that families of such $D(-1)$-pairs with $b = p^i q^j$, where p, q are different odd primes and i, j are positive integers cannot be extended to quadruples in certain rings $\mathbb{Z}[\sqrt{-k}]$, where k depends on p^i, q^i and a
- present the result on non-existence of $D(-1)$-quintuples of a specific form in certain imaginary quadratic rings



Theorem 1.

Let k be a positive integer and $\{a, b\}$ a $D(-1)$-pair in the ring $\mathbb{Z}[\sqrt{-k}]$, with positive integers a, b. If c is an integer and $k \nmid b - a$, then there does not exists a $D(-1)$-triple $\{a, b, c\}$ in $\mathbb{Z}[\sqrt{-k}]$ with a negative integer c. In that case, there does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$ with an integer d.

The case $c < 0$ is possible if and only if $k \mid b - a$ and the equation

$$bx^2 - ay^2 = \frac{b-a}{k} \tag{1}$$

is solvable in integers.



Corollary 1

Let k be a positive integer and $\{a, b, c\}$ a $D(-1)$-triple in the ring $\mathbb{Z}[\sqrt{-k}]$ with positive integers a, b, c.

1) If $k \nmid b - a$ or $k \nmid b - c$ or $k \nmid c - a$, then there does not exist a $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$ with an integer d.
2) If $c = a + b \pm 2r$, where $ab - 1 = r^2$, and $k \mid b - a$, $k \mid b - c$ and $k \mid c - a$, then the existence of a $D(-1)$-quadruple $\{a, b, c, d\}$ in $\mathbb{Z}[\sqrt{-k}]$ with an integer d is possible if the condition $k|4 - 3a^2$ holds.



Theorem 2.

Suppose that p, q are different odd primes, i, j, a, k positive integers such that $k \geq \max\{p^i, q^j\}$, $\gcd(a, pq) = 1$ and $a < p^i q^j$. If $\{a, p^i q^j, c\}$ is a $D(-1)$-triple in the ring $\mathbb{Z}[\sqrt{-k}]$, then c is an integer.

Skech of the proof:

If the set $\{a, p^i q^j, c\}$ be such a $D(-1)$-triple in $\mathbb{Z}[\sqrt{-k}]$, then there exist integers c_1, d_1, x, y, u, v such that $c = c_1 + d_1\sqrt{-k}$ and

$$\begin{aligned} ac - 1 &= (x + y\sqrt{-k})^2, \\ p^i q^j c - 1 &= (u + v\sqrt{-k})^2. \end{aligned} \tag{2}$$



It follows that we have to consider the system of equations

$$\begin{aligned} p^i q^j xy &= auv, \\ p^i q^j (x^2 - ky^2 + 1) &= a(u^2 - kv^2 + 1). \end{aligned} \tag{3}$$

We have shown that this system of equations has no solutions in integers in the case of $xy \neq 0$, which implies that c is an integer.



Example 2

Let $\{2, b, c\}$ is $D(-1)$-triple in the ring $\mathbb{Z}[\sqrt{-k}]$, with $b = p^i q^j$, p, q are different odd primes, $i, j \in \mathbb{N}, k \geq \max\{p^i, q^j\}$ and $c = 2 + b \pm 2r$, where $2b - 1 = r^2$. Then there does not exists the $D(-1)$-quadruple $\{2, b, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$.



Theorem 3. (J. B., Soldo)

Let p, q be different odd primes, a, i, j, k positive integers such that $\gcd(a, pq) = 1$, $a < p^i q^j$ and $ap^i q^j$ is not a square. Further, let $k \geq \max\{p^i, q^j\}$ and $m_k = \dfrac{p^i q^j - a}{k}$. Suppose that either of the following holds:

1) $m_k = 1$ and $a \neq 1$;
2) $m_k = 2$ with $a \neq 1, 5$ and $\gcd(a, 2) = 1$;
3) m_k is a prime and $r \geq am_k$ or $r \geq 2a^2$ with a square-free a, where $ap^i q^j - 1 = r^2$.

If $\{a, p^i q^j, c\}$ is a $D(-1)$-triple in $\mathbb{Z}[\sqrt{-k}]$, then c is a positive integer.



Skech of the proof:

We consider a $D(-1)$-triple $\{a, p^i q^j, c\}$ in $\mathbb{Z}[\sqrt{-k}]$ with $c \in \mathbb{Z}$.
According to Theorem 1, $c < 0$ holds only if $m_k \in \mathbb{Z}$ and

$$p^i q^j x^2 - a y^2 = m_k, \tag{4}$$

is solvable in integers.

1) $m_k = 1$ and $a \neq 1$

Lemma 1. (Grelak, Grytczuk)

Let $U > 1$, V be positive integers such that $\gcd(U, V) = 1$ and $D = UV$ is not a square of a positive integer. Moreover, let (u_0, w_0) denote the least positive integer solution of Pell's equation $u^2 - Dw^2 = 1$. Then equation $Ux^2 - Vy^2 = 1$ has a solution in positive integers x, y if and only if

$$2U | u_0 + 1 \quad \text{and} \quad 2V | u_0 - 1.$$

2) $m_k = 2$ with $a \neq 1, 5$

Lemma 2. (Grelak, Grytczuk)

Let U, V be positive integers such that $\gcd(U, V) = \gcd(U, 2) = \gcd(V, 2) = 1$ and $D = UV$ is not a square of a positive integer. Let (u_0, w_0) denote the least positive integer solution of Pell's equation $u^2 - Dw^2 = 1$. Then equation $Ux^2 - Vy^2 = 2$ has a solution in positive integers x, y if and only if

$$U | u_0 + 1 \quad \text{and} \quad V | w_0 - 1.$$



3 i) $r \geq am_k$

Lemma 3. (Fujita)

Let N and K be integers with $1 < |N| \leq K$. Then the Pellian equation

$$x^2 - (K^2 + 1)y^2 = N$$

has no primitive integer solution (x, y).

We transform the equation (4) to

$$y_1^2 - ap^i q^j x^2 = -am_k, \tag{5}$$

and have shown that this equation has no primitive solution.

3 ii) $r \geq 2a^2$



Lemma 4.

Let d be a positive non-square integer and a, s positive integers. If there exists a primitive solution (x, y) of the equation

$$ax^2 - dy^2 = \delta s, \quad \delta = \pm 1, \tag{6}$$

then there exists an integer l satisfying $l^2 \equiv ad \pmod{s}$ and $0 \leq l \leq \frac{s}{2}$ such that the equation

$$av^2 - du^2 = \frac{l^2 - ad}{\delta s} \tag{7}$$

has s a primitive solution.



We have shown that the equation (4) has a primitive solution if the equation

$$p^i q^j x^2 - a y^2 = \frac{a^2 - a p^i q^j}{m_k} = -ak \tag{8}$$

has a primitive solution.

This equation leads to the equation of the form

$$y^2 - a p^i q^j x_0^2 = k \tag{9}$$

which has no primitive solution.

Theorem 4

Suppose that p, q are different odd primes, a, i, j, k positive integers such that $\gcd(a, pq) = 1$, $a < p^i q^j$ and $ap^i q^j$ is not a square. Moreover, let $k \geq \max\{p^i, q^j\}$ and $m_k = \dfrac{p^i q^j - a}{k}$. Let us further assume that either of the following holds:

1) $m_k = 1$ and $a \neq 1$;
2) $m_k = 2$ with $a \neq 1, 5$ and $\gcd(a, 2) = 1$;
3) m_k is a prime and $r \geq am_k$ or $r \geq 2a^2$ with a square-free a, where $ap^i q^j - 1 = r^2$;
4) m_k is not an integer.

Then, there does not exist a $D(-1)$-quadruple of the form $\{a, p^i q^j, c, d\}$ in the ring $\mathbb{Z}[\sqrt{-k}]$.

By assuming that the set $\{a, p^i q^j, c, d\}$ is a $D(-1)$-quadruple in $\mathbb{Z}[\sqrt{-k}]$, we know that $c, d \in \mathbb{N}$. The statement follows from Theorem 1.

Examples of $D(-1)-$quadruples



Proposition 1 (Dujella, J. B., Soldo)

Let m, n be a positive integers and $b = n^2 + 1$. If $m|n$ and $k = m^2$, then there exist infinitely many $D(-1)$-quadruples of the form $\{1, b, -c, d\}$, $c, d > 0$ in $\mathbb{Z}[\sqrt{-k}]$.

About $D(-1)-$quintuples...



Theorem 3. (J. B., Soldo)

Let m, n, k be a positive integers and $b = n^2 + 1$. Suppose that either of the following holds:

1) $k = m^2$;
2) $k \nmid n^2$.

There does not exists a $D(-1)$-quintuple of the form $\{1, b, -c, d, e\}$ with integers $c, d > 0, e$ in $\mathbb{Z}[\sqrt{-k}]$.



Thank you for your attention!