Extensions of the Triangular $D(3)$-pair $\{3, 6\}$

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Representation Theory XIX, Dubrovnik

June 25, 2025



This work was supported by the Croatian Science Foundation under the project number HRZZ IP-2022-10-5008.

Long history of the original problem

- A set consisting of m positive integers such that the product of any two of its distinct elements increased by 1 is a perfect square is called **a Diophantine m-tuple**.
- **Diophantus of Alexandria** first studied this problem.
  - The first rational Diophantine quadruple $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$.
- How large such set can be?

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  - ...with rational numbers, integers...
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  - Recently, there are some interesting variants.

Variant of the problem with the triangular numbers

- Triangular numbers - $\Delta_k = \frac{k(k+1)}{2}$, $k \in \mathbb{N}$
  - $0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136...$

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Definition 1

Let n be an integer. A set $\{a_1, \ldots, a_m\}$ of m distinct positive integers with $a_i a_j + n$ a triangular number for $1 \leq i < j \leq m$ is called a Triangular Diophantine m-tuple with the property $D(n)$ or simply a Triangular $D(n)$-m-tuple.

- The case $n = 0$ is not significantly different from the case with $n \neq 0$.
- The cardinality of such sets?

Recently solved problems

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- Jukić Bokun and Soldo [2024, prepint] - the only Triangular $D(-1)$-triples $\{1, 2, c\}$, with $c = 2^n p$, n a non-negative integer and p a prime, have $c \in \{11, 46, 352, 11936\}$. No triangular $D(-1)$-quadruple of the form $\{1, 2, c, d\}$ exists.

Our research

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  - ...some other specific cases...
  - ...k of special form, i.e. $k = 2l^2 \pm l$ where $4l \pm 1$ is prime...

Expectations?

- A simple computer search for finding Triangular $D(1)$-quadruples $\{a, b, c, d\}$ in the range $1 \leq a \leq 30$, $a < b \leq 30^2$, $b < c \leq 30^3$, and $c < d \leq 30^4$.

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    - The quadruples seem to have a sparser occurrence than the "classical" diophantine quadruples.
    - Because of the similarity of the methods, we expect a finite upper bound (probably 4) for the size of Triangular Diophantine sets.

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- We transform the problem of extending such a triple to solving a system of simultaneous Pellian equations.
- We search for intersections of the obtained binary recurrence sequences, using some congruences and applying Baker's theory on linear forms in logarithms of algebraic numbers.
- At the end of the proof, we use a variation of Baker-Davenport reduction.

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- We had to eliminate the first three elements of the sequence $(c_k)_{k\in\mathbb{N}}$ which could be element c in a Triangular $D(3)$-quadruple $\{3, 6, c, d\}$.
- There are more possibilities for fundamental solutions of the Pellian equations - to deal with them we had to use some different approaches.

Triangular $D(3)$-triples

- Assume that $\{3, 6, c\}$, $c \neq 0$, is a Triangular $D(3)$-triple where

$$\begin{aligned} 3c + 3 &= \frac{x(x+1)}{2}, \\ 6c + 3 &= \frac{y(y+1)}{2} \end{aligned}$$

hold for some positive integers x and y.

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  hold for some positive integers x and y.

- By taking $X_c = 2x + 1$ and $Y_c = 2y + 1$, we get

$$\begin{aligned} 24c + 25 &= X_c^2, & (1) \\ 48c + 25 &= Y_c^2 & (2) \end{aligned}$$

  and then the Pellian equation

$$Y_c^2 - 2X_c^2 = -25. \quad (3)$$

- All solutions (Y_k, X_k), $k \geq 0$, of (3) are given with

$$Y_k + X_k\sqrt{2} = (5 + 5\sqrt{2})(T_k + U_k\sqrt{2}), \quad k \geq 0,$$

where (T_k, U_k) are given by

$$\begin{aligned} T_0 &= 1, \quad T_1 = 3, \quad T_{k+2} = 6T_{k+1} - T_k, \quad k \geq 0, \\ U_0 &= 0, \quad U_1 = 2, \quad U_{k+2} = 6U_{k+1} - U_k, \quad k \geq 0. \end{aligned}$$

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- From (1), by $X_k = 5(U_k + T_k)$, we get

$$c = c_k = \frac{25((U_k + T_k)^2 - 1)}{24}. \tag{4}$$

  - The terms in (4) are $c_k = 0, 25, 875, 29750, 1010650...$, divisible by 25.
  - For $C = C_k = \frac{c_k}{25}$ we obtain $C_k = 0, 1, 35, 1190, 40426...$, which is the sequence of generalized pentagonal numbers which multiplied by 2 are again generalized pentagonal numbers.

System of Pellian equations

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- Then, for an integer $d > c$ equations

$$24d + 25 = X^2, \tag{5}$$
$$48d + 25 = Y^2, \tag{6}$$
$$8Cd + 1 = Z^2 \tag{7}$$

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- Eliminating d from equations (5)-(7), the system

$$3Z^2 - CX^2 = 3 - 25C, \tag{8}$$
$$6Z^2 - CY^2 = 6 - 25C \tag{9}$$

of Pellian equations arises.

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- By Lemma 1, we may suppose $c \geq 1010650$ ($C \geq 40426$).

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are $(T, 4)$ and $(S, 2)$ where, from (1) and (2),

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- If $X = 1$:
  - $3C + 1 = \square$, so $(3C + 1)(24C + 1)(48C + 1) = \tau_0^2$, for some suitable integer τ_0.
  - This leads to the elliptic equation for which we search for integral points in Magma.
  - We find that it is possible ony for small values of C. ...

Solving the system of Pellian equations

- Equations (8) and (9) have infinitely many solutions in natural numbers given by

$$\begin{aligned} Z\sqrt{3} + X\sqrt{C} &= (Z_0\sqrt{3} + X_0\sqrt{C})(T + 4\sqrt{3C})^m, \quad m \geq 0, \\ Z\sqrt{6} + Y\sqrt{C} &= (Z_1\sqrt{6} + Y_1\sqrt{C})(S + 2\sqrt{6C})^n, \quad n \geq 0, \end{aligned}$$

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- For any solution (X, Y, Z) of the system (8)-(9), we have

$$Z = V_m = W_n, \tag{15}$$

  for some non-negative integers m and n, where the sequences $(V_m)_{m\geq 0}$ and $(W_n)_{n\geq 0}$ are given by

$$\begin{aligned} V_0 &= Z_0, \quad V_1 = TZ_0 + 4CX_0, \quad V_{m+2} = 2TV_{m+1} - V_m, \\ W_0 &= Z_1, \quad W_1 = SZ_1 + 2CY_1, \quad W_{n+2} = 2SW_{n+1} - W_n. \end{aligned}$$

- We prove the estimates

$$\begin{aligned} 0 < \quad X_0^2 &\leq \frac{24(25C-3)}{T-1}, \\ Z_0^2 &\leq \frac{(T-1)(25C-3)}{6}, \\ 0 < \quad Y_1^2 &\leq \frac{12(25C-6)}{S-1}, \\ Z_1^2 &\leq \frac{(S-1)(25C-6)}{12}. \end{aligned}$$

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- Since $C \geq 40426$, we have

$$\begin{aligned} 0 < \quad X_0 &< 9.31\sqrt[4]{C}, \\ |Z_0| &< 0.38C, \\ 0 < \quad Y_1 &< 7.83\sqrt[4]{C}, \\ |Z_1| &< 0.23C. \end{aligned}$$

- We try to find solutions of the equation $Z = V_m = W_n$, which lead to the extension of the Triangular $D(3)$-triple $\{3, 6, c\}$ to a quadruple.

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- We use some congruences proved by using some binary recurrence relations and previously mentioned expressions:

$$\begin{aligned} V_{2m} &\equiv V_0 \pmod{8C}, \\ V_{2m+1} &\equiv V_1 \pmod{8C}, \\ W_{2n} &\equiv W_0 \pmod{4C}, \\ W_{2n+1} &\equiv W_1 \pmod{4C}, \\ Z_0^2 &\equiv 1 \pmod{8C}, \\ Z_1^2 &\equiv 1 \pmod{4C}, \\ X_0^2 &\equiv 1 \pmod{24}, \\ Y_1^2 &\equiv 1 \pmod{24}. \end{aligned}$$

Fundametal solutions of the Pellian equations

- Firstly, we want to find all fundamental solutions (Z_0, X_0) and (Z_1, Y_1) of the equations (8) and (9), respectively, which lead to the extension of a Triangular $D(3)$-triple $\{3, 6, c\}$ to a quadruple.

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  - We separate the cases depending on the parity of indices m and n in the equation $Z = V_m = W_n$.
  - By using previously mentioned congruences, estimates, algorithms, expressions and algebraic manipulations, we prove:

Lemma 2

The fundamental solutions of the equations (8) and (9), for which extensions of the Triangular $D(3)$-triple $\{3, 6, c\}$ are possible, are given by

- $V_{2m} = W_{2n}$, $Z_0 = Z_1 = \pm 1, X_0 = 5, Y_1 = 5$,
- $V_{2m+1} = W_{2n}$, $X_0 = 13, Y_1 = 5, Z_0 = \pm T, Z_1 = \pm 1$.

Linear forms in logarithms

- Using expressions for $Z\sqrt{3} + X\sqrt{C}$ and $Z\sqrt{6} + Y\sqrt{C}$, and the fact that $Z = V_m = W_n$, we get

$$\left(Z_0 + \frac{X_0\sqrt{3C}}{3}\right)(T + 4\sqrt{3C})^m + \left(Z_0 - \frac{X_0\sqrt{3C}}{3}\right)(T - 4\sqrt{3C})^m$$
$$= \left(Z_1 + \frac{Y_1\sqrt{6C}}{6}\right)(S + 2\sqrt{6C})^n + \left(Z_1 - \frac{Y_1\sqrt{6C}}{6}\right)(S - 2\sqrt{6C})^n.$$

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- By using suitable substitutions, Lemma 1... we prove:

Lemma 3

If $V_m = W_n$ holds for $m, n \neq 0$, then

$$\begin{aligned} 0 &< m\log(T + 4\sqrt{3C}) - n\log(S + 2\sqrt{6C}) + \log\frac{Z_0 + \frac{X_0\sqrt{3C}}{3}}{Z_1 + \frac{Y_1\sqrt{6C}}{6}} \\ &< 25.636693(T + 4\sqrt{3C})^{-2m}. \end{aligned}$$

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- Using Lemma 2 and Lemma 3, we prove:

Lemma 4

If $V_m = W_n$ holds for some $m, n \neq 0$, then $m < n < 1.31m$.

Lower bound on indices m and n

We use the congruences proved by induction based on reccurence relations for V_m and W_n:

$$\begin{aligned}
V_{2m} &\equiv (96m^2C + 1)Z_0 + 8mTCX_0 \pmod{384C^2},\\
V_{2m+1} &\equiv (96m(m+1)C + 1)TZ_0 + 4(2m+1)CX_0 \pmod{384C^2},\\
W_{2n} &\equiv (48n^2C + 1)Z_1 + 4nSCY_1 \pmod{384C^2},\\
W_{2n+1} &\equiv (48n(n+1)C + 1)SZ_1 + 2(2n+1)CY_1 \pmod{192C^2}.
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- Using Lemma 2 and these congruences, we prove:

Lemma 5

If $V_m = W_n$ and $mn \neq 0$, then $m > \sqrt{C}$.

Proof of Theorem 1

- We combine Lemma 4 with the following theorem.

Theorem 2 [Baker and Wüstholz]

Let $\Lambda \neq 0$ be linear form in logarithms of ℓ algebraic numbers $\alpha_1, ..., \alpha_\ell$ with rational integer coefficients $b_1, ..., b_\ell$. Then

$$\log \Lambda \geq -18(\ell+1)!\ell^{\ell+1}(32d)^{\ell+2}h'(\alpha_1)\cdots h'(\alpha_\ell)\log(2\ell d)\log B,$$

where $B = \max\{|\ b_1\ |, ..., |\ b_\ell\ |\}$, and d is the degree of the number field generated by $\alpha_1, ..., \alpha_\ell$. Here

$$h'(\alpha) = \max\left\{h(\alpha), \frac{|\log\alpha|}{d_\alpha}, \frac{1}{d_\alpha}\right\},$$

and $h(\alpha)$ denotes the standard Weil logarithmic height of α, where d_α is the degree of the algebraic number α.

- We choose

$$\alpha_1 = T + 4\sqrt{3C}, \quad \alpha_2 = S + 2\sqrt{6C}, \quad \alpha_3 = \frac{Z_0 + \frac{X_0\sqrt{3C}}{3}}{Z_1 + \frac{Y_1\sqrt{6C}}{6}},$$

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  $\ell = 3$, $B = n$, and $d = 4$ in order to apply Theorem 2 for the linear form which appears in Lemma 4.
- Using $C \geq 40426$ and Lemma 2, we compute:
  - $h'(\alpha_1) = \frac{\log(T+4\sqrt{3C})}{2} < 0.4 \log C$,
  - $h'(\alpha_2) = \frac{\log(S+2\sqrt{6C})}{2} < 0.39 \log C$,
  - $h'(\alpha_3) \leq \frac{1}{4} \log \left(36 \left(\frac{25C}{6} - 1 \right)^2 \cdot \frac{(T + \frac{13\sqrt{3C}}{3})^2}{\frac{25C}{6} - 1} \right) < \log C.$

- In Lemma 3, we have

$$\begin{aligned} \log(25.636693(T + 4\sqrt{3C})^{-2m}) &< \log(T + 4\sqrt{3C})^{-2m+1} \\ &< 0.74(-2m + 1)\log C. \end{aligned}$$

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- By Theorem 2 we obtain

$$-\log \Lambda \leq 18 \cdot 4! \cdot 3^4 \cdot 128^5 \cdot 0.4 \cdot 0.39 \cdot \log^3 C \cdot \log(24) \cdot \log n. \tag{17}$$

Final steps of the proof

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Proposition 1

Let $k > 29$ be an integer. Then the Triangular $D(3)$-triple $\{3, 6, c_k\}$ cannot be extended to a quadruple by a larger element.

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- We have a contradiction, because $m \geq \sqrt{C} \geq \sqrt{40426} > 201$ if $m, n \neq 0$.
- The cases where $m = 0$ or $n = 0$ are not possible, because then we would have $d = 0$ or d is not an integer.

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Thank you for your attention!