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$D(\pm 1)$-quintuples in imaginary quadratic fields

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Representation Theory XIX
Conference dedicated to
Andrej Dujella, Ivica Gusić and Borka Jadrijević
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Joint work in progress with Yasutsugu Fujita

Terminology

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For a commutative ring with unity R and $n \in R$ we call $D(n)$–tuple any subset of $R \setminus \{0\}$ with the property that the product of any two of its elements is a perfect square minus n.

One also speaks of $D(n)$–m-tuple when m is the cardinality of the subset.

When $m = 2$ (3, 4, 5, resp.), instead of $D(n)$–m-tuple we speak of n–pair (n-triple, n-quadruple, n-quintuple, resp.)

Object of study

$D(\pm 1)$-quintuples

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$D(\varepsilon)$-sets in the ring $R := \mathbb{Z}[\omega]$, with $\varepsilon := \pm 1$ and

$$\omega := \begin{cases} \sqrt{-D} & \text{if } -D \equiv 2, 3 \pmod 4, \\ \dfrac{1+\sqrt{-D}}{2} & \text{if } -D \equiv 1 \pmod 4. \end{cases}$$

for some square-free positive integer D

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for some square-free positive integer D

Note invariance of this property to two transformations: multiplication by -1 and taking complex conjugate of each element $u + v\omega \mapsto -u - v\omega$,

$$\overline{u + v\omega} \mapsto \begin{cases} u - v\omega & \text{if } -D \equiv 2, 3 \pmod 4, \\ u + v - v\omega & \text{if } -D \equiv 1 \pmod 4. \end{cases}$$

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Multiplication by $i = \sqrt{-1}$ transforms a $D(\varepsilon)$–m-set in $\mathbb{Z}[i]$ into a $D(-\varepsilon)$–m-set

Main problem

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How large a $D(n)$-set can be?

Short answer: no idea unless R is $\mathbb{Z}$ or one of few polynomial rings over $\mathbb{R}$ or $\mathbb{C}$.

Explicit construction of integer $D(1)$–sets

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Euler's idea: try to enlarge a given / known such set

Arkin–Hoggatt–Strauss, Gibbs 1979 $c_\pm = a + b \pm 2r$, if $s,\ t \in \mathbb{N}$ verify $ac + 1 = s^2$, $bc + 1 = t^2$, put

$$d_\pm = a + b + c + 2abc \pm 2rst$$

Then $\{a, b, c_\pm\}$ and $\{a, b, c, d_\pm\}$ are $D(1)$–sets in $\mathbb{Z}$ (provided $c_- d_- \neq 0$) called regular

Same formulæ valid for arbitrary R

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Same formulæ valid for arbitrary R

Indirect approach: solving systems of Pellian equations "works" for arbitrary n and m

Previous works

Bayad–Filipin–Togbé 2016 $\{k, 4k+4, 9k+6, d\}$ is a $D(1)$–quadruple in $\mathbb{Z}[\sqrt{-1}]$ for some $k \neq 0, -1$
$\Longrightarrow d = 144k^3 + 240k^2 + 124k + 20$

Previous works

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Dujella–Jukić Bokun–Soldo 2018 infinitely many $D(-1)$–quadruples of the form $(1, r^2+1, -c, d)$ in $\mathbb{Z}[\sqrt{-1}]$ with $r,\ c,\ d > 0$

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Adžaga–Filipin–Franušić 2022 $k \in \mathbb{Z}[\sqrt{-1}]$ with $|k| \geq 5 \cdot 10^{37}$ and $\operatorname{Re} k \neq 0 \Longrightarrow$ the $D(1)$–triple $(k-1, k+1, 16k^3 - 4k)$ can be extended only by $d = 4k$ or $d = 64k^5 - 48k^3 + 8k$

Bounds on the cardinality of $D(\varepsilon)$-sets

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C–Fujita 2024 there exists no $D(\varepsilon)$-m-tuple in R with $m > 24$

Difficulties

- no total ordering on R compatible with the ring operations; the absolute value is a pale substitute

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- no total ordering on R compatible with the ring operations; the absolute value is a pale substitute
- integers whose absolute values are irrational
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- high number of quadruples with entries of small absolute value

$D(-1)$-quadruples in Gaussian integers

$D(\pm 1)$-quintuples

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a	b	c	d	$\lvert d\rvert^2$
i	$-2i$	$2-i$	$-12+4i$	160
1	$1-2i$	$1+2i$	-15	225
$1-i$	$1+i$	4	-20	400
1	2	5	-24	576
i	$2-i$	$4+2i$	$12-36i$	1440
2	$-1-2i$	$-1+2i$	-40	1600
1	$1+2i$	$4+4i$	$28-36i$	2080
$2i$	$-2-2i$	$2-2i$	$60i$	3600
1	-3	$-2+4i$	$-32+56i$	4160
1	-3	5	65	4225

$D(-1)$-quadruples with $|a| \leq |b| \leq |c| \leq |d|$. Here, $i = \sqrt{-1}$.

$D(-1)$-quadruples in $\mathbb{Z}[\sqrt{-D}]$, $D > 1$

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The same result no matter what $D \mod 4$
(our search covered the range $|d| \leq 200$)

$D(-1)$-quadruples in $\mathbb{Z}[\sqrt{-D}]$, $D > 1$

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The same result no matter what $D \mod 4$ (our search covered the range $|d| \leq 200$)

This suggests an obvious conjecture

$D(1)$-quadruples in $\mathbb{Z}[\sqrt{-D}]$

a	b	c	d	$\lvert d\rvert^2$	D
-1	1	-3	3	9	2
-1	1	-12ω	12ω	288	
$1-\omega$	$1+\omega$	-2	-24	576	
1	-3	$-2+2\omega$	$16-20\omega$	1056	
1	-3	3	-33	1089	
$-1+\omega$	$1+\omega$	4ω	-36ω	2592	
$1+\omega$	-2	$-3+3\omega$	$64+8\omega$	4224	
2	-3	$-1-2\omega$	$20+44\omega$	10080	5
-1	1	-20ω	20ω	2400	6
-1	1	-15	15	225	14
-1	1	-35	35	1225	34
-1	1	-63	63	3969	62

Here, $\omega=\sqrt{-D}$, $D\equiv 1,2 \pmod 4$ and $D>1$.

$D(1)$-quadruples in $\mathbb{Z}[\omega]$, $\omega = (1+\sqrt{-3})/2$

a	b	c	d	$\lvert d\rvert^2$
-1	1	$4-8\omega$	$-4+8\omega$	48
-1	$1+\omega$	$2-\omega$	-8	64
-1	$2-\omega$	$1-3\omega$	$8+8\omega$	192
$-1+\omega$	ω	$7-14\omega$	$-8+16\omega$	192
ω	$-2+\omega$	$-4+4\omega$	$20-4\omega$	336
ω	$-1+2\omega$	$-1+5\omega$	$-16-8\omega$	448
ω	$-1-2\omega$	$3-3\omega$	$-8-32\omega$	1344
-1	$2-\omega$	-8	$49-35\omega$	1911
-2	2	$2-4\omega$	$-28+56\omega$	2352
ω	$-1-2\omega$	$-5+\omega$	$-40+56\omega$	2496
$1-2\omega$	$-1+2\omega$	4	56	3136

$D(1)$-quadruples in $\mathbb{Z}[\omega]$, $\omega = (1+\sqrt{-7})/2$

a	b	c	d	$\lvert d\rvert^2$
-1	1	-8	8	64
1	$-2-\omega$	$-3+\omega$	24	576
-1	$3-\omega$	$2-3\omega$	$8+24\omega$	1408
ω	$-2+\omega$	$-4+4\omega$	$52-20\omega$	2464
$-1-\omega$	$1+\omega$	$-4+2\omega$	$24+36\omega$	4032
-1	$-1+3\omega$	$-6+5\omega$	$80+48\omega$	14848
-1	1	$48-96\omega$	$-48+96\omega$	16128
ω	$2+\omega$	$4+4\omega$	$-116+76\omega$	16192
-2	4	4ω	$4-120\omega$	28336
ω	$-2-3\omega$	$4-4\omega$	$-60-108\omega$	33408

Fermat's $D(1)$-quadruple $(1, 3, 8, 120)$ omitted

$D(1)$-quadruples in $\mathbb{Z}[\omega]$, $\omega = (1+\sqrt{-D})/2$

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a	b	c	d	$\lvert d\rvert^2$	D
-1	1	$8-16\omega$	$-8+16\omega$	960	15
-1	1	-24	24	576	23
-1	1	$12-24\omega$	$-12+24\omega$	5040	35
-1	1	-48	48	2304	47
-1	1	-80	80	6400	79
-1	1	-120	120	14400	119
-1	1	-168	168	28224	167

Here, $D \geq 11$. Fermat's $D(1)$-quadruple $(1, 3, 8, 120)$ omitted in each ring

Nonextendible $D(1)$-quadruples

Theorem Let $a \in R$ with all of $-a+1$, $a+1$ and $-a^2+1$ nonzero squares in R. Then, the quadruple $(-1, 1, -a, a)$ cannot be extended to a $D(1)$-quintuple in R.

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Otherwise, $(-1, 1, -a, a, b)$ is a $D(1)$-quintuple for some $b \in R$, so let x, y, z, w satisfying $-b+1 = x^2$, $b+1 = y^2$, $-ab+1 = z^2$ and $ab+1 = w^2$. Put $f(X) = 4X^4 - 4(a^2+1)X^2 + 4a^2$, $g(X) = 2X^2 - a^2 - 1$.

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Then, $(g(X))^2 - f(X) = (a^2-1)^2$, $f(ab) = (2axyzw)^2$, so $|2a^2b^2 - a^2 - 1| = |g(ab)| \leq |a^2-1|^2$ and $|b| < |a|$.

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Then, $(g(X))^2 - f(X) = (a^2-1)^2$, $f(ab) = (2axyzw)^2$, so $|2a^2b^2 - a^2 - 1| = |g(ab)| \leq |a^2-1|^2$ and $|b| < |a|$.

The same argument applied to the quintuple $(-1, 1, -b, b, a)$ gives $|a| < |b|$.

Concrete examples

Infinite families:

- $D(1)$-quadruple $(-1, 1, -D-1, D+1)$ in the ring $\mathbb{Z}[\sqrt{-D}\,]$ for $D = 4u^2 - 2$ (u positive integer)
- quadruple $(-1, 1, -4k, 4k)$ in the ring $\mathbb{Z}[\omega\,]$ with $\omega = \frac{1+\sqrt{1-4k}}{2}$
- quadruple $(-1, 1, 4u - 8u\omega, -4u + 8u\omega)$ in the ring $\mathbb{Z}[\omega\,]$ with $\omega = \frac{1+\sqrt{1-4u^2}}{2}$

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- quadruple $(-1, 1, -4k, 4k)$ in the ring $\mathbb{Z}[\omega\,]$ with $\omega = \frac{1+\sqrt{1-4k}}{2}$
- quadruple $(-1, 1, 4u - 8u\omega, -4u + 8u\omega)$ in the ring $\mathbb{Z}[\omega\,]$ with $\omega = \frac{1+\sqrt{1-4u^2}}{2}$

Sporadic examples:

- quadruple $(1-\sqrt{-2}, 1+\sqrt{-2}, -2, -24)$ in $\mathbb{Z}[\sqrt{-2}\,]$
- $(2, -1-2i, -1+2i, -40)$ in $\mathbb{Z}[\sqrt{-1}\,]$
- The triple $(-1, -\omega, -1+\omega)$ cannot be extended to a $D(-1)$-quadruple in $\mathbb{Z}[\omega]$, where $\omega = (1+\sqrt{-3}\,)/2$

Simultaneous approximations to square roots close to 1

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Jadrijević–Ziegler 2006 $a_1, a_2, T \in R \setminus \{0\}$ with $a_1 \neq a_2$, $|T| > M := \max\{|a_1|, |a_2|\}$, and T divisible by $a_1 a_2$ in R. Put

$$\theta_1 = \sqrt{1 + \frac{a_1}{T}} \quad \text{and} \quad \theta_2 = \sqrt{1 + \frac{a_2}{T}}$$

Then there are positive, explicitly given λ, L, C such that if $L > 1$, then

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > C|q|^{-\lambda}$$

for all $p_1, p_2, q \in R$.

Estimates for parameters

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When applying the previous result to a solution of the system of Pellian equations generated by extending a triple (a, b, c) with $|a| \leq |b| \leq |c|$, one obtains estimates for the involved parameters by assuming $|c| \geq \sigma |b|^{\alpha}$ for suitable positive real numbers α, σ.

Estimates for parameters

When applying the previous result to a solution of the system of Pellian equations generated by extending a triple (a, b, c) with $|a| \leq |b| \leq |c|$, one obtains estimates for the involved parameters by assuming $|c| \geq \sigma |b|^{\alpha}$ for suitable positive real numbers α, σ.

The aim is to bound from above λ by a value Λ as small as possible.

We obtained satisfactory bounds when $|b - a| \geq |a|$.

Gap Principle

If a quadruple (a, b, c, d) with $|a| \leq |b| \leq |c| \leq |d|$ satisfies an inequality $|c| \geq \sigma |b|^{\alpha}$ found as explained above, then $|d| < \rho |c|^{\beta}$ for certain positive real numbers ρ, β.

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Adžaga $\beta = 50$ C–Fujita $\beta = 45$

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An upper bound for the size m of a $D(\varepsilon)$–set is then obtained by combining such a result with a lower bound for the largest absolute value in a quadruple in terms of the smallest one.

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We provide such a lower bound by showing that the largest absolute value of an entry in an irregular triple is greater than the product of the other two absolute values, provided that all of them are sufficiently large.

Irregular triples

Pumping Lemma Let (a, b, c) be an irregular triple with $|c| \geq |b| \geq |a|$. Assume that at least one of the following conditions holds:

(α) $|a| = \sqrt{2}$ and $|b| \geq 19$;

(β) $|a| = \sqrt{3}$ and $|b| \geq 11$;

(γ) $|a| \geq 2$ and $|b| \geq 9$;

(δ) $|a| \geq 3$ and $|b| \geq 6$;

(ε) $|a| \geq 4$.

Then,

$$|c| \geq 4|ab| - 2.5|b| - 2.5|a|.$$

In particular, $|c| > 2.1|a||b|$ for $|a| > 1$.

Unfinished business

From Pumping Lemma one obtains that if b_1, b_2, ..., b_k is a $D(\varepsilon)$-tuple with all triples formed by successively indexed terms irregular, then $|b_k| > |b_3|^{F_{k-2}}|b_2|^{F_{k-3}}$ $(k \geq 4)$

Unfinished business

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With the current numerical values for Gap Principle, it is seen that in the first situation the bound on cardinality of $D(\varepsilon)$-sets is less than 24

Homework

Open question Is it true that if b_1, b_2, ..., b_k is a $D(\varepsilon)$-tuple with $|b_1| \leq |b_2| \leq \cdots \leq |b_k|$ and (b_1, b_2, b_3) regular, then all triples b_j, b_{j+1}, b_{j+2} with $2 \leq j \leq k-2$ are irregular?

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