Induction from two linked segments with one half border and cuspidal reducibility

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F local non-archimedean field of characteristic different from 2

$$J_n = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix} \in GL(n, F).$$

Symplectic group of rank $n \geq 1$ is a group

$$G_n = Sp(2n, F) = \left\{ g \in GL(2n, F) : \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} g^{\tau} \begin{bmatrix} 0 & -J_n \\ J_n & 0 \end{bmatrix} = g^{-1} \right\},$$

where τ is the transposition with respect to the second diagonal.
Fix $B \leq G_n$ Borel subgroup of upper triangular matrices.
Parabolic subgroups $B \leq P_s$ are parametrized by ordered partitions of n:

$$P_s \cong M_s N_s \longleftrightarrow s = (n_1, ..., n_k), \quad n_1 + \cdots + n_k = n, \ n_i \geq 0,$$
$$M_s \cong GL(n_1, F) \times \cdots \times GL(n_{k-1}, F) \times G_{n_k},$$

M_s is the Levi subgroup of P_s.

G_n is an l-grupa, every neighbourhood of the identity contains an open compact subgroup.
Representation of an l-group on a complex vector space is smooth if the stabiliser of every vector is an open subgroup.

$R(G_n)$ free Abelian group generated by classes of irreducible representations of G_n.

$$R(G) = \oplus_{n \geq 0} R(G_n).$$

$R_0^+(G)$ is $\mathbb{Z}_{\geq 0}$ sublattice generated by classes of irreducible representations.
$\pi_1, \pi_2 \in R(G)$

$$\pi_1 \leq \pi_2 \Leftrightarrow \pi_2 - \pi_1 \in R_0^+(G).$$

Functor of the normalized parabolic induction

$$\delta_1 \times \cdots \times \delta_k \rtimes \tau = \mathrm{Ind}_{M_s}^{G_n}(\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau)$$

normalized Jacquet module

$$\mathrm{r}_s(\sigma) = \mathrm{r}_{M_s}(\sigma) = \mathrm{Jacq}_{M_s}^{G_n}(\sigma).$$

Frobenius reciprocity

$$\mathrm{Hom}_{G_n}(\sigma, \mathrm{Ind}_{M_s}^{G_n}(\pi)) = \mathrm{Hom}_{M_s}(\mathrm{Jacq}_{M_s}^{G_n}(\sigma), \pi).$$

$\wedge$ is a contravariant exact functor on category of smooth representations of G_n

$$\mathrm{Alg}\ G_n \xrightarrow{\wedge} \mathrm{Alg}\ G_n,$$

so

$$\overset{\wedge}{\pi} \cong \pi, \quad \pi \in \mathrm{Irr}\ G_n,$$

for $\delta_i \in \mathrm{Irr}\ GL(n_i, F)$, $i = 1, ..., k$, $m = n_1 + \cdots + n_k$ and $\tau \in \mathrm{Irr}\ G_{n-m}$ we have

$$(\delta_1 \times \cdots \times \delta_k \rtimes \tau)^\wedge \cong \widetilde{\delta_1} \times \cdots \times \widetilde{\delta_k} \rtimes \tau,$$

where $\widetilde{\ }$ denotes contragredient.

$\rho \in Irr_{c,u}GL(m_\rho, F)$ cuspidal and unitarizable.
$\nu = |\det|_F$, $x, y \in \mathbb{R}$, $y - x + 1 \in \mathbb{Z}_{\geq 0}$.

$$\Delta = [\nu^x\rho, \nu^y\rho] = \{\nu^x\rho, ..., \nu^y\rho\} \quad \text{is called a segment.}$$

$e(\Delta) = e(\delta(\Delta) = \frac{x+y}{2}$.

$$\delta(x, y) = \delta([\nu^x\rho, \nu^y\rho]) \hookrightarrow \nu^y\rho \times \cdots \times \nu^x\rho.$$

For $\widetilde{\Delta} = [\nu^{-y}\rho, \nu^{-x}\rho]$ we have $\delta([\nu^x\rho, \nu^y\rho])^\sim \cong \delta(-y, -x)$.

Let $\delta_i = \delta(\Delta_i), e_i = e(\delta_i), i = 1, 2$.

$\delta_1 \times \delta_2$ reduces $\Leftrightarrow \Delta_1 \cup \Delta_2$ is segment and $\Delta_1 \not\subseteq \Delta_2, \Delta_2 \not\subseteq \Delta_1$.

In case of reducibility and $e_1 \geq e_2$ we have

$$\delta(\Delta_1 \cup \Delta_2) \times \delta(\Delta_1 \cap \Delta_2) \hookrightarrow \delta_1 \times \delta_2 \twoheadrightarrow L(\delta_1 \times \delta_2) = L(\delta_1, \delta_2).$$

Langlands classification

$$\pi \qquad \longleftrightarrow \qquad \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \twoheadrightarrow \pi,$$

$\pi \in IrrGL(n, F)$,
$e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$.

$$\sigma \qquad \longleftrightarrow \qquad \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \tau \twoheadrightarrow \sigma$$

$\sigma \in IrrG_n$
$e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$
τ tempered representation of $G_{n'}$

Mœglin-Tadić classification of discrete series

$$\sigma \quad \longleftrightarrow \quad (\mathrm{Jord}, \sigma_{cusp}, \epsilon)$$

σ discrete series of G_n, $n \in \mathbb{N}$, $\rho \in Irr_{c,u} GL(m_\rho, F)$ self-dual, $a \in \mathbb{N}$.

$$\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$$

is irreducible for all a of one parity. For a of the other parity, it reduces, except for finite exceptions.

$\mathrm{Jord}(\sigma) = \{(a, \rho)$ is an exception $\}$ is a finite set.
$\mathrm{Jord}_\rho(\sigma) = \{a \in \mathbb{N} \mid (a, \rho) \in \mathrm{Jord}\}$,
$a_- = \max\{a' \in \mathrm{Jord}_\rho \mid a' < a\}$.

σ_{cusp} is patrial cuspidal support of σ, if it is irreducible, cuspidal and

$$\sigma \hookrightarrow \pi \rtimes \sigma_{cusp}.$$

for some $\pi \in Irr GL(m_\pi, F)$.

$\epsilon : D_\epsilon \subseteq \text{Jord} \cup (\text{Jord} \times \text{Jord}) \to \{\pm 1\}$

$$\begin{aligned} D_\epsilon =& \{((a,\rho),(a',\rho')) \in \text{Jord} \times \text{Jord} \mid \rho \cong \rho' \text{ and } a \neq a'\} \cup \\ & \{(a,\rho) \in \text{Jord} \mid a \in 2\mathbb{N} \text{ or } a \in 2\mathbb{N}-1 \text{ and } \rho \rtimes \sigma_{cusp} \text{ reduces}\}. \end{aligned}$$

$$\epsilon(a,\rho)\epsilon(a_-,\rho)^{-1} = 1 \iff \sigma \hookrightarrow \delta([\nu^{-(a_- -1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma'$$

For $a \in \min\text{Jord}_\rho(\sigma)$ even

$$\epsilon(a,\rho) = 1 \iff \exists \sigma', \sigma \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma'$$

Geometrical lemma

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} \text{s.s.}(r_{(k,n-k)}(\sigma)).$$

$$\mu^*(\delta([\nu^x\rho, \nu^y\rho]) \rtimes \sigma) = \sum_{\delta' \otimes \sigma' \leq \mu^*(\sigma)} \sum_{i=0}^{y-x+1} \sum_{j=0}^{i}$$

$$\delta([\nu^{i-y}\rho, \nu^{-x}\rho]) \times \delta([\nu^{y+1-j}\rho, \nu^y\rho]) \times \delta' \otimes \delta([\nu^{y+1-i}\rho, \nu^{y-j}\rho]) \rtimes \sigma'.$$

From now
σ cuspidal representation of G_n,
$\rho \in Irr_{c,u} GL(m_\rho, F)$ self-dual
$\nu^{\frac{1}{2}} \rho \rtimes \sigma$ reduces
$a, b, c \in \mathbb{Z} + \frac{1}{2}$ $\frac{1}{2} \leq a < b < c$.
Decompose

$$\delta(\tfrac{1}{2}, a) \times \delta(-b, c) \rtimes \sigma$$
$$\delta(\tfrac{1}{2}, b) \times \delta(-a, c) \rtimes \sigma$$
$$\delta(\tfrac{1}{2}, c) \times \delta(-a, b) \rtimes \sigma$$

We have discrete series

$$\begin{aligned}
\sigma_a &\hookrightarrow \delta(\tfrac{1}{2}, a) \rtimes \sigma \twoheadrightarrow L(\delta(\tfrac{1}{2}, a) \rtimes \sigma), \\
\sigma^+_{b,c} + \sigma^-_{b,c} &\hookrightarrow \delta(-b, c) \rtimes \sigma \twoheadrightarrow L(\delta(-b, c) \rtimes \sigma), \\
\sigma^+_{b,c} &\hookrightarrow \delta(\tfrac{1}{2}, b) \rtimes \sigma_c \\
\sigma^+_{b,c,a} + \sigma^-_{b,c,a} &\hookrightarrow \delta(-b, c) \rtimes \sigma_a \twoheadrightarrow L(\delta(-b, c) \rtimes \sigma_a) \\
\sigma^+_{a,b,c} + \sigma^-_{a,b,c} &\hookrightarrow \delta(-a, b) \rtimes \sigma_c \twoheadrightarrow L(\delta(-a, b) \rtimes \sigma_c) \\
\sigma^+_{b,c,a} &= \sigma^+_{a,b,c}
\end{aligned}$$

Standard intertwining operators

$$\begin{aligned}\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma &\cong \delta(\tfrac{1}{2},a)\times\delta(-b,c)\rtimes\sigma\\ &\to \delta(\tfrac{1}{2},a)\times\delta(-c,b)\rtimes\sigma\\ &\cong \delta(-c,b)\times\delta(\tfrac{1}{2},a)\rtimes\sigma\\ &\to \delta(-c,b)\times\delta(-a,-\tfrac{1}{2})\rtimes\sigma.\end{aligned}$$

Image

$$L(\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma)$$

Kernels

$$\delta(\tfrac{1}{2},a)\times\sigma^{+}_{b,c}+\delta(\tfrac{1}{2},a)\times\sigma^{-}_{b,c},$$

$$\delta(-c,b)\times\sigma_a.$$

$$\delta(\tfrac{1}{2},a)\times\sigma^{+}_{b,c}+\delta(\tfrac{1}{2},a)\times\sigma^{-}_{b,c}\hookrightarrow\delta(\tfrac{1}{2},a)\times\delta(-b,c)\rtimes\sigma$$
$$\sigma^{+}_{b,c,a}+\sigma^{-}_{b,c,a}\hookrightarrow\delta(-b,c)\times\sigma_a\hookrightarrow\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma$$

$$\mu^*(\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma)\underset{MAX}{\geq}2\cdot\delta(-b,c)\times\delta(\tfrac{1}{2},a)\otimes\sigma$$

$$\pi\hookrightarrow\delta(\tfrac{1}{2},a)\times\sigma^{+}_{b,c}\hookrightarrow\delta(\tfrac{1}{2},a)\times\delta(\tfrac{1}{2},b)\rtimes\tau\cong\delta(\tfrac{1}{2},b)\times\delta(\tfrac{1}{2},a)\rtimes\tau$$
$$\hookrightarrow\delta(a+1,b)\rtimes\tau'.$$

$\pi\in\{\sigma^{+}_{b,c,a},\sigma^{-}_{b,c,a}\},\epsilon_\pi(a)=\epsilon_\pi(b)\implies\pi\cong\sigma^{+}_{b,c,a}$.
We have exact sequences

$$\sigma^{\pm}_{b,c,a}\hookrightarrow\delta(\tfrac{1}{2},a)\rtimes\sigma^{\pm}_{b,c}\twoheadrightarrow L(\delta(\tfrac{1}{2},a)\rtimes\sigma^{\pm}_{b,c}).$$

Note $\mu^*(\sigma^{\pm}_{b,c,a})\geq\delta(\tfrac{1}{2},a)\rtimes\sigma^{\pm}_{b,c}$

Teorem. Representation $\delta(-b,c) \times \delta(\frac{1}{2},a) \rtimes \sigma$ has a filtration $\{V_i\}$

$$V_3/V_2 \cong L(\delta(-b,c) \times \delta(\tfrac{1}{2},a) \rtimes \sigma)$$
$$V_2/V_1 \cong L(\delta(\tfrac{1}{2},a) \rtimes \sigma^+_{b,c}) + L(\delta(\tfrac{1}{2},a) \rtimes \sigma^-_{b,c}) + L(\delta(-b,c) \rtimes \sigma_a)$$
$$V_1 \cong \sigma^+_{b,c,a} + \sigma^-_{b,c,a}$$

Now decompose

$$\psi = \delta(-a, c) \times \delta(\tfrac{1}{2}, b) \rtimes \sigma \cong \\ \delta(\tfrac{1}{2}, b) \times \delta(-a, c) \rtimes \sigma.$$

Standard intertwinings

$$\delta(-a,c)\times\delta(\tfrac{1}{2},b)\rtimes\sigma$$
$$\xrightarrow{f_0}\delta(\tfrac{1}{2},b)\times\delta(-a,c)\rtimes\sigma$$
$$\xrightarrow{f_1}\delta(\tfrac{1}{2},b)\times\delta(-c,a)\rtimes\sigma$$
$$\xrightarrow{f_2}\delta(-c,a)\times\delta(\tfrac{1}{2},b)\rtimes\sigma$$
$$\xrightarrow{f_3}\delta(-a,c)\times\delta(-b,\tfrac{1}{2})\rtimes\sigma.$$

$$\delta(\tfrac{1}{2},b)\times\delta(-a,c)\rtimes\sigma$$
$$\xrightarrow{g_0}\delta(-a,c)\times\delta(\tfrac{1}{2},b)\rtimes\sigma$$
$$\xrightarrow{g_1}\delta(-a,c)\times\delta(-b,-\tfrac{1}{2})\rtimes\sigma$$
$$\xrightarrow{g_2}\delta(-b,-\tfrac{1}{2})\times\delta(-a,c)\rtimes\sigma$$
$$\xrightarrow{g_3}\delta(-b,-\tfrac{1}{2})\times\delta(-c,a)\rtimes\sigma.$$

Let $K_i = \mathrm{Ker}\ f_i$, and $H_i = \mathrm{Ker}\ g_i$. We have

$$K_0 \cong H_0 = \{0\},$$
$$K_1 \cong H_3{}^\wedge \cong \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^+_{a,c} + \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \sigma^-_{a,c},$$
$$K_2 \cong H_2^\wedge \cong \delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma,$$
$$K_3 \cong H_1^\wedge \cong \delta([\nu^{-c}\rho, \nu^a\rho]) \rtimes \sigma_b,$$

$$\mathrm{Im}(f_3 \circ \cdots \circ f_0) \cong L(\psi) \cong \mathrm{Im}(g_3 \circ \cdots \circ g_0),$$

In $R(G)$

$$\forall i \quad K_i \leq \psi \leq K_1 + K_2 + K_3 + L(\psi).$$

(Muić) In $R(G)$

$$\delta(-a, c) \rtimes \sigma_b = \sigma^+_{b,c,a} + L(\delta(-a, c) \rtimes \sigma_b) + L(\delta(-a, b) \rtimes \sigma_c) + L(\delta(-b, c) \rtimes \sigma_a).$$

Cuspidal support: $\sigma^+_{b,c,a}$, $\sigma^-_{b,c,a}$ $\sigma^-_{a,b,c}$ only possible d.s.

$$\mu^*(\delta(\tfrac{1}{2},b)\times\delta(-a,c)\rtimes\sigma)\geq 1\cdot\delta(\tfrac{1}{2},a)\otimes\sigma^+_{b,c}+1\cdot\delta(\tfrac{1}{2},a)\otimes\sigma^-_{b,c},$$
$$\mu^*(\delta(\tfrac{1}{2},b)\rtimes\sigma^+_{a,c})\geq 1\cdot\delta(-a,b)\otimes\sigma_c+1\cdot\delta(\tfrac{1}{2},a)\otimes\sigma^+_{b,c}+$$
$$0\cdot\delta(\tfrac{1}{2},a)\otimes\sigma^-_{b,c}.$$
$$\mu^*(\delta(\tfrac{1}{2},b)\rtimes\sigma^-_{a,c})\geq 0\cdot\delta(-a,b)\otimes\sigma_c+0\cdot\delta(\tfrac{1}{2},a)\otimes\sigma^-_{b,c}.$$

Corollary. All d.s. with multiplicities

$$\delta(\tfrac{1}{2},b)\times\delta(-a,c)\rtimes\sigma\geq\sigma^+_{b,c,a}+\sigma^-_{b,c,a},$$
$$\delta(\tfrac{1}{2},b)\rtimes\sigma^+_{a,c}\geq\sigma^+_{b,c,a},$$
$$\delta(\tfrac{1}{2},b)\rtimes\sigma^-_{a,c}\geq 0.$$

Other subquotients

Lemma. $\mu^*(L(\delta(-a,b) \rtimes \sigma_c)) \geq \delta(\frac{1}{2},a) \otimes L(\delta(\frac{1}{2},b) \rtimes \sigma_c)$.

Corollary.

$$\begin{aligned}
\delta(\tfrac{1}{2},b) \rtimes \sigma^+_{a,c} =& \sigma^+_{a,b,c} + L(\delta(-a,b) \rtimes \sigma_c) + \\
& L(\delta(\tfrac{1}{2},a) \rtimes \sigma^+_{b,c}) + L(\delta(\tfrac{1}{2},b) \rtimes \sigma^+_{a,c}), \\
\delta(\tfrac{1}{2},b) \rtimes \sigma^-_{a,c} =& L(\delta(\tfrac{1}{2},a) \rtimes \sigma^-_{b,c}) + L(\delta(\tfrac{1}{2},b) \rtimes \sigma^-_{a,c}).
\end{aligned}$$

The remaining subquotient that appears in two kernels satisfies $\mu^*(L(\delta(-b,c) \rtimes \sigma_a)) \geq \delta(\frac{1}{2},a) \otimes L(\delta(-b,c) \rtimes \sigma)$.

Theorem. In $R(G)$

$$\begin{aligned}\delta(\tfrac{1}{2},b)\times\delta(-a,c)\rtimes\sigma =& L(\delta(-a,c)\times\delta(\tfrac{1}{2},b)\rtimes\sigma)+\\ & L(\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma)\\ & +\\ & L(\delta(-a,c)\rtimes\sigma_b)+\\ & L(\delta(-a,b)\rtimes\sigma_c)+\\ & L(\delta(-b,c)\rtimes\sigma_a)\\ & +\\ & L(\delta(\tfrac{1}{2},a)\rtimes\sigma^+_{b,c})+L(\delta(\tfrac{1}{2},b)\rtimes\sigma^+_{a,c})+\\ & L(\delta(\tfrac{1}{2},a)\rtimes\sigma^-_{b,c})+L(\delta(\tfrac{1}{2},b)\rtimes\sigma^-_{a,c})\\ & +\\ & \sigma^+_{b,c,a}+\sigma^-_{b,c,a}.\end{aligned}$$

Decomposing kernels (in bold)

$$L(\delta(\tfrac{1}{2},a)\rtimes\sigma^{+}_{b,c})+L(\delta(-a,b)\rtimes\sigma_c)\hookrightarrow \boldsymbol{\delta(\tfrac{1}{2},b)\rtimes\sigma^{+}_{a,c}}/\sigma^{+}_{a,b,c}\twoheadrightarrow L(\delta(\tfrac{1}{2},b)\rtimes\sigma^{+}_{a,c}),$$

$$L(\delta(\tfrac{1}{2},a)\rtimes\sigma^{-}_{b,c})\hookrightarrow \boldsymbol{\delta(\tfrac{1}{2},b)\rtimes\sigma^{-}_{a,c}}\twoheadrightarrow L(\delta(\tfrac{1}{2},b)\rtimes\sigma^{-}_{a,c}),$$

$$\sigma^{+}_{b,c,a}+L(\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma)+\sigma^{-}_{b,c,a}\hookrightarrow \boldsymbol{\delta(-c,b)\times\delta(\tfrac{1}{2},a)\rtimes\sigma}/L(\delta(-b,c)\rtimes\sigma_a)\twoheadrightarrow L(\delta(\tfrac{1}{2},a)\rtimes\sigma^{+}_{b,c})+L(\delta(\tfrac{1}{2},a)\rtimes\sigma^{-}_{b,c}),$$

$$L(\delta(-b,c)\rtimes\sigma_a)+L(\delta(-a,b)\rtimes\sigma_c)\hookrightarrow \boldsymbol{\delta(-c,a)\rtimes\sigma_b}/L(\delta(-a,c)\rtimes\sigma_b)\twoheadrightarrow\sigma^{+}_{a,b,c}.$$

Let $k_0 = h_0 = \{0\}$, for all $i \geq 1$

$$k_i = K_i \cap Im(f_{i-1} \circ \cdots \circ f_0), \quad h_i = H_i \cap Im(g_{i-1} \circ \cdots \circ g_0).$$

In $R(G)$

$$k_i = \lfloor K_i - k_{i-1} - \cdots - k_0 \rfloor_{R_0^+(G)}.$$

where $\pi \in R(G), \pi = \sum_i m_i \pi_i$, $m_i \in \mathbb{Z}$, $\pi_i \ncong \pi_j \in IrrG$ for $i \neq j$, define

$$\lfloor \pi \rfloor_{R_0^+(G)} = \sum_{\{i | m_i \geq 0\}} m_i \pi_i.$$

In $R(G)$

$$k_1 = K_1, \quad k_2 = \lfloor K_2 - k_1 \rfloor_{R_0^+(G)}, \quad k_3 = \lfloor K_3 - k_2 - k_1 \rfloor_{R_0^+(G)},$$
$$h_1 = H_1, \quad h_2 = \lfloor H_2 - h_1 \rfloor_{R_0^+(G)}, \quad h_3 = \lfloor H_3 - h_2 - h_1 \rfloor_{R_0^+(G)}.$$

Two filtrations of ψ

$$L(\delta(\tfrac{1}{2},a)\rtimes\sigma^{+}_{b,c})+L(\delta(-a,b)\rtimes\sigma_c)+L(\delta(\tfrac{1}{2},b)\rtimes\sigma^{-}_{a,c})\hookrightarrow$$

$$\boldsymbol{k_1}/(\sigma^{+}_{a,b,c}+L(\delta(\tfrac{1}{2},a)\rtimes\sigma^{-}_{b,c}))\twoheadrightarrow L(\delta(\tfrac{1}{2},b)\rtimes\sigma^{+}_{a,c}),$$

$$\boldsymbol{k_2}/L(\delta(-b,c)\rtimes\sigma_a)\cong\sigma^{-}_{b,c,a}+$$

$$L(\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma),$$

$$\boldsymbol{k_3}\cong L(\delta(-a,c)\rtimes\sigma_b),$$

$$L(\delta(-b,c)\rtimes\sigma_a)+L(\delta(-a,b)\rtimes\sigma_c)\hookrightarrow$$

$$\boldsymbol{h_1}/\sigma^{+}_{a,b,c}\twoheadrightarrow L(\delta(-a,c)\rtimes\sigma_b),$$

$$\boldsymbol{h_2}/(L(\delta(\tfrac{1}{2},a)\rtimes\sigma^{+}_{b,c})+L(\delta(\tfrac{1}{2},a)\rtimes\sigma^{-}_{b,c}))\cong$$

$$\sigma^{-}_{b,c,a}+L(\delta(-b,c)\times\delta(\tfrac{1}{2},a)\rtimes\sigma),$$

$$\boldsymbol{h_3}\cong L(\delta(\tfrac{1}{2},b)\rtimes\sigma^{+}_{a,c})+L(\delta(\tfrac{1}{2},b)\rtimes\sigma^{-}_{a,c}).$$

Theorem. Let $\psi = \delta([\nu^{-a}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{b}\rho]) \rtimes \sigma$. Define

$$\begin{aligned}
W_1 =& \sigma^{+}_{b,c,a} + L(\delta(\tfrac{1}{2}, a) \rtimes \sigma^{-}_{b,c}),\\
W_2 =& L(\delta(\tfrac{1}{2}, a) \rtimes \sigma^{+}_{b,c}) + L(\delta(-a, b) \rtimes \sigma_c) +\\
& L(\delta(\tfrac{1}{2}, b) \rtimes \sigma^{-}_{a,c}) + L(\delta(-b, c) \rtimes \sigma_a),\\
W_3 =& L(\delta(\tfrac{1}{2}, b) \rtimes \sigma^{+}_{a,c}) + L(\delta(-a, c) \rtimes \sigma_b) +\\
& \sigma^{-}_{b,c,a} + L(\delta(-b, c) \times \delta(\tfrac{1}{2}, a) \rtimes \sigma),\\
W_4 =& L(\psi).
\end{aligned}$$

Then there exists a sequence
$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 = \psi$, such that

$$V_i / V_{i-1} \cong W_i, \quad i = 1, \ldots, 4.$$

Finally we decompose $\delta(\frac{1}{2}, c) \times \delta(-a, b) \rtimes \sigma$.

$$\delta(\tfrac{1}{2}, c) \times \delta(-a, b) \rtimes \sigma$$
$$\downarrow f_1$$
$$\delta(-a, b) \times \delta(\tfrac{1}{2}, c) \rtimes \sigma$$
$$\downarrow f_2$$
$$\delta(-a, b) \times \delta(-c, -\tfrac{1}{2}) \rtimes \sigma$$
$$\downarrow f_3$$
$$\delta(-c, -\tfrac{1}{2}) \times \delta(-a, b) \rtimes \sigma$$
$$\downarrow f_4$$
$$\delta(-c, -\tfrac{1}{2}) \times \delta(-b, a) \rtimes \sigma$$

For all i let $K_i = \mathrm{Ker}\ f_i$. We have

$$K_1 \cong \delta(\tfrac{1}{2}, b) \times \delta(-a, c) \rtimes \sigma,$$
$$K_2 \cong \delta(-a, b) \rtimes \sigma_c,$$
$$K_3 \cong \delta(-c, b) \times \delta(\tfrac{1}{2}, a) \rtimes \sigma,$$
$$K_4 \cong \delta(-c, -\tfrac{1}{2}) \rtimes \sigma^+_{a,b} + \delta(-c, -\tfrac{1}{2}) \rtimes \sigma^-_{a,b}.$$

In $R(G)$:

$$\forall i, \quad K_i \leq \psi \leq K_1 + K_2 + K_3 + K_4 + L(\psi).$$

Theorem. Representation $\delta(\frac{1}{2}, c) \times \delta(-a, b) \rtimes \sigma$ has a filtration $\{V_i\}$:

$$
\begin{aligned}
V_1 \cong & \sigma^+_{b,c,a} + L(\delta(\tfrac{1}{2}, a) \rtimes \sigma^-_{b,c}),\\
V_2/V_1 \cong & L(\delta(\tfrac{1}{2}, a) \rtimes \sigma^+_{b,c}) + L(\delta(\tfrac{1}{2}, b) \rtimes \sigma^-_{a,c}) +\\
& L(\delta(-b, c) \rtimes \sigma_a) + L(\delta(-a, b) \rtimes \sigma_c),\\
V_3/V_2 \cong & \sigma^-_{b,c,a} + \sigma^-_{a,b,c} + L(\delta(\tfrac{1}{2}, b) \rtimes \sigma^+_{a,c}) + L(\delta(-a, c) \rtimes \sigma_b) +\\
& L(\delta(-b, c) \times \delta(\tfrac{1}{2}, a) \rtimes \sigma),\\
V_4/V_3 \cong & L(\delta(\tfrac{1}{2}, b) \times \delta(-a, c) \rtimes \sigma) + L(\delta(\tfrac{1}{2}, c) \rtimes \sigma^+_{a,b}) +\\
& L(\delta(\tfrac{1}{2}, c) \rtimes \sigma^-_{a,b}),\\
V_5/V_4 \cong & L(\psi).
\end{aligned}
$$

Thank you!