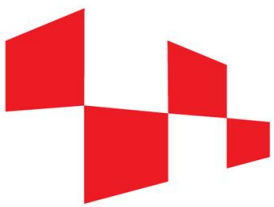
# Locally analytic vectors in Banach space representations of $SL_2(\mathbb{Q}_p)$

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Joint work with Matthias Strauch

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# Overview

In our previous work, we provided a classification of absolutely irreducible admissible unitary Banach space representations of

$$H = SL_2(\mathbb{Q}_p).$$

based on the $p$-adic Langlands correspondence for

$$G = GL_2(\mathbb{Q}_p)$$

We now focus on the structure of these representations, particularly on the space of locally analytic vectors.

We consider the case where the restricted representation splits into two components.

## $p$-adic Langlands correspondence for $GL_2(E)$

$E$ is a finite extension of $\mathbb{Q}_p$. Let

$$\psi : \mathcal{G}_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL_2(E),$$

be an absolutely irreducible representation. Attached to $\psi$ by Colmez' $p$-adic Langlands correspondence is the unitary Banach space representation $\Pi(\psi)$ of $G$. The map

$$\psi \mapsto \Pi(\psi)$$

induces a bijection between the set of equivalence classes of absolutely irreducible continuous two-dimensional $E$-representations of $\mathcal{G}_{\mathbb{Q}_p}$ and absolutely irreducible non-ordinary admissible unitary $E$-Banach space representations of $GL_2(\mathbb{Q}_p)$.

# From $GL_2(\mathbb{Q}_p)$ to $SL_2(\mathbb{Q}_p)$

Given

$$\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E) \quad \text{and} \quad \Pi = \Pi(\psi),$$

we consider the corresponding projective representation

$$\overline{\psi} : \mathcal{G}_{\mathbb{Q}_p} \to PGL_2(E)$$

and the restriction

$$\Pi|_H.$$

Set

$$X(\psi) = \{\chi \in \mathcal{G}^*_{\mathbb{Q}_p} \mid \chi \otimes \psi \cong \psi\}.$$

- The cardinality of $X(\psi)$ divides 4.
- If $X(\psi)$ has cardinality one, $\psi$ is called *primitive*.
- If $X(\psi)$ has cardinality two, $\psi$ is called *simply-imprimitive*.
- If $X(\psi)$ has cardinality one, $\psi$ is called *triply-imprimitive*.

# From $GL_2(\mathbb{Q}_p)$ to $SL_2(\mathbb{Q}_p)$

## Theorem 1

*Let $\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$ be an absolutely irreducible continuous representation and $\Pi = \Pi(\psi)$. Denote by $\overline{\psi} : \mathcal{G}_{\mathbb{Q}_p} \to PGL_2(E)$ the corresponding projective representation. Then, after possibly replacing $E$ by a finite extension, we have*

(i) *$\Pi|_H$ is either absolutely irreducible or it decomposes as a direct sum of two absolutely irreducible representations. Let $r$ be the number of components of $\Pi|_H$ (counting multiplicities).*

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(ii) *Let $S_{\overline{\psi}}$ be the centralizer in $PGL_2(\overline{E})$ of the image of $\overline{\psi}$. Then $S_{\overline{\psi}}$ is finite, with cardinality 1,2, or 4.*

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$$r = |S_{\overline{\psi}}| \leq 2.$$

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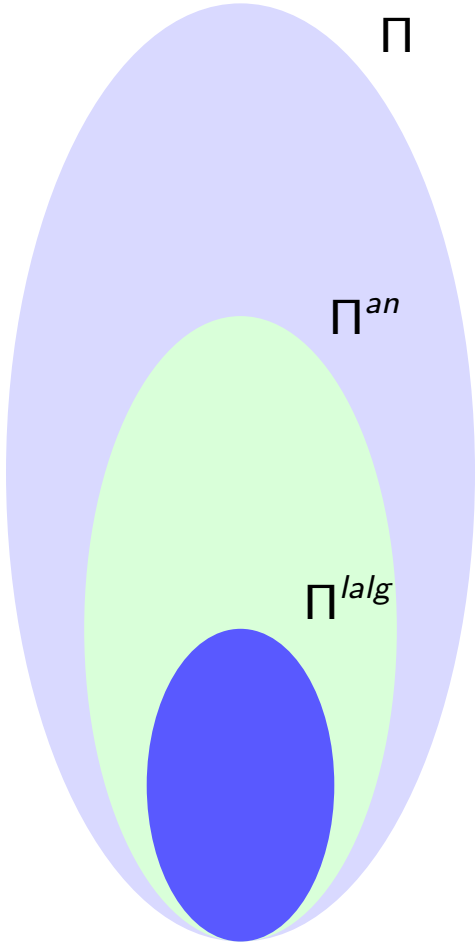
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(iv) *If $\psi$ is triply-imprimitive, then $|S_{\overline{\psi}}| = 4$ and $\Pi$ restricts to $H$ with multiplicity two. More specifically, $\Pi|_H$ decomposes as a direct sum of two equivalent irreducible representations.* □

# The structure of $\Pi(\psi)$




$\Pi = \Pi(\psi)$ the irreducible admissible Banach space representation attached to $\psi$

$\Pi^{an}$ = locally analytic vectors in $\Pi$

$\Pi^{lalg}$ = locally algebraic vectors in $\Pi$

$$\Pi^{lalg} = \sigma(\psi) \otimes \Pi^{alg}$$

$\sigma(\psi)$ smooth

$\Pi^{alg}$ algebraic

$\Pi^{lalg} \neq 0$ if and only if $\psi$ is de Rham with distinct Hodge-Tate weights

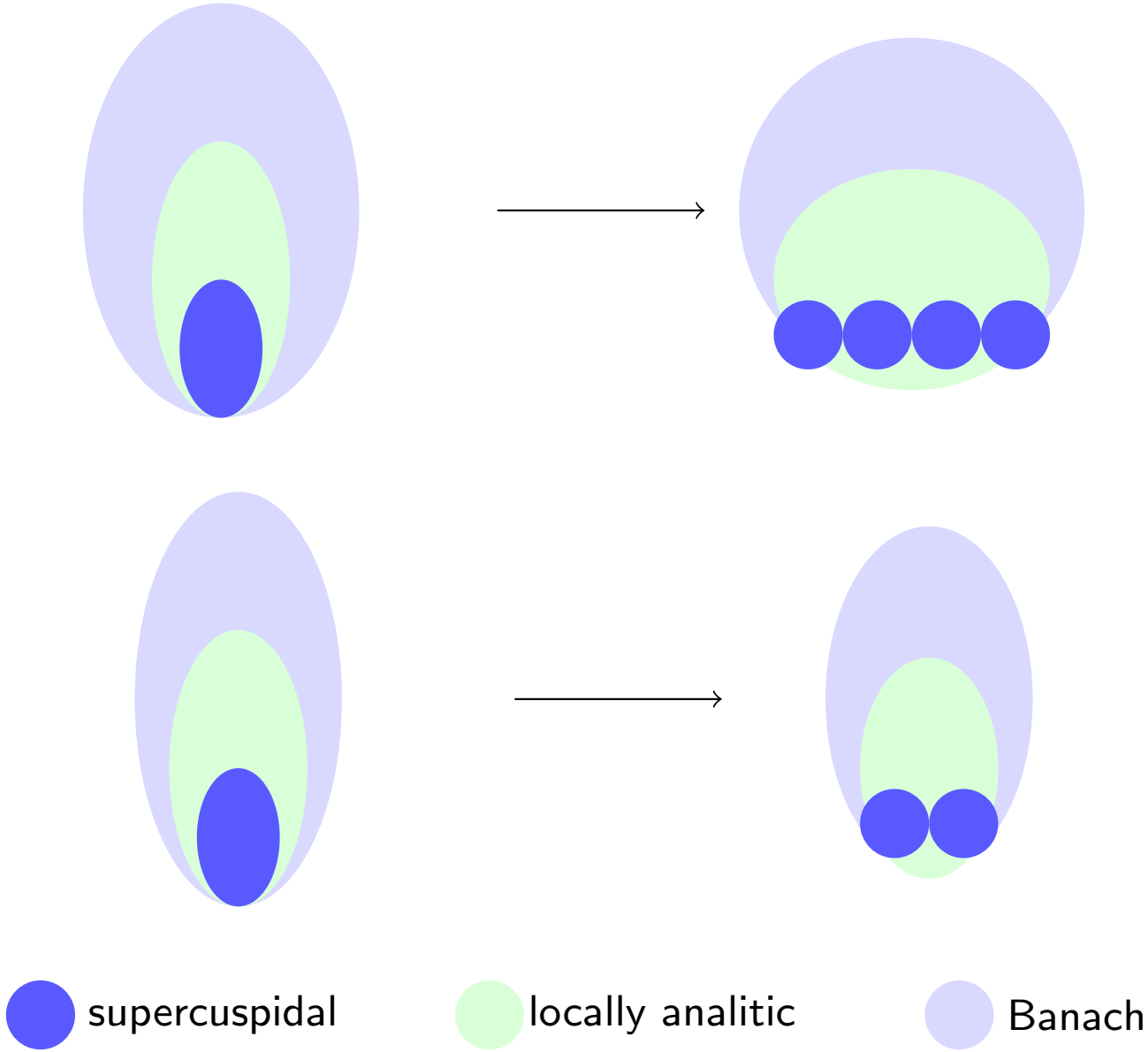
Suppose that $\psi$ is de Rham with distinct Hodge-Tate weights. Then

- If $\psi$ is trianguline, then $\sigma(\psi)$ is a principal series or the Steinberg representation.
- If $\psi$ is not trianguline, then $\sigma(\psi)$ is supercuspidal.

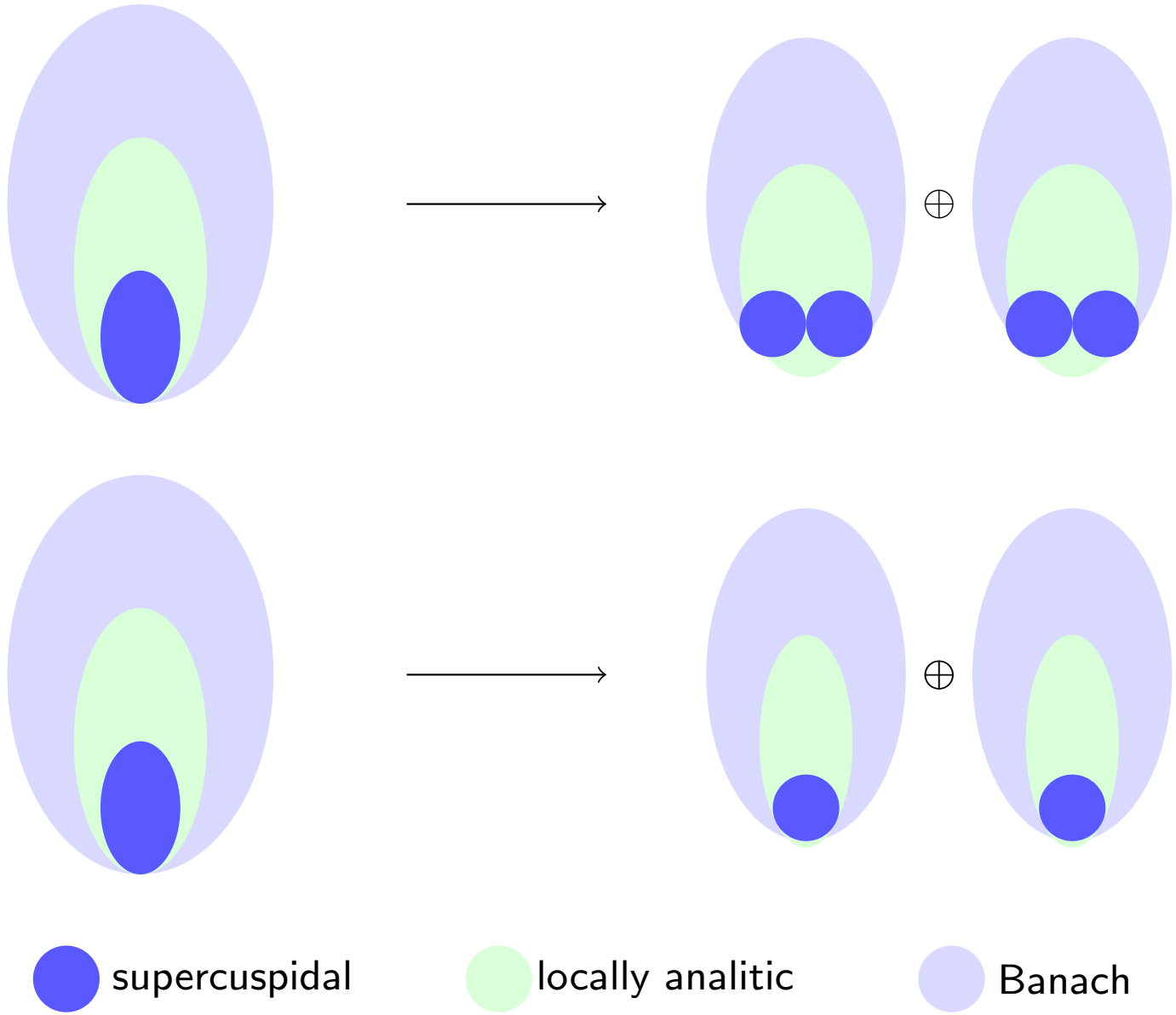
# The Non-trianguline Case

$\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$ is an absolutely irreducible de Rham representation with distinct Hodge-Tate weights
which is not trianguline over any finite extension of $E$.

# Restriction from $GL_2(\mathbb{Q}_p)$ to $SL_2(\mathbb{Q}_p)$: Standard non-trianguline cases



# Restriction from $GL_2(\mathbb{Q}_p)$ to $SL_2(\mathbb{Q}_p)$: Special non-trianguline cases



# Structure of $\Pi^{\mathrm{an}}$ in the Non-trianguline Case

Colmez, Dospinescu, and Niziol described $\Pi^{\mathrm{an}}$ using the $p$-adic cohomology of the Drinfeld tower. As in their work, we assume that $\psi$ is non-trianguline, with Hodge-Tate weights 0 and 1. We consider a rigid analytic space defined on Drinfeld coverings and prove that it decomposes as a disjoint union of two $H \times W_F$-invariant rigid analytic spaces.

Assume that $|S_{\overline{\psi}}| = 2$. Then, there is a quadratic extension $F/\mathbb{Q}_p$ such that

$$\psi = \mathrm{ind}_{\mathcal{G}_F}^{\mathcal{G}_{\mathbb{Q}_p}}(\chi_1)$$

We set

$$H^F = \{g \in G \mid \det(g) \in N_{F/\mathbb{Q}_p}(F^\times)\}, \quad D^F = \{x \in D^\times \mid \mathrm{Nrd}(x) \in N_{F/\mathbb{Q}_p}(F^\times)\},$$

which are subgroups of index two in $G$ and $D^\times$, respectively.

# Structure of $\Pi^{an}$ in the Non-trianguline Case

For a compact open subgroup $\Delta \subseteq D^\times$, define

$$\mathcal{M}_\Delta = \mathcal{M}_\infty/\Delta.$$

In the context of Drinfeld's $p$-adic upper half plane coverings, $\mathcal{M}_\Delta$ is a pro-rigid analytic space constructed as a quotient of an infinite-level covering space by a compact open subgroup $\Delta$.
$\mathcal{M}_\Delta$ is equipped with the action of $G \times D^\times \times W_{\mathbb{Q}_p}$

**Lemma 2**

*There is a compact open normal subgroup $\Delta \subset \mathcal{O}_D^\times$ such that*

$$\mathcal{M}_\Delta = \mathcal{M}_{\Delta,1}^F \sqcup \mathcal{M}_{\Delta,2}^F$$

*is the disjoint union, on the level of rigid analytic spaces, of $\mathcal{M}_{\Delta,1}^F$ and $\mathcal{M}_{\Delta,2}^F$. Each subspace $\mathcal{M}_{\Delta,i}^F$ is stable under the action of $H^F \times D^F \times W_F$.*

# Decomposition of $\Pi^{\mathrm{an}}|_H$ in the Non-trianguline Case

## Theorem 3

*Let $\Pi = \Pi(\psi)$, and $\pi = \Pi^{\mathrm{lalg}}$. Set $\rho = JL(\pi)$, and let $\Delta \subset \mathcal{O}_D^\times \cap \ker(\rho)$ be any compact open subgroup.*

*There is a canonical isomorphism of topological vector spaces*

$$\iota : (\Pi^{\mathrm{an}})'_b \xrightarrow{\simeq} \mathrm{Hom}_{E[W_{\mathbb{Q}_p}]}\left(\psi, H^1(\mathcal{M}_\Delta, E(1))^\rho\right)$$

*which is equivariant for the actions of $G$ and $H$. Furthermore, if $|S_{\overline{\psi}}| = 2$, and $F$ and $\chi_1$ are as above, then we have the following $H$-equivariant decomposition*

$$(\Pi^{\mathrm{an}})'_b \cong (\Pi'_{\mathrm{an}})_1 \oplus (\Pi'_{\mathrm{an}})_2,$$

*where*

$$(\Pi'_{\mathrm{an}})_i := \mathrm{Hom}_{E[W_F]}\left(\chi_1, H^1(\mathcal{M}^F_{\Delta,i}, E(1))^{\rho_1}\right),$$

*and*

$$\Pi^{\mathrm{an}}|_H \cong \Pi^{\mathrm{an}}_1 \oplus \Pi^{\mathrm{an}}_2,$$

*where $\Pi^{\mathrm{an}}_i := ((\Pi'_{\mathrm{an}})_i)'_b$ is a closed and admissible locally analytic subrepresentation of $\Pi^{\mathrm{an}}|_H$.*

# The Trianguline Case

$\psi : \mathcal{G}_{\mathbb{Q}_p} \to GL_2(E)$ is an absolutely irreducible trianguline and de Rham with distinct Hodge-Tate weights

In this case, the locally analytic vectors in $\Pi = \Pi(\psi)$ are built from locally analytic principal series representations:

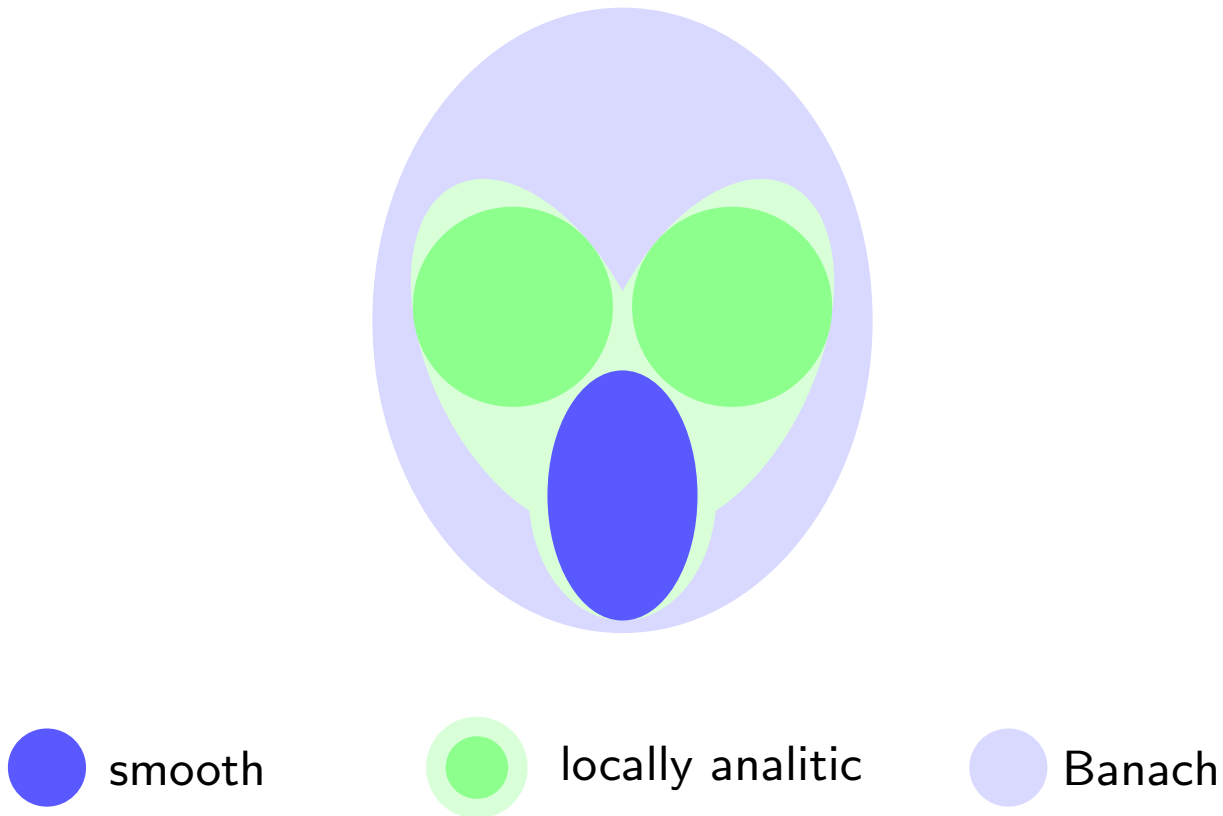
$$B^{\mathrm{an}}(\delta_1, \delta_2) = \mathrm{Ind}_P^G(\delta_2 \otimes \delta_1 \chi_{\mathrm{cyc}}^{-1})^{\mathrm{an}} ,$$

# Structure of $\Pi$ in the Trianguline Case

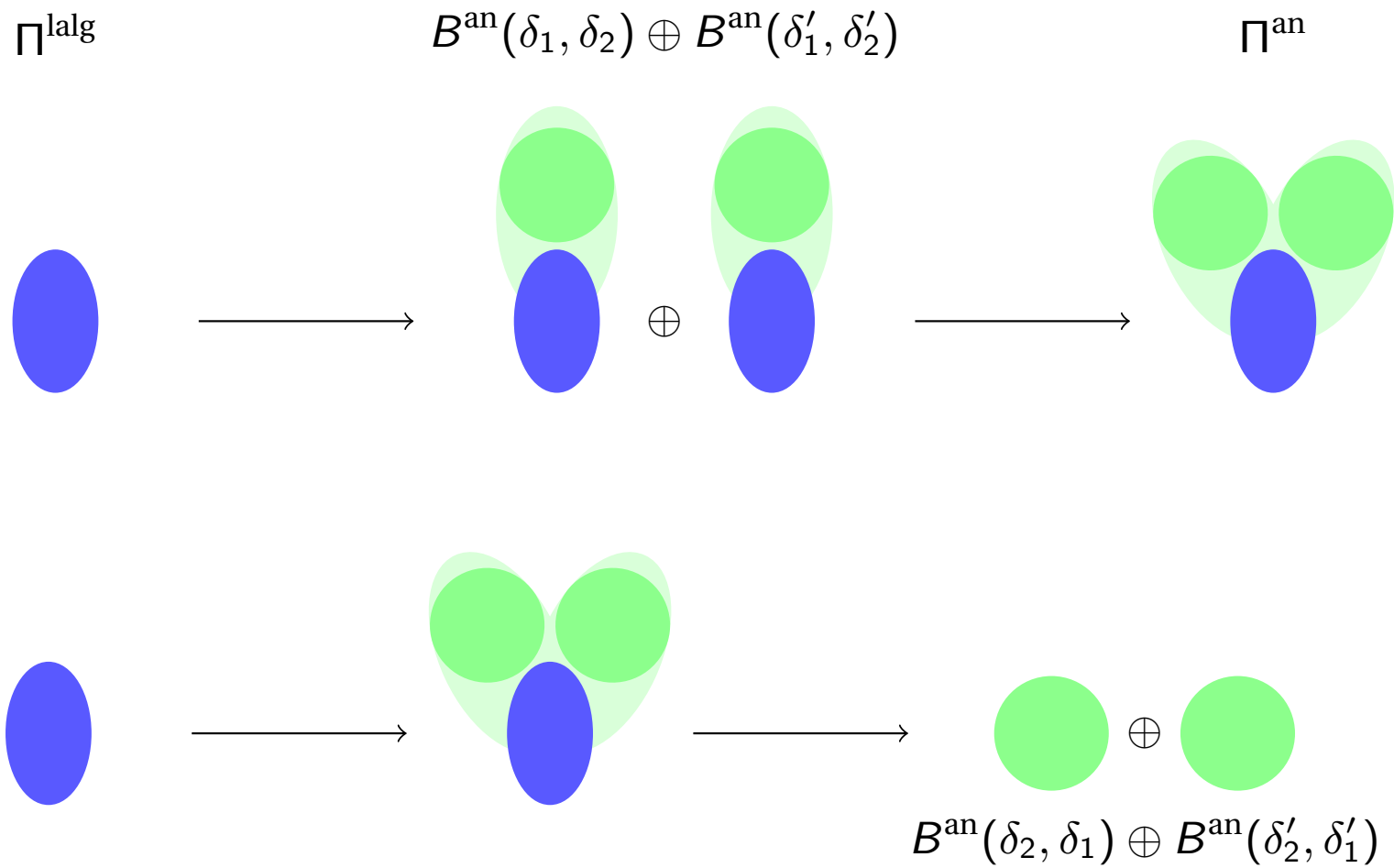
By [Colmez 2014], we have the following exact sequences

$$0 \longrightarrow \Pi^{\mathrm{lalg}} \longrightarrow B^{\mathrm{an}}(\delta_1, \delta_2) \oplus B^{\mathrm{an}}(\delta_1', \delta_2') \longrightarrow \Pi^{\mathrm{an}} \longrightarrow 0,$$

$$0 \longrightarrow \Pi^{\mathrm{lalg}} \longrightarrow \Pi^{\mathrm{an}} \longrightarrow B^{\mathrm{an}}(\delta_2, \delta_1) \oplus B^{\mathrm{an}}(\delta_2', \delta_1') \longrightarrow 0.$$



# Structure of $\Pi^{an}$ in pictures



# The restriction of $\Pi$ to $H$

## Theorem 4

*Let $s = (\delta_1, \delta_2, \infty) \in \mathcal{S}_*^{\mathrm{cris}}$ and $\Pi = \Pi(\psi(s))$. Suppose that $\Pi|_H = \Pi_1 \oplus \Pi_2$ decomposes as a direct sum of two absolutely irreducible representations.*

(i) *There are integers $c_1 \geq c_2$, a smooth character $\tau$ of $\mathbb{Q}_p^\times$, and a non-trivial quadratic character $\mathrm{sgn}_\theta$ such that*

$$\delta_1(x)\chi_{\mathrm{cyc}}^{-1} = x^{-c_2}|x|^{-\frac{1}{2}}\tau(x) \quad \text{and} \quad \delta_2(x) = x^{-c_1}|x|^{\frac{1}{2}}\mathrm{sgn}_\theta(x)\tau(x).$$

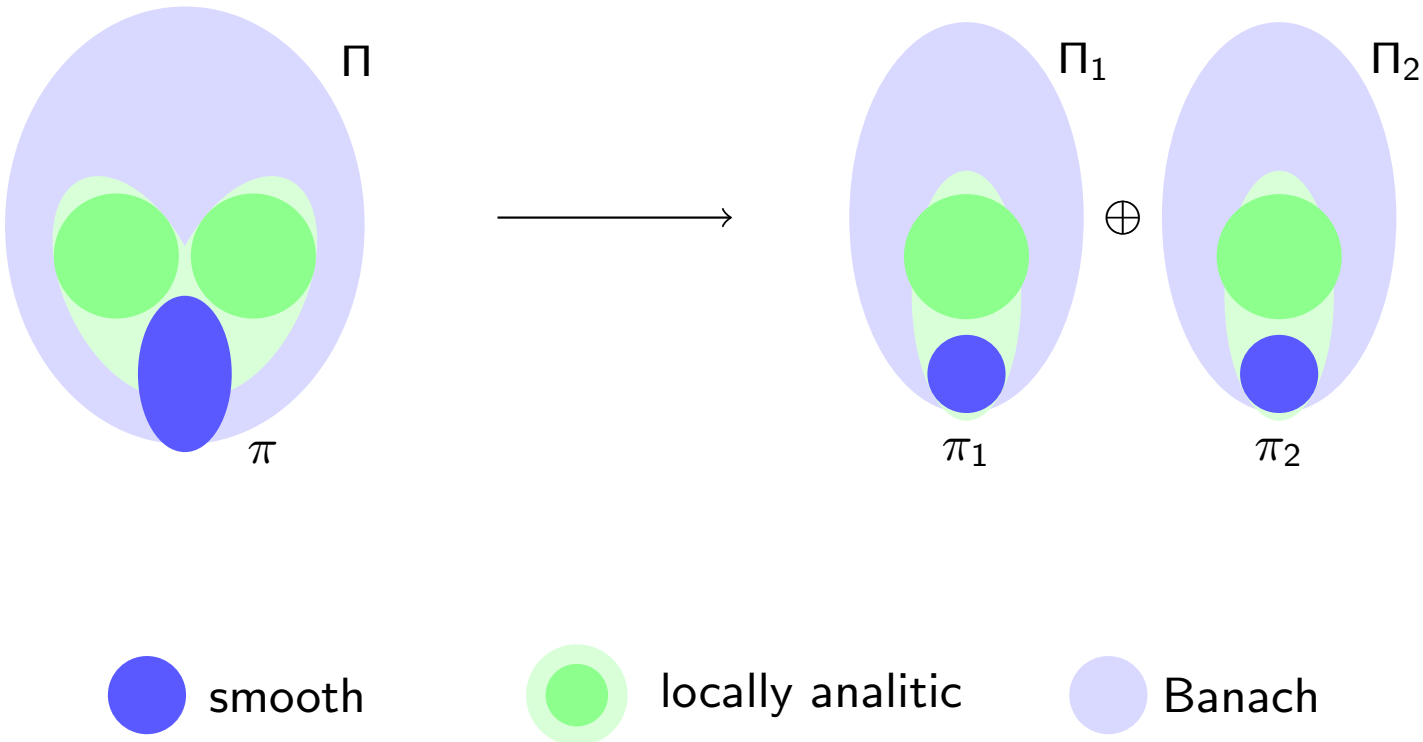
(ii) *Let $W = \Pi^{\mathrm{an}} = B^{\mathrm{an}}(\delta_1, \delta_2) \oplus_{\Pi^{\mathrm{lalg}}} B^{\mathrm{an}}(\delta_1', \delta_2')$ and $\eta' = \alpha^{c_1 - c_2 + 1}(\delta_2 \otimes \delta_1\chi_{\mathrm{cyc}}^{-1})$. The restriction of $W$ to $H$ decomposes as $W|_H = W_1 \oplus W_2$, where for $i = 1, 2$, $W_i$ fits in the following exact sequence of $H$-modules*

$$0 \to W_i^{\mathrm{lalg}} \to W_i \to \mathrm{Ind}_{P_H}^H(\eta')^{\mathrm{an}} \to 0.$$

*The $H$-modules $W_i^{\mathrm{lalg}}$ and $\mathrm{Ind}_{P_H}^H(\eta')^{\mathrm{an}}$ are irreducible and $W_1^{\mathrm{lalg}} \not\cong W_2^{\mathrm{lalg}}$. Moreover, for any $g \in G$ such that $\mathrm{sgn}_\theta(\det(g)) = -1$ we have $g \cdot W_1 = W_2$ and $g \cdot W_2 = W_1$.*

# The restriction of $\Pi$ to $H$ in pictures

$\Pi|_H$ decomposes as follows:



# The End

Hvala!