Regular Higher Power Diophantine Triples

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Diophantine m-tuples

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Dujella proved that the size m of an integer Diophantine m-tuple is at most 5 and in fact later it was proved that an integer Diophantine quintuple is impossible.
Finally, we note that parametric families of rational Diophantine sextuples have been found, but not one example of a septuple.

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- for $k = 2$, $m \leq 4$,
- for $k = 3$, $m \leq 7$,
- for $k = 4, 5$, $m \leq 5$,
- for $6 \leq k \leq 176$, $m \leq 4$,
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No other higher power integer triples are known. (There are three more triples if we allow negative numbers for $k = 3$, namely $(1, -2, 63)$, $(4, -7, 31)$, $(36, -37, 35973)$.

Standard Notation

Let us fix the usual notation for Diophantine triples $\{a, b, c\}$, namely

$$ab + 1 = r^k,$$
$$ac + 1 = s^k,$$
$$bc + 1 = t^k.$$

Newer results

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$$\{a, b, c\} = \left\{ -\frac{1}{r^{k/2}}, \frac{r^k - 1}{r^{k/2}}, r^{k/2} \right\}$$

for k even, and similarly

$$\{a, b, c\} = \left\{ -\frac{1}{r^{(k-1)/2}}, \frac{r^k - 1}{r^{(k+1)/2}}, r^{(k+1)/2} \right\}.$$

for k odd.

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There exist infinitely many cubic Diophantine quadruples.

The authors demonstrate this by finding a rational parametrization for such quadruples.

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Definition

A higher power Diophantine triple (for even k) is *regular* if $a = s^{k/2} - r^{k/2}$ (or equivalently, $c = a + b + 2r^{k/2}$).

In particular, we will concentrate on the case $k = 4$ and try to find some interesting examples of families of triples.

The Equation for Regular Quartic Triples

From the first two equations, take $b = \frac{r^4-1}{a}$, $c = \frac{s^4-1}{a}$. The third equation becomes

$$\frac{r^4-1}{a} \cdot \frac{s^4-1}{a} + 1 = t^4.$$

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In general, choosing random numbers a, r, this will give an equation of genus 3. But if we take a regular triple i.e. $a = s^2 - r^2$, then this reduces to

$$\left(\frac{s^2r^2-1}{s^2-r^2}\right)^2 = t^4.$$

Upon taking roots, we get an elliptic curve in s, t (for fixed r).

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Note that for $r=337$ and $s=339$, then $t=3107$ and vice-versa. This gives the already known integer triple. Unfortunately, for the remaining pair, the corresponding t is not an integer.

Fixing the difference between s and r

Note that in our only integer solution we have $s - r = 2$. Set $s = r + \alpha$. Then our main equation becomes

$$t^2 = \frac{s^2 r^2 - 1}{s^2 - r^2} = \frac{r^2(r+\alpha)^2 - 1}{\alpha(2r+\alpha)}$$
$$= \frac{\left((2r+\alpha)^2 - \alpha^2 + 4\right)\left((2r+\alpha)^2 - \alpha^2 - 4\right)}{16\alpha(2r+\alpha)}$$

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Note that the only way cancellation can occur is if $\alpha = \pm 2$.

A very special case

Then for $\alpha = 2$, one gets the elliptic curve

$$t^2 = \frac{1}{4}(r+1)(r^2+2r-1).$$

This a curve of rank 1 with torsion $\mathbb{Z}/2\mathbb{Z}$. The only integer point that gives a non-trivial solution is $r = 337$ giving $t = 3107$.

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Note: The above elliptic curve can be written as $y^2 = x^3 - 2x$ or $y^2 = 2x^4 - 1$, giving the relation $239^2 + 1 = 2 \cdot 169^2$. This is used in certain Machin-like formulas. Also, $3107 = 13 \cdot 239$.

A (complex) sidenote

Note that, if one is willing to extend the problem to Gaussian integers, $\alpha = \pm 2i$ can be chosen in

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so that cancellation occurs. This again leads to the curve just mentioned (but over $\mathbb{Q}(i)$). Now it has rank 2 and one can find a couple Gaussian integer triples, namely

$$(28+4i, 42+24i, 140+52i),$$
$$(15-10i, 15+10i, 16i).$$

The Generic Case

Now we consider the case with generic α (r is a parameter here).

$$\frac{r^2(r+\alpha)^2-1}{\alpha(2r+\alpha)}=t^2.$$

[1]Note the form $y^2=x(x+(r^2-1)^2)(x+(r^2+1)^2)$ which guarantees a rational point of order 4. But a rational point of order 8 is impossible.

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where $x=rs=\alpha r+r^2$. And in Weierstrass form as[1]

$$E_r \quad : \quad y^2=(x-2r^2)(x+2r^2)(x-r^4-1).$$

This curve generically has rank 0.

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The Generic Case Weierstrass Equation

For a given r, if we find a point (x, y) on E_r it will correspond to

$$s = \frac{r(x-2)}{x-2r^4}, \quad t = \frac{4r^4+2r^2y-x^2}{2(2r^6-2r^2x+2r^2+y)r}$$

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It is to be noted that adding torsion points on the elliptic curve will correspond to changing the signs of s, t, taking reciprocals and switching s and t.

The Search for Good r

We conducted a computer search for r such that the curve has non-zero rank. With this we certainly found many (a, b, c) with small denominators, but never integers.

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- for $r = 10864$, where $s = -236052992/18817$ and $t = 236052993/10864$,
- for $r = 40545$, where $s = -1643897025/35113$ and $t = 3287794051/40545$.

Comparatively, for other r of similar size, the denominators were usually about[2] size of r^2.

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Continued fraction for $\sqrt{3}$

Note the relation between the denominators appearing in the previous slide

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$$p^2 - 3r^2 = 1.$$

From integers to rationals via $\sqrt{3}$

If we assume then that r is such that $r^2 = (p^2 - 1)/3$, we will find an extra point on E_r, namely $(r^2 + 1, pr(1 - r^2))$.[3]

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$$r = \frac{2u}{3 - u^2},$$

with corresponding s, t given by

$$\pm\frac{u^4 - 9}{8u^2}, \pm\frac{2u(u^2 - 3)}{u^4 + 2u^2 + 9}$$

as well as their reciprocals.

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as well as their reciprocals. It turns out that r can be taken to be a square in infinitely many ways.

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$$(u^2+1)^2r^4-(u^2-1)^2=\square$$

In Weierstrass form this is

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This has the generic point $P=(2(u^2+1)(u-1)^2,4(u^2+1)^2(u-1)^2)$. This gives trivial solutions with $r=1$, but $2P=(4u^2,4u(u^4+1))$ gives

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$$r=\frac{u^4+1}{2u(u^2+1)},\quad s=\frac{u^4+1}{2u(u^2-1)}$$

Choosing $\frac{1+\sqrt{3}-\sqrt{2\sqrt{3}}}{2}<u<1+\frac{1+\sqrt{3}+\sqrt{2\sqrt{3}}}{2}$, will give positive a,b,c.

Even more parametrizations

Another option is to consider when $a = s^2 - r^2 = k\square$. (For example, when $k = 2$, we obtain $(r, s) = (337, 339)$. Then $s = \frac{ku^2+1}{ku^2-1} r$

$$(ku^2 + 1)^2 r^4 - (ku^2 - 1)^2 = k\square$$

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$$(ku^2 + 1)^2 r^4 - (ku^2 - 1)^2 = k\square$$

In Weierstrass form this is

$$y^2 = x^3 + 4k^2(k^2u^4 - 1)^2 x.$$

This has a point of infinite order $P = (4k^2u^2, 4k^2u(k^2u^4 + 1))$. When $k = 1$, this was a double point.

Even more parametrizations

Another option is to consider when $a = s^2 - r^2 = k\square$. (For example, when $k = 2$, we obtain $(r, s) = (337, 339)$. Then $s = \frac{ku^2+1}{ku^2-1} r$

$$(ku^2 + 1)^2 r^4 - (ku^2 - 1)^2 = k\square$$

In Weierstrass form this is

$$y^2 = x^3 + 4k^2(k^2u^4 - 1)^2 x.$$

This has a point of infinite order $P = (4k^2u^2, 4k^2u(k^2u^4 + 1))$. When $k = 1$, this was a double point. But here it's not and so it gives trivial r!

Even More Parametrizations

Still, we can take $2P$ and arrive at some wild parametrization with

$$r = \frac{(ku^2+1)^4 - 12k^2u^4}{(ku^2-1)^4 - 12k^2u^4}$$

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giving complicated a, b, c. Note that a simple example would be given when $ku^2 - 1 = 1$, which leads to $k = 2\square$. Finally, I note that writing $\beta = \frac{kt^2+1}{kt^2-1}$ i.e. $s = \beta r$, we get

$$r = \frac{\beta^4 + 6\beta^2 - 3}{1 + 6\beta^2 - 3\beta^4},$$

so we may have started with this as well.

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- What about similar results for $D(n)$-triples?

The End

Thank you for your attention.