Recent progress on tensor categories of vertex operator algebras

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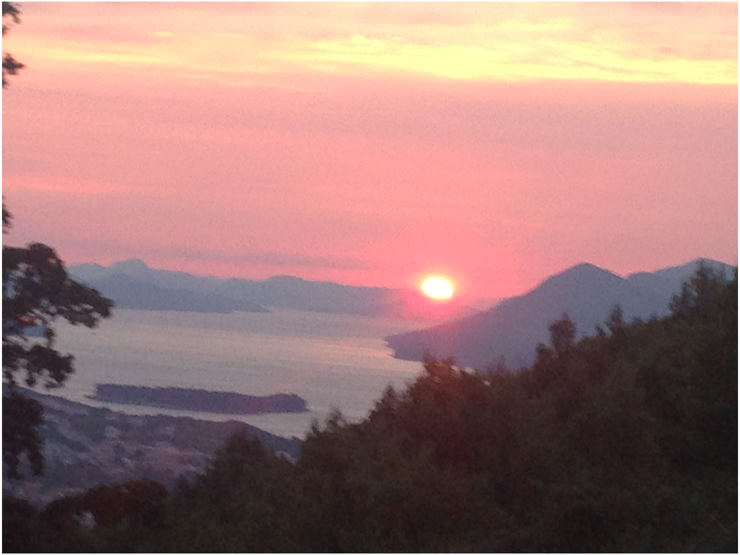
Sunset in Dubrovnik, June 26, 2013



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- Rigidity: Differential equations.
- Modularity (non-degeneracy).

Rational vertex operator algebras

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Question: Whether there is a braided tensor category structure on an appropriate V-module category if V is NOT rational or NOT C_2-cofinite? There are lots of such VOAs!

Finite length modules

- Finite length: W is of *finite length* if there is a filtration

$$0 = W_0 \subset W_1 \subset \cdots \subset W_{n-1} \subset W_n = W$$

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- The category of finite length modules is closed under the operation of direct sum, taking submodules, quotient, and contragradient duals.
- It might not be closed under taking fusion products, and the *convergence and extension property* required by Huang-Lepowsky-Zhang is also not clear.

C_1-cofinite modules

- C_1-cofiniteness condition: Let

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- It might not be closed under the operation of taking submodules and contragradient duals.
- If the category of lower bounded C_1-cofinite modules is the same as the category of finite length modules, then the category of finite length modules has braided vertex tensor category structure.

Existence of vertex tensor categories

For the following (non-rational) VOAs, the category of lower bounded C_1-cofinite modules is the same as the category of finite length modules:

- Universal and simple Virasoro VOAs at all central charges (Creutzig-Jiang-Orosz-Ridout-Y., 2021);

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- Singlet algebras (Creutzig-McRae-Y., 2023);
- **Conjecture**: Under minor assumptions on the VOA, the category of lower bounded C_1-cofinite modules is the same as the category of finite length modules.

Vertex operator algebras associated to $\widehat{\mathfrak{g}}$

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$$V_{\widehat{\mathfrak{g}}}(k, \lambda) := U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}\otimes\mathbb{C}[t]\oplus\mathbb{C}\mathbf{k})} E^\lambda,$$

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- If $k \neq -h^\vee$, $V_{\widehat{\mathfrak{g}}}(k, 0)$ and $L_{\widehat{\mathfrak{g}}}(k, 0)$ have vertex operator algebra structures. We denote them by $V_k(\mathfrak{g})$ and $L_k(\mathfrak{g})$.

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- $k \in \mathbb{N}$, $L_k(\mathfrak{g})$ is rational and C_2-cofinite, called *Wess-Zumino-Witten models*.

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Tensor categories of affine vertex operator algebras

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Theorem (Creutzig-Huang-Y., 2018)

Assume k is admissible, i.e., $k + h^\vee = \frac{q}{p}$, $q \geq h^\vee$ if $(r^\vee, q) = 1$, and $q \geq h$ if $(r^\vee, q) = r^\vee$. Then KL_k has BTC structure.

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Theorem (Creutzig-Y., 2021)

If $V_{\widehat{\mathfrak{g}}}(k, \lambda)$ is of finite length, then KL_k has BTC structure.

Kazhdan-Lusztig category

The general theorem provides a few NEW families of $KL_k(\mathfrak{g})$ with BTC structures.

- k ***generic***, generalized Verma modules are irreducible: $\mathfrak{sl}_2$, $k = -2 + \frac{1}{p}$ for $p \geq 1$ (Creutzig).

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- k ***collapsing level*** for minimal W-algebra, KL_k is semisimple:

$\mathfrak{g}$	A_ℓ	$A_{2\ell-1}$	B_ℓ	C_ℓ	D_ℓ	$D_{2\ell-1}$	E_6	E_7	F_4
k	-1	$-\ell$	-2	$-1-\frac{\ell}{2}$	-2	$-2\ell+3$	-4	-6	-3

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- k ***collapsing level*** for non-minimal W-algebra, KL_k is semisimple:
  $\mathfrak{g} = A_{2n-1}$, $k = -\frac{2n+1}{2}$
  (Adamović-Creutzig-Perše-Vukorepa, 2022).

- k such that minimal W-algebra $W_k(\mathfrak{g}, \theta)$ is C_2-cofinite.
  $\mathfrak{g} = D_4, E_6, E_7, E_8$, $k \geq \frac{-h^\vee}{6} - 1$, $k \in \mathbb{Z}$;
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- Many more to study.

Tensor category for VOA extensions

Let V be a VOA, and $\mathcal{C}$ be a category of V-modules with a natural vertex tensor category structure.

- A vertex operator (super)algebra extension $V \subset A$ is equivalent to a commutative associative algebra in $\mathcal{C}$ (Huang-Kirillov-Lepowsky, 2015);

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- The induction functor $\mathcal{C} \to \mathcal{C}_A^{loc}$ is a braided tensor functor (Creutzig-Kanade-McRae, 2017).
- Usually, we need to replace $\mathcal{C}$ in the above theorems by the direct limit completion of $\mathcal{C}$ (Creutzig-McRae-Y., 2021).

VOA extensions: singlet algebra

- VOA extensions: $L(c_{1,p}, 0) \subset$ *Singlet algebra* $\mathcal{M}(p) \subset$ *Triplet algebra* $\mathcal{W}(p)$ (Adamović 2003, 2005; Adamović-Milas 2007, 2008, 2009, 2017; Creutzig-Gainutdinov-Runkel, 2020, Creutzig-Milas, 2014,2017; Creutzig-Ridout-Wood, 2014, etc.).

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- The category of finite length modules for $\mathcal{M}(p)$ has a ribbon category structure (Creutzig-McRae-Y., 2023).

Extensions of singlet algebra times Heisenberg

- The vertex algebra $\mathcal{B}_p$:

$$\mathcal{B}_p = \begin{cases} \beta\gamma\text{-ghost vertex algebra} & \text{if } p = 2 \\ L_{-4/3}(\mathfrak{sl}_2) & \text{if } p = 3 \\ \text{subregular } W\text{-algebra of } \mathfrak{sl}_{p-1} & \text{if } p \geq 3. \end{cases}$$

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- The algebra $\mathcal{S}_p$: the simple principal W-superalgebra of $\mathfrak{sl}_{p-1|1}$ at level $-(p-2) + \frac{p}{p-1}$ (Creutzig-McRae-Y., 2023).

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- Level 1 affine superalgebras.

Affine vertex superalgebra $V_k(\mathfrak{gl}(1|1))$

- Lie superalgebra $\mathfrak{gl}(1|1)$ is spanned by

$$N = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \psi^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

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- Representation theory of affine superalgebra $V_k(\mathfrak{gl}(1|1))$ (Creutzig-Ridout 2013).
  - Let $\widehat{V}_{n,e}$ be the generalized Verma module whose top space is the Verma $\mathfrak{gl}(1|1)$-module generated by a highest weight vector v such that

$$N \cdot v = (n + 1/2)v, \quad E \cdot v = ev, \quad \psi^+ \cdot v = 0.$$

  - Irreducible modules for $V_k(\mathfrak{gl}(1|1))$:
    Typical modules: $\widehat{V}_{n,e}$ for $e/k \notin \mathbb{Z}$
    Atypical modules: $\widehat{A}_{n,e}$ for $e/k \in \mathbb{Z}$

- Fusion products:

$$\widehat{A}^k_{n,\ell k} \boxtimes \widehat{A}^k_{n',\ell' k} \cong \widehat{A}^k_{n+n'-\varepsilon(\ell,\ell'),(\ell+\ell')k}$$

$$\widehat{V}^k_{n,e} \boxtimes \widehat{V}^k_{n',e'} \cong \begin{cases} \widehat{V}^k_{n+n'+\frac{1}{2},e+e'} \oplus \widehat{V}^k_{n+n'-\frac{1}{2},e+e'} & \text{if} \quad (e+e')/k \notin \mathbb{Z} \\ \widehat{P}^k_{n+n'+\varepsilon((e+e')/k),e+e'} & \text{if} \quad (e+e')/k \in \mathbb{Z} \end{cases}.$$

$KL_k(\mathfrak{gl}(1|1))$

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- $\widehat{P}_{n,e}$ is a projective cover of $\widehat{A}_{n,e}$ with Loewy diagram:

$$\widehat{P}_{n,e}: \quad \begin{array}{ccccc} & & \widehat{A}_{n,e} & & \\ & \nearrow & & \nwarrow & \\ \widehat{A}_{n-1,e} & & & & \widehat{A}_{n+1,e} \\ & \nwarrow & & \nearrow & \\ & & \widehat{A}_{n,e} & & \end{array}$$

Rigidity of $KL_k(\mathfrak{gl}(1|1))$

Theorem (Creutzig-McRae-Y., 2021)

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First derive *super Knizhnik-Zamolodchikov equation*:

$$z(1-z)\phi''(z) + [(4\Delta+1) - (8\Delta+1)z]\phi'(z)$$
$$+\ 4\Delta^2 z^{-1}\phi(z) + 2\Delta(2\Delta-1)(1-z)^{-1}\phi(z)$$
$$+\ [(e/k)^2 - 16\Delta^2]\phi(z) = 0.$$

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Substitute $f(z) = z^{2\Delta}(1-z)^{2\Delta}\phi(z)$, then $f(z)$ satisfies

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Connection formula of solutions to above equation gives the rigidity composition:

$$f_{\widehat{V}_{n,e}} = \frac{\sin(\pi e/k)}{\pi e/k} \neq 0 \quad \text{if } e/k \notin \mathbb{Z}.$$

Thank you for listening to my talk!