Duality structures on tensor categories coming from vertex operator algebras

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Overview of HLZ tensor structure [Huang-Lepowsky-Zhang]

Definition: Fusion product (vertex algebra tensor product)

Let V be a vertex operator algebra and let $M,\ N$ be V-modules. The fusion product of M and N is a V-module $M \boxtimes N$ together with an intertwiner $\mathcal{Y}^{M\boxtimes N} \in \binom{M\boxtimes N}{M,\ N}$ satisfying the following universal property:
For any module X and any intertwining operator $\mathcal{I} \in \binom{X}{M,\ N}$, there exists a unique $f \in \mathrm{Hom}_V(M \boxtimes N, X)$

$$\begin{array}{ccc} M \otimes_{\mathbb{C}} N & \xrightarrow{\ \mathcal{Y}^{M,N}\ } & M \boxtimes N\{z\} \\ & \searrow^{\mathcal{I}} & \downarrow{\scriptstyle \exists! f} \\ & & X\{z\} \end{array}$$

Due to being characterised by a universal property, tensor products are unique, if they exist. The construction (existence) of tensor products is hard because vertex operator algebras do not admit free modules.

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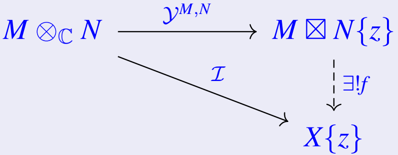

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Overview of HLZ tensor structure [Huang-Lepowsky-Zhang]

Let V be a vertex operator algebra and let $\mathcal{C}$ be a category of V-modules to which HLZ theory applies.

- For module homomorphimsms $f : M \to U$, $g : N \to W$, the morphism $f \boxtimes g$ is uniquely characterised by $(f \boxtimes g) \circ \mathcal{Y}^{M\boxtimes N} = \mathcal{Y}^{U\boxtimes W} \circ (f \otimes g)$
- V is the tensor identity and the unit isomorphisms are uniquely characterised by
  $\ell_M \left(\mathcal{Y}^{V,M}(a,z)m\right) = Y^M(a,z)m$ and
  $r_M \left(\mathcal{Y}^{M,V}(m,z)a\right) = e^{zL_{-1}} Y^M(a,-z)m$.
- associativity isomorphisms (hardest part!)
  $A_{M,N,R} \left(\mathcal{Y}^{M,N\boxtimes R}(m,x_1)\mathcal{Y}^{N,R}(n,x_2)r\right) = \mathcal{Y}^{M\boxtimes N,R}(\mathcal{Y}^{M,N}(m,x_1-x_2)n,x_2)r$
  All analytic details hidden.
- Braiding isomorphisms uniquely characterised by
  $c_{M,N} \left(\mathcal{Y}^{M,N}(m,x)n\right) = e^{xL_{-1}}\mathcal{Y}^{N,M}(n,e^{i\pi}x)m$

Features of tensor categories from vertex algebra modules

A category of modules over a vertex operator algebra need not

- be semisimple,
- be finite,
- have a self-contragredient tensor unit (the vertex operator algebra),
- have a simple tensor unit,
- have a left exact tensor product (right exactness is automatic),
- be rigid.

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Example: rational vertex operator algebras

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Example: Heisenberg algebra (no lattice)

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Example: bosonic ghosts or affine algebras at admissible non-integral levels

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Example: $(1,p)$-triplet, $p \geq 2$

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Example: (p, q)-triplet, $p, q \geq 2$

Rigidity

Since rigidity is a property (unique, if it exists) it is in practice very hard to verify and there are only a few families of non-rational examples for which this has been done.

- $(1,p)$-triplet [Tsuchiya-SW]
- Bosonic ghosts [Allen-SW]
- Virasoro at generic central charges [Creutzig-Jiang-Orosz Hunziker-Ridout-Yang]
- $(1,p)$-singlet [Creutzig-McRae-Yang]
- B_p and related algebras [Creutzig-McRae-Yang]
- $\mathfrak{gl}(1|1)$ [Creutzig-McRae-Yang]

It would be helpful to have a weaker notion of duality that is a structure rather than a property.

Grothendieck-Verdier or $*$-autonomous categories

Definition: Grothendieck-Verdier (GV) category

Let $\mathcal{C}$ be a monoidal category. An object $K \in \mathcal{C}$ is called *dualising* if

1. The functor $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(- \otimes Y, K)$ is representable, that is, $\exists GY \in \mathcal{C}$ such that
$$\varpi_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, K) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(X, GY).$$
Note this defines a contravariant functor $G : Y \mapsto GY$.
2. The functor characterised above is an anti-equivalence.
Let be G^{-1} be a choice of quasi-inverse.

(Ribbon) Grothendieck-Verdier structure on categories of vertex operator algebra modules

Theorem

Let V be a vertex operator and let $\mathcal{C}$ be a category of V-modules to which HLZ theory applies (in particular $V \in \mathcal{C}$, $\mathcal{C}$ is closed under contragredients and taking contragredients is involutive). Then

1. V^* is a dualising object with $X \to X^*$ as dualising functor.
2. $(\mathcal{C}, K)$ is ribbon Grothendieck-Verdier. (The V-module braiding c and twist θ satisfy $\theta_X = G^{-1}(\theta_{GX})$ and $\theta_{X \otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y)$.)

(Ribbon) Grothendieck-Verdier structure on categories of vertex operator algebra modules

Proof

Recall the natural isomorphism

$$\mathrm{Hom}_V(X \boxtimes Y, Z) \cong \binom{Z}{X,\ Y} \to \binom{Y^*}{X,\ Z^*} \cong \mathrm{Hom}_V(X \boxtimes Z^*, Y^*).$$

$$\mathcal{Y} \mapsto \text{“}x \otimes \zeta \mapsto \zeta(\mathcal{Y}(e^{zL_1}(-z^{-2})^{L_0}x, z^{-1})-)\text{”}$$

Set $Z = V^*$, then

$$\mathrm{Hom}_V(X \boxtimes Y, V^*) \cong \mathrm{Hom}_V(X \boxtimes V^{**}, Y^*) \cong \mathrm{Hom}_V(X, Y^*).$$

Heisenberg example

Let

- $F_0^\rho, \rho \in \mathbb{R}$ be the rank 1 Heisenberg vertex algebra, with conformal vector $(\frac{1}{2}a_{-1}^2 + \rho a_{-2})\left|0\right\rangle, \rho \in \mathbb{R}$ and central charge $c = 1 - 12\rho^2$.
- Choose $\mathcal{C} = F_0 - \text{mod}$ to be (semisimply) generated by F_μ, $\mu \in \mathbb{R}$.

Then

- $F_\mu \boxtimes F_\nu \cong F_{\mu+\nu}$, $\mu, \nu \in \mathbb{R}$.
- $c_{\mu,\nu} = \mathrm{e}^{\mathrm{i}\pi\mu\nu}$.
- $\theta_\mu = \mathrm{e}^{\mathrm{i}\pi\mu(\mu-2\rho)}$.
- $F_\mu^* \cong F_{2\rho-\mu}$.

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- $\theta_\mu = \mathsf{e}^{\mathsf{i}\pi\mu(\mu-2\rho)}$. Depends on choice of conformal structure.
- $F_\mu^* \cong F_{2\rho-\mu}$. Depends on choice of conformal structure.

Bimodule example

Let A be a finite dimensional algebra and (A, A)-mod the category of finite dimensional (A, A)-bimodules. Then

- (A, A)-mod is monoidal with the tensor product characterised by the coequaliser

$$M \otimes_{\mathbb{C}} A \otimes_{\mathbb{C}} N \longrightarrow M \otimes_{\mathbb{C}} N \underset{\rho}{\overset{\lambda}{\rightrightarrows}} M \otimes_A N \longrightarrow 0.$$

- $A^* = \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C})$ is a dualising object for (A, A)-mod, with the vector space dual as dualising functor.

Known features of Grothendieck-Verdier categories

Proposition: [Boyarchenko-Drinfeld, Barr]

Let $(\mathcal{C}, \otimes)$ be monoidal.

1. The full subcategory of dualising objects is a torsor over the invertible objects.
2. Dualising functors need not be monoidal, but their squares are.
3. For any $X, Y \in \mathcal{C}$, $X \otimes -$ and $- \otimes Y$ admit right adjoints (internal homs):

$$\mathrm{Hom}_{\mathcal{C}}(Y, \underline{\mathrm{Hom}}^{\mathrm{l}}(X,Z)) \cong \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}^{\mathrm{r}}(Y,Z)),$$

$$\underline{\mathrm{Hom}}^{\mathrm{l}}(X,Z) = G^{-1}(GZ \otimes X), \qquad \underline{\mathrm{Hom}}^{\mathrm{r}}(Y,Z) = G(Y \otimes G^{-1}Z).$$

4. $\mathcal{C}$ admits a second monoidal product $X \bullet Y = G(G^{-1}Y \otimes G^{-1}X)$, which admits left adjoints and internal cohoms

$$\mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{coHom}}^{\mathrm{l}}(Y,X),Z)) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y \bullet Z)$$
$$\cong \mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{coHom}}^{\mathrm{r}}(X,Z),Y),$$

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The second tensor product for bimodules

Recall the (A, A)-bimodule tensor product characterised by the coequaliser

$$M \otimes_{\mathbb{C}} A \otimes_{\mathbb{C}} N \longrightarrow M \otimes_{\mathbb{C}} N \underset{\rho}{\overset{\lambda}{\rightrightarrows}} M \otimes_A N \longrightarrow 0.$$

Set $M = Y^*$, $N = X^*$ and take the dual of the sequence above (and identify $(C \otimes_{\mathbb{C}} D)^* \cong D^* \otimes_{\mathbb{C}} C^*$. This is the equaliser

$$X \otimes_{\mathbb{C}} A^* \otimes_{\mathbb{C}} Y \longleftarrow X \otimes_{\mathbb{C}} Y \underset{\lambda^t}{\overset{\rho^t}{\leftleftarrows}} X \otimes^A Y \longleftarrow 0.$$

That is, $X \otimes^A Y \cong (Y^* \otimes_A X^*)^*$ is the second GV tensor product. It is also the cotensor product of (A, A)-bicomodules.

Motivations from the triplet algebra

In [Gaberdiel-Runkel-SW '09,'10, '12] bulk CFT constructions were studied for the triplet algebra $W = W(2,3)$ at $c = 0$.

- W is C_2-cofinite, 13 simple modules. [Adamovic-Milas]
- W admits the non-split exact sequence [Feigin-Gainutdinov-Semikhatov-Tipunin]

  $$0 \longrightarrow S(2) \longrightarrow W \longrightarrow S(0) \longrightarrow 0.$$

  In particular, $W^* \ncong W$.
- $S(0)$ is not flat [Gaberdiel-Runkel-SW '09].
- The projective cover of $S(0)$ is not rigid [Gaberdiel-Runkel-SW '10].
- Boundary CFT requires associative boundary algebras with non-degenerate evaluations, injective coevaluations. This distinguishes a subcategory $\mathcal{B}$ of objects satisfying $\forall X, Y \in B$, $X \otimes Y^* \cong X \bullet Y^*$. [Gaberdiel-Runkel-SW '09]

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Distributors

- It is claimed in the literature that GV categories are the same as linear distributive categories with a negation.
- Unfortunately, the literature on this is sparse and poorly codified.
- Linear distributive categories with a negation admit distributors $\partial^l : X \otimes (Y \bullet Z) \to (X \otimes Y) \bullet Z$ and $\partial^r : (X \bullet Y) \otimes Z \to X \bullet (Y \otimes Z)$. These have interesting properties such as mixed associator pentagons.

$$
\begin{array}{ccccc}
 & & (V \otimes X) \otimes (Y \bullet Z) & & \\
 & \swarrow^{\partial^l} & & \searrow^{\alpha} & \\
((V \otimes X) \otimes Y) \bullet Z & & & & V \otimes (X \otimes (Y \bullet Z)) \\
\downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle \partial^l} \\
(V \otimes (X \otimes Y)) \bullet Z & & \xleftarrow{\quad \partial^l \quad} & & V \otimes ((X \otimes Y) \bullet Z)
\end{array}
$$

Distributors

Lemma [Shimizu]

Let $\mathcal{C}$ be linear monoidal with left module categories $(\mathcal{M}, \triangleright)$, $(\mathcal{N}, \triangleright)$ and let $F : \mathcal{M} \to \mathcal{N}$ be a linear functor with right adjoint $G : \mathcal{N} \to \mathcal{M}$. Then the oplax $\mathcal{C}$-module structures on F are in bijection with lax $\mathcal{C}$-module structures on G. That is

$$\{\text{lax}: \; X \triangleright G(Y) \to G(X \triangleright Y)\} \leftrightarrow \{\text{oplax}: F(X \triangleright Y) \to X \triangleright F(Y)\}$$

Definition

Let $(\mathcal{C}, K)$ be GV and consider the adjoint pair of $\mathcal{C}$-module functors $R_U(-) = - \otimes U : \mathcal{C} \to \mathcal{C}$ and $\underline{\mathrm{Hom}}^{\mathrm{r}}(U, -) : \mathcal{C} \to \mathcal{C}$. The associator of $\mathcal{C}$ is a strong module functor structure on R_U.
Let ${}^{r}\delta^{U}_{X,Y} : X \otimes \underline{\mathrm{Hom}}^{\mathrm{r}}(U, Y) \to \underline{\mathrm{Hom}}^{\mathrm{r}}(U, X \otimes Y)$ be the corresponding lax $\mathcal{C}$-module structure on $\underline{\mathrm{Hom}}^{\mathrm{r}}(U, -)$. The lax $\mathcal{C}$-module structure ${}^{l}\delta^{U}$ on $\underline{\mathrm{Hom}}^{\mathrm{l}}(U, -)$ is characterised similarly.

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Theorem [Fuch-Schaumann-Schweigert-SW]

The lax module structures on internal homs are distributors, that is,

$$\partial^{r}_{X,Y,Z} = {}^{l}\delta^{X^*}_{Y,Z}, \qquad \partial^{l}_{X,Y,Z} = {}^{r}\delta^{Z^*}_{X,Y}.$$

Distinguishing certain monoidal subcategories

Proposition [Fuch-Schaumann-Schweigert-SW]

Let $(\mathcal{C}, K)$ be GV and let $X \in \mathcal{C}$. Then

1. The lax module functor $\underline{\mathrm{Hom}}^{\mathrm{r}}(X, -)$ is strong if and only if X has a right $\otimes$-dual $X^{\vee}$. In this case then $X^{\vee} = \underline{\mathrm{Hom}}^{\mathrm{r}}(X, 1)$ and $\underline{\mathrm{Hom}}^{\mathrm{r}}(X, -) \cong - \otimes X^{\vee}$ as module functors.
2. The lax module functor $\underline{\mathrm{Hom}}^{\mathrm{l}}(X, -)$ is strong if and only if X has a left $\otimes$-dual ${}^{\vee}X$. In this case then ${}^{\vee}X = \underline{\mathrm{Hom}}^{\mathrm{l}}(X, 1)$ and $\underline{\mathrm{Hom}}^{\mathrm{l}}(X, -) \cong {}^{\vee}X \otimes -$ as module functors.
3. The oplax module functor $\underline{\mathrm{coHom}}^{\mathrm{r}}(X, -)$ is strong if and only if X has a right $\bullet$-dual $X^{\blacktriangledown}$. In this case then $X^{\blacktriangledown} = \underline{\mathrm{coHom}}^{\mathrm{r}}(X, 1)$ and $\underline{\mathrm{coHom}}^{\mathrm{r}}(X, -) \cong - \bullet X^{\blacktriangledown}$ as module functors.
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Consequences for bimodules

Proposition

Let A be a finite dimensional algebra and M a finite dimensional bimodule. The following are equivalent.

1. $\underline{\mathrm{Hom}}^{\mathrm{r}}(M, -)$ is a strong module functor.
2. M has a right-$\otimes_A$ dual.
3. M is projective as a right A-module.
4. M^* is injective as a left A-module.
5. For all bimodules X, Y, the distributor

   $$X \otimes_A (Y \otimes^A M^*) \to (X \otimes_A Y) \otimes^A M^*$$

   is an isomorphism.

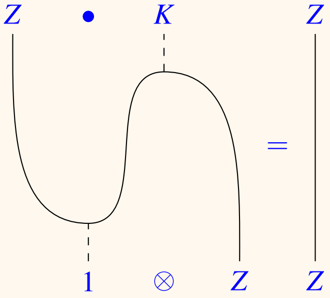
Likewise for the left versions of the above statements.

Zig Zag relations

Let $Z \in \mathcal{C}$ and let $\eta^Z,\ \varepsilon^Z$ be the unit and counit of the adjunction $- \otimes Z \dashv \underline{\mathrm{Hom}}^{\mathrm{r}}(Z, -)$. Consider the components

$$\mathrm{coev}_Z = \eta^Z_1 : 1 \to \underbrace{\underline{\mathrm{Hom}}^{\mathrm{r}}(Z, 1 \otimes Z)}_{Z \bullet GZ},$$

$$\mathrm{ev}_Z = \varepsilon^Z_K : \underbrace{\underline{\mathrm{Hom}}^{\mathrm{r}}(Z, K) \otimes Z}_{GZ \otimes Z} \to K.$$



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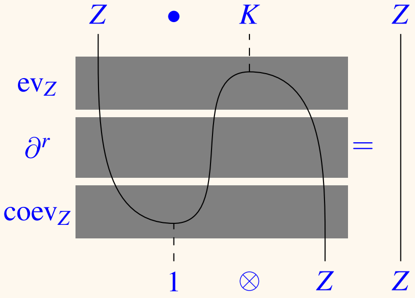
$$\mathrm{coev}_Z = \eta^Z_1 : 1 \to \underbrace{\underline{\mathrm{Hom}}^{\mathrm{r}}(Z, 1 \otimes Z)}_{Z \bullet GZ},$$

$$\mathrm{ev}_Z = \varepsilon^Z_K : \underbrace{\underline{\mathrm{Hom}}^{\mathrm{r}}(Z, K) \otimes Z}_{GZ \otimes Z} \to K.$$



Thank you!