Reflective forms on orthogonal groups and their expansions at 1-dimensional cusps

Janik Wilhelm
jt. with N. Scheithauer and T. Driscoll-Spittler

Technische Universität Darmstadt

Representation Theory XVIII, Dubrovnik
27 June 2023

The **modular discriminant**

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \eta(\tau)^{24} \qquad \text{is a}$$

- modular form for $\begin{cases} \mathrm{SL}_2(\mathbb{Z}). \\ \text{an integral subgroup } \Gamma \subset \mathrm{O}_{1,2}(\mathbb{R}). \end{cases}$
- global section of a line bundle on $\begin{cases} \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}. \\ \Gamma \backslash \mathcal{H}. \end{cases}$

We study a **generalisation to $\mathbf{O}_{n,2}(\mathbb{R})$** of modular forms regarding

- **classification results** for reflective forms,
- **the geometry** of the complex space $\Gamma \backslash \mathcal{H}$.

A **lattice** L is a free $\mathbb{Z}$-module of finite rank together with a non-degenerate bilinear form $(\cdot,\cdot)$ with values in $\mathbb{Q}$.

A lattice ...

- has **signature** (n_+, n_-), if n_+ (n_-) is max. dim. of a positive-definite (negative-...) subspace of $L \otimes_{\mathbb{Z}} \mathbb{Q}$.
- has **dual lattice** $L' := \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} : (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$.
- is called **even**, if (x, x) is even for all $x \in L$.

Note: If L even, then $L \subset L'$.

Let $(L, (\cdot,\cdot))$ be an even lattice. Then $D = L'/L$ together with $\mathrm{q}(x + L) := (x, x)/2 \mod \mathbb{Z}$ is the **discriminant form** of L.

An even lattice L has …

level $N \in \mathbb{Z}_{>0}$ if N is minimal s.t. $N\mathrm{q} = 0$.

genus $II_{n_+,n_-}(D)$ if it has signature (n_+, n_-) and $L'/L \cong D$.

Example:

Lattice $A_1 := \mathbb{Z}v$, $(v, v) = 2$, is even lattice with genus $II_{1,0}(2_1^{+1})$.

Setting:

- an even lattice L of signature $(n, 2)$,
- the vector space $V := L \otimes_{\mathbb{Z}} \mathbb{Q}$ and its complexification $V(\mathbb{C})$,
- the **projective domain**

$$\mathcal{H} := \{[z] \in P(V(\mathbb{C})) \mid (z, z) = 0, \, (z, \bar{z}) < 0\}^{+},$$

- the affine cone $\tilde{\mathcal{H}} \subset V(\mathbb{C})$ lying above $\mathcal{H}$,
- a finite-index subgroup $\Gamma \subset \mathrm{O}(L)^{+}$.

An **automorphic form of weight** k **and character** χ **for** Γ is a meromorphic function $\Psi : \tilde{\mathcal{H}} \to \mathbb{C}$ s.t.

- $\Psi(tz) = t^{-k}\Psi(z)$ for all $t \in \mathbb{C}^*$ and
- $\Psi(\varphi z) = \chi(\varphi)\Psi(z)$ for all $\varphi \in \Gamma$.

How can we construct an automorphic form?

Singular theta lift due to Richard Borcherds.

Ingredients: an even lattice L of signature $(n, 2)$ and
a function $F = \sum_{\gamma \in D} F_\gamma e^\gamma : \mathbb{H} \to \mathbb{C}[D]$ s.t.

- F is holomorphic,
- F is a **modular form for the Weil rep.** of weight $1 - n/2$,
- $[F_\gamma](-m) \in \mathbb{Z}$ for $m \in \mathbb{Q}_{>0}$.

Output: the automorphic product Ψ_F.

Example: Δ is lift of $12\Theta_{A_1}$ on $L = A_1(-1) \oplus II_{1,1}$.

Theorem

There are exactly 11 *regular even lattices of signature* $(n, 2)$, $n > 2$ *and even, splitting* $II_{1,1} \oplus II_{1,1}$ *which carry a reflective automorphic product* Ψ_F *of singular weight.*

Theorem

There are exactly 11 ***regular*** *even lattices of signature $(n, 2)$, $n > 2$ and even, splitting $II_{1,1} \oplus II_{1,1}$ which carry a reflective automorphic product Ψ_F of singular weight.*

regular Sufficient isotropic elements in L'/L.

Theorem

There are exactly 11 *regular even lattices of signature* $(n, 2)$, $n > 2$ *and even,* ***splitting*** $II_{1,1} \oplus II_{1,1}$ *which carry a reflective automorphic product* Ψ_F *of singular weight.*

regular	Sufficient isotropic elements in L'/L.
splitting $II_{1,1} \oplus II_{1,1}$	A decomposition $L = K \oplus II_{1,1} \oplus II_{1,1}$ exists.

Theorem

There are exactly 11 regular even lattices of signature $(n, 2)$, $n > 2$ and even, splitting $II_{1,1} \oplus II_{1,1}$ which carry a ***reflective*** *automorphic product Ψ_F of singular weight.*

regular	Sufficient isotropic elements in L'/L.
splitting $II_{1,1} \oplus II_{1,1}$	A decomposition $L = K \oplus II_{1,1} \oplus II_{1,1}$ exists.
reflective	Condition on coefficients of F. Morally: The zeros of Ψ_F lie on hyperplanes defining reflections of L.

Theorem

There are exactly 11 *regular even lattices of signature* $(n, 2)$, $n > 2$ *and even, splitting* $II_{1,1} \oplus II_{1,1}$ *which carry a reflective automorphic product* Ψ_F *of* ***singular weight***.

regular	Sufficient isotropic elements in L'/L.
splitting $II_{1,1} \oplus II_{1,1}$	A decomposition $L = K \oplus II_{1,1} \oplus II_{1,1}$ exists.
reflective	Condition on coefficients of F. Morally: The zeros of Ψ_F lie on hyperplanes defining reflections of L.
singular weight	Smallest non-trivial weight.

Theorem (continued)

These 11 *lattices are*

n	L
26	$II_{26,2}$
18	$II_{18,2}(2_{II}^{+10})$
14	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$, $II_{14,2}(3^{-8})$
12	$II_{12,2}(2_2^{+2}4_{II}^{+6})$
10	$II_{10,2}(2_{II}^{+6}3^{-6})$, $II_{10,2}(5^{+6})$
8	$II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$, $II_{8,2}(2_1^{+1}4_1^{+1}8_{II}^{+4})$, $II_{8,2}(7^{-5})$
6	$II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$

The corresponding automorphic product is ***unique up to*** $\boldsymbol{O}(L)^+$.

Why at most 11 lattices?

Starting point: **infinite** number of lattices.

n	N	N_H	N_E	cusp forms	candidate
⋮	⋮	⋮	⋮	⋮	⋮
14	2	5	—	—	—
	3	5	2	$\eta_{1^6 3^6}\theta_{A_2}^2$	$II_{14,2}(3^{-8})$
	4	35	20	$\eta_{1^8 2^8}$	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$
⋮	⋮	⋮	⋮	⋮	⋮

Endpoint: **11** candidates.

n	N	N_H	N_E	cusp forms	candidate
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
14	2	5	–	–	–
	3	5	2	$\eta_{1^6 3^6}\theta_{A_2}^2$	$II_{14,2}(3^{-8})$
	4	35	20	$\eta_{1^8 2^8}$	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Aim: Establish **dependency of level N on n.**

Strategy:

i) Note $\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^*} F_{a\gamma}$ is modular form for $\Gamma_0(N)$ if γ isotropic.

ii) Since L regular, exists such modular form $g \neq 0$.

iii) Show the pole orders of g are small.

iv) Apply valence formula to g.

Outcome (simplified):

$$\frac{n-2}{24}\left(\prod_{p|N} p^{\nu_p(N)-1}\right)\left(\prod_{p||N}\frac{p}{2}\right) \leq 1.$$

n	N	N_H	N_E	cusp forms	candidate
⋮	⋮	⋮	⋮	⋮	⋮
14	2	5	–	–	–
	3	5	2	$\eta_{1^6 3^6}\theta_{A_2}^2$	$II_{14,2}(3^{-8})$
	4	35	20	$\eta_{1^8 2^8}$	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$
⋮	⋮	⋮	⋮	⋮	⋮

Aim: List for admissible (n, N) the **lattices** L **splitting** $II_{1,1} \oplus II_{1,1}$.

Strategy:

i) Observe L is uniquely determined by its genus $II_{n,2}(D)$.

ii) Since $L = K \oplus II_{1,1} \oplus II_{1,1}$, consider genera of type $II_{n-2,0}(D)$.

iii) List such genera.

iv) Eliminate genera of non-regular lattices.

Outcome: 474 lattices.

n	N	N_H	N_E	cusp forms	candidate
⋮	⋮	⋮	⋮	⋮	⋮
14	2	5	–	–	–
	3	5	2	$\eta_{1^6 3^6}\theta_{A_2}^2$	$II_{14,2}(3^{-8})$
	4	35	20	$\eta_{1^8 2^8}$	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$
⋮	⋮	⋮	⋮	⋮	⋮

Aim: Filter list by **pairing with Eisenstein series**.

Strategy:

i) Reflectivity: F_γ has pole at $\infty \implies q(\gamma) = 1/d$ and $d\gamma = 0$.

ii) Define the **singular sets**

$$M_d := \{\gamma \in D_{d,1/d} : F_\gamma \text{ has pole at } \infty\}.$$

iii) Choose Eisenstein series E such that $(F, \overline{E})$ has weight 2.

iv) Evaluate condition (simplified): $\sum_{d|N} e_d \, |M_d| = 2 - n$.

Outcome: 132 lattices.

n	N	N_H	N_E	cusp forms	candidate
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
14	2	5	–	–	–
	3	5	2	$\eta_{1^6 3^6}\theta_{A_2}^2$	$II_{14,2}(3^{-8})$
	4	35	20	$\eta_{1^8 2^8}$	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Aim: Filter list by **pairing with cusp forms**.

Strategy:

i) Construct classical cusp forms of weight $1 + n/2$.

ii) Lift to **symmetric modular forms** G for dual Weil rep.

iii) Repeat previous step with G instead of E.

Outcome: The 11 lattices.

Why at most 11 reflective forms?

Recall the singular sets $M_d = \{\gamma \in D_{d,1/d} : F_\gamma \text{ has pole at } \infty\}$.

If N squarefree, then we observe $|M_d| = |D_{d,1/d}|$. Done!
If not, situation is more complicated.

Case $II_{14,2}(2_{II}^{-10}4_{II}^{-2})$:

i) Observe $|M_1| = |D_{1,1}|$ and $|M_4| = |D_{4,1/4}|$ but $|M_2| < |D_{2,1/2}|$.

ii) Lift cusp forms to **non-symmetric modular forms**.

iii) Exploit conditions on M_2 to prove its uniqueness mod $O(L)$.

How to construct these 11 reflective forms

$II_{14,2}(3^{-8})$	$F_{1/\eta_{1^6 3^6},0}$
$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$	$F_{\theta_{0_0}/\Delta,0} - F_{\theta_{0_B}/\Delta,0} + \frac{1}{12}\sum_{\gamma\in M_2} F_{h,\gamma}$

Observation: 11 lattices all related to **Leech lattice** Λ. Namely, if L is such a lattice, there is $g \in \mathrm{O}(\Lambda)$ of order n s.t.

$$L = \Lambda_N^g \oplus II_{1,1} \oplus II_{1,1}(n^2/N), \quad \Lambda_N^g \subset \Lambda^g.$$

Approach: Lift modular forms coming from the Leech lattice.

If N is squarefree and g has cycle shape $\prod_{d|N} d^{b_d}$:

$$f(\tau) = \prod_{d|N} 1/\eta(d\tau)^{b_d} \quad \mapsto \quad F_{f,0} \quad \mapsto \quad \Psi.$$

If not, ...

$II_{14,2}(3^{-8})$	$F_{1/\eta_{1^6 3^6},0}$
$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$	$F_{\theta_{0_0}/\Delta,0} - F_{\theta_{0_B}/\Delta,0} + \frac{1}{12}\sum_{\gamma\in M_2} F_{h,\gamma}$

Expectations: We want the Borcherds input F to

- be reflective and satisfy the constraints on the M_d.
- satisfy $F_0 = \sum_{k|N}\sum_{d|k} \frac{\mu(k/d)}{k}\frac{\theta_{\Lambda^{g,d}}}{\eta_{g^d}}$.
- have integral principal part.

Procedure: Set $F = 0$ and

i) search for f with useful - in sense above - Fourier expansions.

ii) replace F with $F + sF_{f,\gamma}$.

iii) check whether F is reflective. If not, continue with i).

What are the cusps for automorphic forms?

Setting (reminder):

- an even lattice L of signature $(n, 2)$, $n > 2$ and even,
- the vector space $V := L \otimes_{\mathbb{Z}} \mathbb{Q}$ and its complexification $V(\mathbb{C})$,
- the projective domain $\mathcal{H} \subset P(V(\mathbb{C}))$,
- a finite-index subgroup $\Gamma \subset \mathrm{O}(L)^+$.

The quotient space $\Gamma \backslash \mathcal{H}$ can be compactified through the **Baily-Borel compactification** by adding rational cusps.

dim. of cusp $\mathcal{C}$	rep. $U \subset V$	relationship
0-dim.	isotropic, 1-dim.	$\mathcal{C} = P(U(\mathbb{C}))$
1-dim.	isotropic, 2-dim.	$\mathcal{C} = P(U(\mathbb{C})) \backslash (\text{0-dim. cusps})$

Aim: Identify **invariants** of cusps useful for classification.

Observation: Let $\mathcal{C}$ be cusp of $\Gamma\backslash\mathcal{H}$ represented by $U \subset V$.

- The subgroup
$$H := (U \cap L')/(U \cap L)$$
is an invariant up to Γ.
- The lattice
$$K := (U^{\perp} \cap L^{H})/(U \cap L^{H}), \quad L^{H} \supset L$$
is an invariant up to isomorphism.

Definition: Say that cusp $\mathcal{C}$ has **type** H and **orbit lattice** K.

Aim: List $(k-1)$-dimensional cusps of $\Gamma\backslash\mathcal{H}$ of type 0.

Follow the following **algorithm**.

i) Choose sublattice $K \subset L$ for each class in $II_{n-k,2-k}(D)$.

For each such K,

ii) fix k-dim. isotropic subspace U_K of $K^\perp \otimes_{\mathbb{Z}} \mathbb{Q}$ and

iii) representatives $\{\varphi_K^{(1)}, \varphi_K^{(2)}, \ldots\}$ of quotient $\overline{\Gamma}\backslash \mathrm{O}(D)/\overline{\mathrm{O}(K)}$.

Then collection $\{\varphi_K^{(i)}(U_K)\}$ **represents the cusps of type** 0.

Consequence: There is a unique 0-dimensional cusp of type 0.

Aim: Better understand the 1-dimensional cusps of type 0 of $O(L, F)^+\backslash\mathcal{H}$ for the 11 reflective automorphic forms.

Strategy: Calculate the **Fourier-Jacobi expansion at $\mathcal{C}$.**

Result: The first coefficient is

$$\kappa\eta(\tau)^{n-2}\prod_{\alpha\in R^+}\frac{\vartheta(-(\alpha,\omega),\tau)}{\eta(\tau)}.$$

Aim: Better understand the 1-dimensional cusps of type 0 of $\mathrm{O}(L,F)^+\backslash\mathcal{H}$ for the 11 reflective automorphic forms.

Strategy: Calculate the **Fourier-Jacobi expansion at $\mathcal{C}$**.

Result: The first coefficient is

$$\kappa\eta(\tau)^{n-2}\prod_{\alpha\in R^+}\frac{\vartheta(-(\alpha,\omega),\tau)}{\eta(\tau)}.$$

Theorem

*The sets R only depend on their respective cusp and are **root systems**. They parametrise the 1-dimensional cusps of type 0 of the 11 spaces $O(L,F)^+\backslash\mathcal{H}$ and range exactly over the 69 **root systems in Schellekens' list** and the empty set.*

Observation: Expansion above has **constant term** in one case:

- Lattice $L = II_{26,2}$.
- The unique cusp of type $H = 0$ with orbit lattice $K = \Lambda$.
- Constant coefficient Δ.

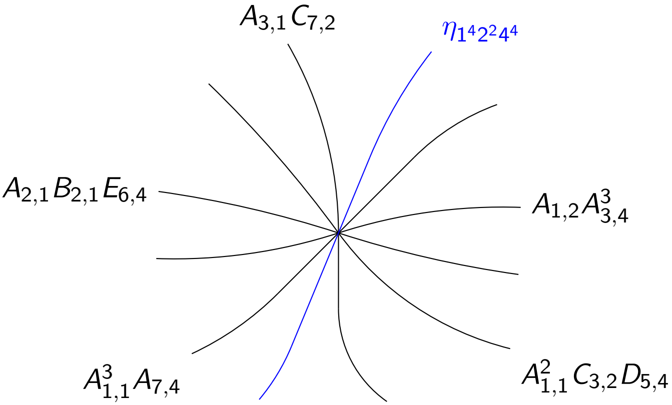
Aim: Construct cusp with constant term for all reflective forms.

Recall relationship of 11 lattices L with Leech lattice Λ:

$$L = \Lambda_N^g \oplus II_{1,1} \oplus II_{1,1}(n^2/N).$$

Outcome: A 1-dimensional cusp $\mathcal{C}$

- of **type** $H \cong \mathbb{Z}/n\mathbb{Z}$ with **orbit lattice** $K = \Lambda^g$.
- with **constant coefficient** η_g up to constant.



The 1-dim. cusps of type 0 and the "special" cusp for $L = II_{12,2}(2_2^{+2}4_{II}^{+6})$.

An Application: Holomorphic VOAs of central charge 24

Aim: Classify semisimple V_1-spaces using automorphic products.

Starting point: Holomorphic VOA V of central charge 24 with

$$V_1 = \mathfrak{g}_{1,k_1} \oplus \dots \oplus \mathfrak{g}_{m,k_m}.$$

Strategy:

i) Define $M = \bigoplus_{i=1}^{m} Q_i^\vee(k_i)$, where Q_i **coroot lattice** of $\mathfrak{g}_i$.

ii) Observe $\chi_V = \mathrm{tr}_V\, e^{2\pi i z_0}\, q^{L_0 - 1}$ **Jacobi form** of lattice index M.

iii) Note $\chi_V = \sum_{\gamma \in K'/K} F_\gamma \theta_\gamma$ for some $K \supset M$.

iv) Show Lie algebra $\mathfrak{g}(V)$ is **generalised Kac-Moody algebra**.

v) Deduce Ψ_F is **reflective form of singular weight**.

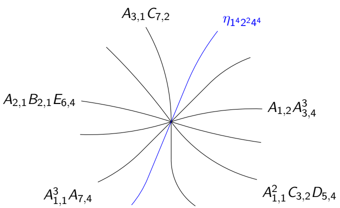
Theorem

Let V be a holomorphic VOA of central charge 24 with semisimple V_1-space.
Assume V is unitary and has regular orbit lattice of even rank.
*Then the automorphic form Ψ_F is one of the **11 reflective automorphic products of singular weight** in our list.*

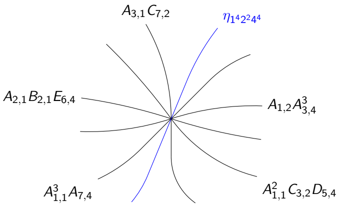
Consequences: The VOA V

- belongs to one of **11 generalised Kac-Moody algebras**.
- has one of the **69** affine structures in **Schellekens' list**.

n	L
26	$II_{26,2}$
18	$II_{18,2}(2_{II}^{+10})$
14	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$, $II_{14,2}(3^{-8})$
12	$II_{12,2}(2_{2}^{+2}4_{II}^{+6})$
10	$II_{10,2}(2_{II}^{+6}3^{-6})$, $II_{10,2}(5^{+6})$
8	$II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$, $II_{8,2}(2_{1}^{+1}4_{1}^{+1}8_{II}^{+4})$, $II_{8,2}(7^{-5})$
6	$II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$



n	L
26	$II_{26,2}$
18	$II_{18,2}(2_{II}^{+10})$
14	$II_{14,2}(2_{II}^{-10}4_{II}^{-2})$, $II_{14,2}(3^{-8})$
12	$II_{12,2}(2_2^{+2}4_{II}^{+6})$
10	$II_{10,2}(2_{II}^{+6}3^{-6})$, $II_{10,2}(5^{+6})$
8	$II_{8,2}(2_{II}^{+4}4_{II}^{-2}3^{+5})$, $II_{8,2}(2_1^{+1}4_1^{+1}8_{II}^{+4})$, $II_{8,2}(7^{-5})$
6	$II_{6,2}(2_{II}^{-2}4_{II}^{-2}5^{+4})$



Thank you for your attention.