New semi-simple categories on affine vertex algebras at non-admissible levels

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Representation Theory XVIII
July 1, 2023

SVEUČILIŠTE U ZAGREBU 1876 PMF 1946 PRIRODOSLOVNO - MATEMATIČKI FAKULTET

UNIVERSITAS STUDIORUM ZAGRABIENSIS MDCLXIX

Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

Projekt KK.01.1.1.01.0004

Projekt je sufinancirala Europska unija iz Europskog fonda za regionalni razvoj. Sadržaj ovog seminara isključiva je odgovornost Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu te ne predstavlja nužno stajalište Europske unije.

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References:

- D. Adamović, O. Perše, I. Vukorepa, *On the representation theory of the vertex algebra* $L_{-5/2}(sl(4))$, Communications in Contemporary Mathematics (2021), 2150104, 42 pp.
- D. Adamović, T. Creutzig, O. Perše, I. Vukorepa, *Tensor category* $KL_k(sl(2n))$ *via minimal affine* W*–algebras at the non-admissible level* $k = -\frac{2n+1}{2}$, arXiv:2212.00704 [math.QA]

Content

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $(\cdot,\cdot)$ be a non-degenerate, symmetric bilinear form on $\mathfrak{g}$.
The **affine Lie algebra** $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined as

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K,$$

where K is central element and Lie algebra structure is given by

$$[x(m), y(n)] = [x, y](m+n) + m\delta_{m,-n}(x, y)K,$$

where $x(m)$ denotes $x \otimes t^m \in \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

We say that M is a $\hat{\mathfrak{g}}$–module of level $k \in \mathbb{C}$ if the central element K acts on M as a multiplication with k.

Affine vertex operator algebra

- $V^k(\mathfrak{g})$ universal affine vertex operator algebra of level k, $k \neq -h^\vee$
- As $\hat{\mathfrak{g}}$–module, we have

$$V^k(\mathfrak{g}) = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{g}\otimes\mathbb{C}[t]+\mathbb{C}K)} \mathbb{C}1.$$

- $L_k(\mathfrak{g})$ simple quotient of $V^k(\mathfrak{g})$.
- Let $V^k(\lambda)$ be the generalized Verma module for $\hat{\mathfrak{g}}$ induced from the irreducible highest-weight $\mathfrak{g}$–module $V(\lambda)$ with highest weight λ.
- Let $L_k(\lambda)$ be its simple quotient.

Summary of previous results

- Non-negative integers levels: Frenkel-Zhu, Li

  The category of $\mathbb{Z}_{\geq 0}$–graded $L_k(\mathfrak{g})$–modules is semi-simple.

- Admissible levels: Adamović-Milas, Dong-Li-Mason, Arakawa, Perše

  The category of $L_k(\mathfrak{g})$–modules which are in the category $\mathcal{O}$ as $\hat{\mathfrak{g}}$–modules is semi-simple.

Negative integer levels which appear in:

- free-field realizations of certain simple affine vertex algebras (Adamović-Perše),
- in the context of affine vertex algebras associated to the Deligne exceptional series (Arakawa-Moreau),
- in the context of collapsing levels for minimal affine $\mathcal{W}$–algebras (Adamović-Kac-Moseneder Frajria-Papi-Perše).

The category $KL_k(\mathfrak{g})$

- $KL_k(\mathfrak{g})$ is semi-simple for generic levels k.
- Arakawa's result implies that $KL_k(\mathfrak{g})$ is semi-simple for k admissible.
- $KL_k(\mathfrak{g})$ is a braided tensor category for k admissible [CHY '18] and in most cases rigity is proven.
- $KL_k(\mathfrak{g})$ is semi-simple when k is a collapsing level for minimal W–algebra or when the minimal affine W-algebra $W_k(\mathfrak{g}, \theta)$ is rational [AKMPP '20].
- At these levels, vertex tensor category is constructed [CY '21].
- There is a notion of collapsing level for non-minimal W-algebras.

Admissible and almost admissible levels

- For $\mathfrak{g} = sl(n)$ level k is admissible if

$$k + n = \frac{p}{q}, \quad p, q \in \mathbb{N}, \ (p, q) = 1, \ p \geq n.$$

- We are interested in levels which are almost admissible, i.e.

$$k = -n + \frac{n-1}{q}, \quad q \in \mathbb{N}, \quad (n-1, q) = 1.$$

- First such example is $V^{-1}(sl(n))$, $n \geq 3$.
- For $q = 2$, we have $L_k(sl(n))$, $k = -\frac{n+1}{2}$, for n even.

On the vertex algebra $L_{-1}(sl(4))$

- D. Adamović and O. Perše determined an explicit formula for the singular vector in $V^{-1}(sl(4))$ and classified irreducible $L_{-1}(sl(4))$–modules in the category $\mathcal{O}$.
- Category $\mathcal{O}$ for $L_{-1}(sl(4))$ is not semi-simple unlike the admissible case.
- Description of the maximal ideal in $V^{-1}(sl(4))$ was obtained using minimal QHR functor H_θ (Arakawa-Moreau).
- Level $k = -1$ is collapsing for $W^k(sl(4), \theta)$ and $W_{-1}(sl(4), \theta) = \mathcal{H}$.
- Category KL_{-1} is semi-simple [AKMPP'20].
- Category KL_{-1} is a rigid braided tensor category [Creutzig-Yang'21].

The vertex algebra $L_{-5/2}(sl(4))$

- New example of non-admissible, half-integer level.
- It appears in conformal embedding [AKMPP '16]

$$L_k(sl(n)) \otimes \mathcal{H} \hookrightarrow L_k(sl(n+1)), \quad k = -\frac{n+1}{2}, \; n \geq 4$$

  where $\mathcal{H}$ denotes the Heisenberg vertex algebra associated to abelian Lie algebra of rank one.
- The level $k = -\frac{n+1}{2}$ is admissible for $\widehat{sl(n+1)}$, for n even.

Theorem

Let $\mathfrak{g} = sl(4)$. The following vector v is a singular vector of weight $-\frac{5}{2}\Lambda_0 - 4\delta + 2\omega_2$ in $V^{-5/2}(\mathfrak{g})$:

$$\begin{aligned} v = {} & e_{\varepsilon_1-\varepsilon_3}(-1)e_{\varepsilon_2-\varepsilon_4}(-3)\mathbf{1} + e_{\varepsilon_1-\varepsilon_3}(-3)e_{\varepsilon_2-\varepsilon_4}(-1)\mathbf{1} + \frac{1}{2}e_{\varepsilon_1-\varepsilon_3}(-2)e_{\varepsilon_2-\varepsilon_4}(-2)\mathbf{1} \\ & - e_{\varepsilon_1-\varepsilon_4}(-1)e_{\varepsilon_2-\varepsilon_3}(-3)\mathbf{1} - e_{\varepsilon_1-\varepsilon_4}(-3)e_{\varepsilon_2-\varepsilon_3}(-1)\mathbf{1} - \frac{1}{2}e_{\varepsilon_1-\varepsilon_4}(-2)e_{\varepsilon_2-\varepsilon_3}(-2)\mathbf{1} \\ & + e_{\varepsilon_2-\varepsilon_4}(-1)e_{\varepsilon_2-\varepsilon_3}(-2)e_{\varepsilon_1-\varepsilon_2}(-1)\mathbf{1} - e_{\varepsilon_2-\varepsilon_4}(-2)e_{\varepsilon_2-\varepsilon_3}(-1)e_{\varepsilon_1-\varepsilon_2}(-1)\mathbf{1} \\ & - e_{\varepsilon_1-\varepsilon_3}(-1)e_{\varepsilon_2-\varepsilon_3}(-2)e_{\varepsilon_3-\varepsilon_4}(-1)\mathbf{1} - 3e_{\varepsilon_1-\varepsilon_3}(-2)e_{\varepsilon_2-\varepsilon_3}(-1)e_{\varepsilon_3-\varepsilon_4}(-1)\mathbf{1} \\ & + 2e_{\varepsilon_1-\varepsilon_2}(-1)e_{\varepsilon_2-\varepsilon_3}(-1)^2e_{\varepsilon_3-\varepsilon_4}(-1)\mathbf{1} - \frac{2}{3}e_{\varepsilon_1-\varepsilon_3}(-1)e_{\varepsilon_2-\varepsilon_4}(-1)h_2(-2)\mathbf{1} + \cdots \end{aligned}$$

The remaining terms can be found in the referenced paper.

Let us denote

$$\widetilde{L}_{-5/2}(\mathfrak{g}) = V^{-5/2}(\mathfrak{g})/\langle v\rangle.$$

Theorem

The complete list of irreducible $\widetilde{L}_{-5/2}(sl(4))$–modules in the category $\mathcal{O}$ is given by

$$\left\{L_{-5/2}(\mu_i(t)) \mid i=1,\ldots,16,\ t\in\mathbb{C}\right\},$$

where:

$$\mu_1(t) = t\omega_1, \qquad \mu_9(t) = -\tfrac{3}{2}\omega_1 + t\omega_3,$$

$$\mu_2(t) = t\omega_3, \qquad \mu_{10}(t) = t\omega_1 - \tfrac{3}{2}\omega_3,$$

$$\mu_3(t) = t\omega_1 + (-t-\tfrac{5}{2})\omega_2, \qquad \mu_{11}(t) = -\tfrac{3}{2}\omega_1 + t\omega_2 + (-t-1)\,\omega_3,$$

$$\mu_4(t) = t\omega_2 + (-t-\tfrac{5}{2})\omega_3, \qquad \mu_{12}(t) = (-t-1)\,\omega_1 + t\omega_2 - \tfrac{3}{2}\omega_3,$$

$$\mu_5(t) = t\omega_1 - \tfrac{3}{2}\omega_2, \qquad \mu_{13}(t) = -\tfrac{1}{2}\omega_1 - \tfrac{1}{2}\omega_2 + t\omega_3,$$

$$\mu_6(t) = -\tfrac{3}{2}\omega_2 + t\omega_3, \qquad \mu_{14}(t) = -\tfrac{1}{2}\omega_1 + t\omega_2 + (-t-\tfrac{3}{2})\omega_3,$$

$$\mu_7(t) = t\omega_1 + (-t-1)\omega_2, \qquad \mu_{15}(t) = t\omega_1 - \tfrac{1}{2}\omega_2 - \tfrac{1}{2}\omega_3,$$

$$\mu_8(t) = t\omega_2 + (-t-1)\omega_3, \qquad \mu_{16}(t) = (-t-\tfrac{3}{2})\omega_1 + t\omega_2 - \tfrac{1}{2}\omega_3.$$

Corollary

The complete list of irreducible $\widetilde{L}_{-5/2}(sl(4))$–modules in the category $KL_{-5/2}$ is given by

$$\left\{L_{-5/2}(t\omega_1) \mid t \in \mathbb{Z}_{\geq 0}\right\} \cup \left\{L_{-5/2}(t\omega_3) \mid t \in \mathbb{Z}_{\geq 0}\right\}.$$

Next goal: Prove the simplicity of $\widetilde{L}_{-5/2}(sl(4))$.

- It turns out that in this case we can not use $\mathcal{W}$–algebra $W^k(sl(4),\theta)$ as in the case $k=-1$.

$\mathcal{W}$–algebra $W^{-5/2}(sl(4), f_{subreg})$

- We use subregular nilpotent element
$$f = f_{subreg} = f_{\varepsilon_2-\varepsilon_3} + f_{\varepsilon_3-\varepsilon_4}.$$
- Let
$$x = \omega_2 + \omega_3$$
be a semisimple element of $sl(4)$ which defines a good grading with respect to f.
- Vertex algebra $W^{-5/2}(sl(4), f)$ is strongly generated by five elements; $J, \bar{L} = L + \partial J, W, G^+, G^-$ having conformal weights 1,2,3,1,3, respectively.
- The OPE formulas are presented by T. Creutzig and A. Linshaw.

Let us denote $\mathfrak{g} = sl(4)$.

Theorem

Level $k = -5/2$ is a collapsing level for $W^k(\mathfrak{g}, f_{subreg})$ and

$$W_{-5/2}(\mathfrak{g}, f_{subreg}) \cong M_J(1),$$

where $M_J(1)$ is the Heisenberg vertex algebra generated by J.

Lemma

The image of singular vector v in $W^{-5/2}(\mathfrak{g}, f_{subreg})$ coincides (up to a non-zero scalar) with the vector G^+.

QHR functor $H_{f_{subreg}}(\cdot)$

Problem: The properties of the QHR functor $H_{f_{subreg}}(\cdot)$ are not presented so explicitly as in the case of the minimal reduction.

- $H_\theta(L_k(\lambda)) \neq 0$ iff $(k\Lambda_0 + \lambda)(\alpha_0^\vee) \notin \mathbb{Z}_{\geq 0}$ [Ar'05]
- In the case $k = -1$ we have $H_\theta(L_{-1}(n\omega_i)) \neq 0$, $i = 1, 3$.

Theorem

For any $n \in \mathbb{Z}_{>0}$ we have:
(P) $H_{f_{subreg}}(L_{-5/2}(n\omega_3)) \neq \{0\}$ and $H_{f_{subreg}}(M) = \{0\}$ for any highest weight $\widetilde{L}_{-5/2}(\mathfrak{g})$–module M in $KL_{-5/2}$ of $\mathfrak{g}$–weight $n\omega_1$.

The proof is based on a construction of singular vectors in generalized Verma modules $V^{-5/2}(n\omega_i)$, $i = 1, 3$, and the description of their submodules $\langle v \rangle \cdot V^{-5/2}(n\omega_i)$.

Simplicity of $\widetilde{L}_{-5/2}(\mathfrak{g})$ and semi-simplicity of $KL_{-5/2}$

Theorem

We have:

(i) $\langle v \rangle$ is the maximal ideal in $V^{-5/2}(sl(4))$, i.e. $L_{-5/2}(sl(4)) \cong V^{-5/2}(sl(4))/\langle v \rangle$.

(ii) The category $KL_{-5/2}$ is semi-simple.

The main idea in the case $k = -5/2$ is to use property (P) and the automorphism σ which interchanges the weights $n\omega_1$ and $n\omega_3$.

Using the similar arguments and exactness of $H_{f_{subreg}}(\cdot)$ in the category $KL_{-5/2}$ we prove that any highest weight module in $KL_{-5/2}$ is irreducible. Then the result of [AKMPP'20] implies that $KL_{-5/2}$ is semi-simple.

Proposition (AKMPP '16)

There is a conformal embedding

$$L_k(sl(m)) \otimes \mathcal{H} \hookrightarrow L_k(sl(m+1)), \quad k = -\frac{m+1}{2},$$

and we have the following decomposition of $L_k(sl(m+1))$ as an $L_k(sl(m)) \otimes \mathcal{H}$–module:

$$L_k(sl(m+1)) = \bigoplus_{i=0}^{\infty} L_k(i\omega_1) \otimes \mathcal{F}_i \ \oplus \ \bigoplus_{i=1}^{\infty} L_k(i\omega_{m-1}) \otimes \mathcal{F}_{-i}.$$

We introduce the following notation for the irreducible $L_k(sl(2n))$–modules in the category $KL_k(sl(2n))$:

$$U_i^{(n)} = L_k(i\omega_1), \ U_{-i}^{(n)} = L_k(i\omega_{2n-1}), \ i \in \mathbb{Z}_{\geq 0}.$$

Proposition

The modules $U_i^{(2)}$ are simple currents with the following fusion rules:

$$U_i^{(2)} \times U_j^{(2)} = U_{i+j}^{(2)}, \quad i, j \in \mathbb{Z}.$$

This means that for $i, j, k \in \mathbb{Z}$:

$$\dim I \binom{U_k^{(2)}}{U_i^{(2)} \quad U_j^{(2)}} = \delta_{i+j,k}.$$

Corollary

$KL_{-5/2}$ is a semi-simple rigid braided tensor category with the fusion rules

$$U_i^{(2)} \boxtimes U_j^{(2)} = U_{i+j}^{(2)} \quad i, j \in \mathbb{Z}.$$

- In the case $L_{-5/2}(sl(4))$ we used very complicated formulas for singular vectors in $V^{-5/2}(sl(4))$ and $V^{-5/2}(n\omega_i)$, $i = 1, 3$, and OPEs in subregular W-algebra.

$\Longrightarrow$ For $KL_k(sl(2n))$, $k = -\frac{2n+1}{2}$, $n \geq 2$ we need a different approach.

- The level $k = -\frac{n+1}{2}$ is admissible for $\widehat{sl(n+1)}$ if and only if n is even.
- Only for even n one expects to get similar vertex tensor category structure as in the case $n = 4$.
- The cases when n is odd are more complicated and not all modules in $KL_k(sl(n))$ appear in the decomposition of above conformal embedding.
- Levels $k = -\frac{n+1}{2}$ are collapsing for W-algebras of certain hook type [AMP'22].

Main theorem

Goal: Using tensor category approach, generalize the results on $KL_{-5/2}(sl(4))$ to $KL_k(sl(2n))$, $k = -\frac{2n+1}{2}$.

Main theorem $KL_k(sl(m))$ is a semi-simple, rigid braided tensor category for all even $m \geq 4$ and $k = -\frac{m+1}{2}$. Modules in $KL_k(sl(m))$ are simple currents, and all of them appear in the decomposition of conformal embedding.

- The argument for the proof is an induction for $m = 2n$.
  A base case is $L_{-5/2}(sl(4))$.

Results on singlet $\mathcal{M}(2)$ [A'03] [AM'17], [CMY'21]:

- The category of all C_1–cofinite $\mathcal{M}(2)$–modules $\mathcal{O}_{\mathcal{M}(2)}$ is a rigid braided tensor category.
- Let $\mathcal{M}_i$, $i \in \mathbb{Z}$ be all atypical $\mathcal{M}(2)$–modules.

  Modules $\mathcal{M}_i$, $i \in \mathbb{Z}$ are simple currents in $\mathcal{O}_{\mathcal{M}(2)}$ with the following fusion rules

$$\mathcal{M}_i \times \mathcal{M}_j = \mathcal{M}_{i+j}, \quad i, j \in \mathbb{Z}.$$

  The conformal weight Δ of the top level of $\mathcal{M}_i$ is

$$\Delta(\mathcal{M}_i) = \frac{|i|(|i|+1)}{2}.$$

Using that $KL_k(sl(m))$ is a braided tensor category and above results on singlet, we obtain:

- $W_{k-1}(sl(m+2),\theta)$ is a simple current extension of $L_k(sl(m)) \otimes \mathcal{H} \otimes \mathcal{M}(2)$, where $\mathcal{H}$ denotes the rank one Heisenberg vertex algebra generated by h.
- For $l = -\frac{m}{m+2}$, we have the following decomposition

$$W_{k-1}(sl(m+2),\theta) = \bigoplus_{i\in\mathbb{Z}} U_i^{(\frac{m}{2})} \otimes \mathcal{F}_i^l \otimes \mathcal{M}_i,$$

where $\mathcal{F}_i^l$ denotes Fock $\mathcal{H}$–module generated by highest weight vector v_i such that

$$h(n)v_i = \delta_{n,0} i v_i \ (n \geq 0).$$

The category $KL_{k-1}(sl(m+2))$ and ordinary $W_{k-1}(sl(m+2),\theta)$–modules

We use the theory of vertex algebra extensions to study ordinary modules for this minimal W–algebra.

Key fact: Let $H_\theta(L_{k-1}(\lambda))$ be the QHR of a simple module in $KL_{k-1}(sl(m+2))$. Then:

$$H_\theta(L_{k-1}(\lambda)) \cong \mathcal{A}_{a,b} = \bigoplus_{i\in\mathbb{Z}} U_i^{(\frac{m}{2})} \otimes \mathcal{F}^l_{a+i} \otimes \mathcal{M}_{b+i},$$

for $b = \frac{m+2}{m} a \in \mathbb{Z}$.

Theorem

(1) Set $\{U_i^{(n)} | \ i \in \mathbb{Z}\}$ provides all irreducible modules in $KL_k(sl(2n))$ and we have the following fusion rules:

$$U_i^{(n)} \times U_j^{(n)} = U_{i+j}^{(n)}, \quad i, j \in \mathbb{Z}.$$

(2) $KL_k(sl(2n))$ is a semi-simple rigid braided tensor category.

(3) For $n \geq 2$, $W_{k-1}(sl(2n+2), \theta)$ is a simple current extension of $L_k(sl(2n)) \otimes \mathcal{H} \otimes \mathcal{M}(2)$, and we have the following decomposition

$$W_{k-1}(sl(2n+2), \theta) = \bigoplus_{i \in \mathbb{Z}} U_i^{(n)} \otimes \mathcal{F}_i \otimes \mathcal{M}_i.$$

This theorem gives a whole family of new examples of semi-simple $KL_k(\mathfrak{g})$ at collapsing levels for non-minimal W-algebras.

Second approach to $W_{k-1}(sl(m+2),\theta)$, $k=-\frac{m+1}{2}$

Let $\mathcal{S}$ be the $\beta\gamma$ vertex algebra (Weyl vertex algebra).

Theorem

Let $m \in \mathbb{Z}_{\geq 0}$, $m \geq 4$. Then

$$W_{k-1}(sl(m+2),\theta) \cong Com(\mathcal{H}_1, L_k(sl(m+1)) \otimes \mathcal{S}),$$

where $\mathcal{H}_1$ is a certain Heinseberg vertex algebra of rank one, and we have the following decomposition

$$W_{k-1}(sl(m+2),\theta) = \bigoplus_{i\in\mathbb{Z}} U_i^{(\frac{m}{2})} \otimes \mathcal{F}_i^l \otimes \mathcal{M}_i$$

with $l = -\frac{m}{m+2}$.

Thank you!