# VOAs associated with intermediate Lie algebra $E_{7+1/2}$

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# An exotic Lie algebra $E_{7+1/2}$

$E_{7+1/2}$ is an intermediate Lie algebra found independently by (Mathur-Muhki-Sen 88) in physics and by (Deligne, Cohen-de Man 96) in mathematics.

1. It has dimension 190
2. Dual Coxeter number 24
3. It is intermediate between $E_7$ and $E_8$
4. Filling a hole in the Deligne-Cvitanović exceptional series
5. Not simple, or semisimple

# Main result

Computational evidence shows the following interesting conjecture[1]

**Conjecture**

*Affine VOA $(E_{7+1/2})_k$ is rational if and only if level $1 \leq k \leq 5$. For these levels, affine VOAs $(E_{7+1/2})_k$ and $(E_7)_k$ have the same number of characters $r(k)$ and satisfy the MLDEs with the same index $l(k)$. More precisely, we have*

| $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $r(k)$ | 2 | 6 | 12 | 25 | 44 |
| $l(k)$ | 0 | 0 | 20 | 160 | 636 |

[1]The possible irrationality of VOA $(E_{7+1/2})_k$ for $k > 5$ was pointed out to us by Arakawa and Kawasetsu from the viewpoint of $W$-algebras. We will provide new evidence from the viewpoint of MLDE.

# Outline

1. Background
2. Known results for VOA $E_{7+1/2}$ at level 1
3. Known results for rank-one $E_{7+1/2}$ instanton VOA at level $-5$
4. VOA $E_{7+1/2}$ at level 2
5. VOA $E_{7+1/2}$ at higher levels
6. Rank-two and higher $E_{7+1/2}$ instanton VOAs
7. A conjectural Weyl dimension formula for $E_{7+1/2}$ representations
8. $E_{7+1/2}$ as a gauge algebra

# How physicists found $E_{7+1/2}$

The modular linear differential equations (MLDE) were proposed by (Eguchi-Ooguri 87) and became a major approach to classify 2d RCFTs.

Consider 2nd order holomorphic MLDE, sometimes called Kaneko-Zagier equation:

$$[D^2 + \mu E_4]\chi = 0.$$

To require the two solutions $\chi$ are the characters of a 2d RCFT, $\mu$ can only take finitely many discrete values (Mathur-Mukhi-Sen 88).

1. $\chi$ is a rank-two weakly holomorphic vector-valued modular form of weight 0 with non-negative Fourier coefficients

$$\chi_0 = q^{-\frac{c}{24}}(1 + a_1 q + a_2 q^2 + \cdots), \quad \chi_h = q^{-\frac{c}{24}+h}(b_0 + b_1 q + b_2 q^2 + \cdots).$$

2. $D$ is the Serre derivative $q\partial_q - \frac{k}{12}E_2$.
3. $E_{2k}$ is the Eisenstein series of weight $2k$.

# How physicists found $E_{7+1/2}$

Table I

| $l$ | $\mu$ | $m_1$ | $c$ | $h$ | Identification |
|---|---|---|---|---|---|
| 96 | $\frac{11}{900}$ | 1 | $\frac{2}{5}$ | $\frac{1}{5}$ | $c=-\frac{22}{5}$ minimal model $(c \leftrightarrow c-24h)$ |
| 90 | $\frac{5}{144}$ | 3 | 1 | $\frac{1}{4}$ | $k=1$ SU(2) WZW model |
| 80 | $\frac{1}{12}$ | 8 | 2 | $\frac{1}{3}$ | $k=1$ SU(3) WZW model |
| 72 | $\frac{119}{900}$ | 14 | $\frac{14}{5}$ | $\frac{2}{5}$ | $k=1$ $G_2$ WZW model |
| 60 | $\frac{2}{9}$ | 28 | 4 | $\frac{1}{2}$ | $k=1$ SO(8) WZW model |
| 48 | $\frac{299}{900}$ | 52 | $\frac{26}{5}$ | $\frac{3}{5}$ | $k=1$ $F_4$ WZW model |
| 40 | $\frac{5}{12}$ | 78 | 6 | $\frac{2}{3}$ | $k=1$ $E_6$ WZW model |
| 30 | $\frac{77}{144}$ | 133 | 7 | $\frac{3}{4}$ | $k=1$ $E_7$ WZW model |
| 24 | $\frac{551}{900}$ | 190 | $\frac{38}{5}$ | $\frac{4}{5}$ | ? |
| 20 | $\frac{2}{3}$ | 248 | 8 | $\frac{5}{6}$ | $\supset k=1$ $E_8$ WZW model |

# How physicists found $E_{7+1/2}$

> Finally we turn our attention to the row corresponding to $m = 190$. We do not know of a theory satisfying the properties given in the table. However, the values of $c$ and $h$ are consistent (in the sense of refs.[16-18]) with a theory of two primary fields $I$ and $\Phi$ with the fusion rules $\Phi \otimes \Phi = I \oplus \Phi$. Given these fusion rules, and the conformal weight of $\Phi$, one can follow the procedure of ref.[18] to derive differential equations for various correlation functions on the plane and on the torus. With the help of these differential equations one can construct a crossing-symmetric four point function on the sphere for this theory in terms of hypergeometric functions[22]. It thus seems quite plausible that there exists a rational conformal field theory corresponding to this row based on some as yet undiscovered symmetry algebra.

Now we know this is the VOA $(E_{7+1/2})_1$. It is dual to Lee-Yang model $M_{\text{eff}}(5,2)$ w.r.t $(E_8)_1$.

# How mathematicians found $E_{7+1/2}$

**Deligne-Cvitanović exceptional series (Deligne 96)**

$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$ .

The dimensions of some representations of the Deligne-Cvitanović exceptional series have uniform formula

$$\dim \mathfrak{g} := \dim(\theta) = \frac{2(5h^\vee - 6)(h^\vee + 1)}{h^\vee + 6},$$

$$\dim \mathfrak{g}^{(2)} := \dim(2\theta) = \frac{5h^{\vee 2}(2h^\vee + 3)(5h^\vee - 6)}{(h^\vee + 6)(h^\vee + 12)}.$$

By explicit computations (Cohen-de Man 96) found 25 such dimension formulas depending on $h^\vee$. (Landsberg-Manivel 06) proved a formula for arbitrary $\dim(k\theta)$.

**A remarkable phenomenon**

In all these dimension formulas, $h^\vee = 24$ gives integer dimensions. For example, $\dim \mathfrak{g} = 190, \dim \mathfrak{g}^{(2)} = 15504...$

# How mathematicians found $E_{7+1/2}$

This indicates there exist an intermediate Lie algebra filling the hole between $E_7$ and $E_8$ in the Deligne-Cvitanović exceptional series.

$$E_7 \subset E_{7+1/2} \subset E_8.$$

It is easy to observe the following representation decompositions for $E_{7+1/2} \subset E_8$

$$\mathbf{248} = \mathbf{190} + \mathbf{57} + \mathbf{1},$$

and for $E_7 \subset E_{7+1/2}$

$$\mathbf{190} = \mathbf{133} + \mathbf{56} + \mathbf{1}, \qquad \mathbf{57} = \mathbf{56} + \mathbf{1}.$$

Intermediate Lie algebras $A_{n-1/2}$, $B_{n-1/2}$, $C_{n-1/2}$, $D_{n-1/2}$ and their representations have been extensively studied by (Shtepin 93-14).

# VOA $(E_{7+1/2})_1$

$(E_{7+1/2})_1$ has central charge $c = \frac{38}{5}$ and two primaries with weights $0, \frac{4}{5}$.

The two characters of $(E_{7+1/2})_1$ are the solution of 2nd order MLDE found in (Mathur-Mukhi-Sen 88), see also (Kaneko-Nagatomo-Sakai 13),

$$[D^2 - \frac{551}{3600} E_4]\chi = 0.$$

The two solutions are

$$\chi_0 = q^{-\frac{19}{60}}(1 + 190q + 2831q^2 + 22306q^3 + 129276q^4 + \dots),$$
$$\chi_{\frac{4}{5}} = q^{\frac{29}{60}}(57 + 1102q + 9367q^2 + 57362q^3 + 280459q^4 + \dots).$$

The VOA was constructed by (Kawasetsu 13) as an intermediate vertex subalgebra of the lattice VOA $V_{E_8}$. It has a nice coset construction

$$(E_{7+1/2})_1 = \frac{(E_7)_1}{M(5,3)} = (E_7)_1 \otimes M_{\rm eff}(5,3).$$

# VOA $(E_{7+1/2})_1$

The two characters of $(E_{7+1/2})_1$ were realized as the $\mathsf{T}_{19}$ Hecke image of minimal model $M(5,2)$, i.e., the Lee-Yang model (Harvey-Wu 18).

**Hecke operator (Harvey-Wu 18)**

$$(E_{7+1/2})_1 = \mathsf{T}_{19} M_{\text{eff}}(5,2).$$

$$\mathsf{T}_{19} \text{ maps central charge } c : \frac{2}{5} \to \frac{38}{5}, \text{ conformal weight } h_1 : \frac{1}{5} \to \frac{4}{5}.$$

One consequence of the Hecke operator is that each character of $(E_{7+1/2})_1$ can be written as a degree 19 homogeneous polynomial of the two Lee-Yang characters.

$$\chi_0 = \phi_1^{19} + 171\phi_1^{14}\phi_2^5 + 247\phi_1^9\phi_2^{10} - 57\phi_1^4\phi_2^{15},$$
$$\chi_{\frac{4}{5}} = \phi_2^{19} - 171\phi_2^{14}\phi_1^5 + 247\phi_2^9\phi_1^{10} + 57\phi_2^4\phi_1^{15}, \text{ with}$$

$$\phi_1 = q^{-\frac{1}{60}} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \qquad \phi_2 = q^{\frac{11}{60}} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

# VOA $(E_{7+1/2})_{-5}$

**4d SCFT/VOA correspondence (Beem-Lemos-Liendo-Peelaers-Rastelli-Rees 15)**

A series of familiar examples are the 4d $\mathcal{N}=2$ $H_{\mathfrak{g}}$ instanton SCFT/VOA $(\mathfrak{g})_{-h^\vee/6-1}$ correspondence for $\mathfrak{g}=A_1\subset A_2\subset D_4\subset E_6\subset E_7\subset E_8$. The Schur index of the 4d SCFT is equal to the VOA vacuum character.

The vacuum character of quasi-lisse VOA $(\mathfrak{g})_{-h^\vee/6-1}$ for the Deligne-Cvitanović exceptional series satisfy an uniform 2nd order MLDE (Arakawa-Kawasetsu 16).

Analogously the $\chi_{vac}$ of $(E_{7+1/2})_{-5}$ is expected to satisfy the 2nd order MLDE

$$\left[D^2-\frac{575}{144}E_4\right]\chi_{vac}=0.$$

It is easy to solve the vacuum character as

$$\chi_{vac}=q^{\frac{25}{12}}(1+190q+15695q^2+783010q^3+27319455q^4+725679750q^5+\dots).$$

# VOA $(E_{7+1/2})_{-5}$

This is a quasi-modular form with exact expression (Kaneko-Koike 03)

$$\chi_{vac} = \frac{E_4'}{240\eta^{10}} P_3\Big(\frac{E_6}{\Delta^{1/2}}\Big) - \eta^2 Q_3\Big(\frac{E_6}{\Delta^{1/2}}\Big),$$

where $P_3(x) = x^3 + 904x$, $Q_3(x) = x^2 + 442$. We can determine the flavored character as

$$1+\mathbf{190}q+(\mathbf{1}+\mathbf{190}+\mathbf{15504})q^2+(2\times\mathbf{1}+\mathbf{2640}+2\times\mathbf{15504}+\mathbf{749360})q^3+\ldots.$$

The other solution of the MLDE is a Log one:

$$\chi_{vac}\log[q] - \frac{f}{5354228880},$$

where

$$f = \frac{E_4}{\eta^{10}} P_3\Big(\frac{E_6}{\Delta^{1/2}}\Big) = q^{-\frac{23}{12}}(1 - 322q + 75279q^2 - 23121118q^3 - \ldots).$$

# VOA $(E_{7+1/2})_k$

**Question**

How to define affine VOA $(E_{7+1/2})_k$ for $k > 1$?

**Hints**

Assuming the general formulas for central charge and conformal weights still hold. For affine VOA $(E_{7+1/2})_k$, we have

$$c_k = \frac{190k}{24+k}, \qquad h_\lambda = \frac{C_2(R_\lambda)}{2(24+k)}.$$

For example, we know from $(E_{7+1/2})_1$ that $C_2(\mathbf{57}) = 40$.

**Caution**

In general, $C_2(R_\lambda)$ is no longer defined by $\langle \lambda + 2\rho, \lambda \rangle$.

# VOA $(E_{7+1/2})_2$

Accidentally, last year we find the characters of $E_{7+1/2}$ at level 2 can be realized as the $\mathsf{T}_{19}$ Hecke image of minimal model $M(13,2)$ (Duan-Lee-KS 22, Section 7.1).

**Hecke operator**

$$(E_{7+1/2})_2 = \mathsf{T}_{19} M_{\text{eff}}(13,2).$$

$$\mathsf{T}_{19} \text{ maps central charge } c : \frac{10}{13} \to \frac{190}{13},$$
$$\text{conformal weight } h_i : 0, \frac{1}{13}, \frac{3}{13}, \frac{6}{13}, \frac{10}{13}, \frac{15}{13} \to 0, \frac{10}{13}, \frac{12}{13}, \frac{18}{13}, \frac{19}{13}, \frac{21}{13}.$$

# Characters of $(E_{7+1/2})_2$

Explicit calculation of the Hecke image gives the following v.v.m.f

$$\chi_0 = q^{-\frac{95}{156}}(1 + 190q + 18335q^2 + 448210q^3 + 6264585q^4 + \dots),$$
$$\chi_{\frac{10}{13}} = q^{\frac{25}{156}}(57 + 10830q + 321575q^2 + 4979330q^3 + 53025295q^4 + \dots),$$
$$\chi_{\frac{12}{13}} = q^{\frac{49}{156}}(190 + 20596q + 537890q^2 + 7761500q^3 + 79066030q^4 + \dots),$$
$$\chi_{\frac{18}{13}} = q^{\frac{121}{156}}(1045 + 48070q + 910955q^2 + 10983690q^3 + 99272435q^4 + \dots),$$
$$\chi_{\frac{19}{13}} = q^{\frac{133}{156}}(2640 + 109155q + 1979610q^2 + 23245740q^3 + 206319480q^4 + \dots),$$
$$\chi_{\frac{21}{13}} = q^{\frac{157}{156}}(1520 + 51395q + 860890q^2 + 9606457q^3 + 82347710q^4 + \dots).$$

We have the following evidences these are indeed the characters of $(E_{7+1/2})_2$:

1. All Fourier coefficients are positive integers
2. Spin-1 currents of vacuum character is dimension 190
3. All initial Fourier coefficients appear as dims of irreps in (Cohen-de Man 96)
4. All six primaries have the correct conformal weights

# MLDE of $(E_{7+1/2})_2$

The six characters of $(E_{7+1/2})_2$ satisfy a 6th order MLDE (Duan-Lee-KS 22)

**MLDE of $(E_{7+1/2})_2$**

$$[D^6+\mu_1E_4D^4+\mu_2E_6D^3+\mu_3E_4^2D^2+\mu_4E_4E_6D+(\mu_5E_4^3+\mu_6E_6^2)]\chi=0,$$
$$\mu_1=-\frac{1225}{1872},\quad \mu_2=\frac{25205}{36504},\quad \mu_3=-\frac{1349885}{3504384},\quad \mu_4=\frac{36703535}{296120448}$$
$$\mu_5=-\frac{57214927525}{4804258148352},\quad \mu_6=-\frac{3824637775}{450399201408}.$$

The $S$-matrix are given in (Duan-Lee-KS 22). An uniform 6th order MLDE for the level 2 affine characters of the Deligne-Cvitanović exceptional series is given in (Lee-KS-Wang 23).

# Coset constructions involving $(E_{7+1/2})_2$

We find two coset constructions involving $(E_{7+1/2})_2$ (Lee-KS-Wang 23)[2]

**Coset A**

$$\frac{(E_8)_2}{(E_{7+1/2})_2} = M_{\text{eff}}(13,4).$$

**Coset B**

$$\frac{(E_{7+1/2})_1 \otimes (E_{7+1/2})_1}{(E_{7+1/2})_2} = M_{(D_6,A_{12})}(13,10).$$

These remind us of the familiar maverick cosets (Dunbar-Joshi 93)

$$\frac{(E_8)_2}{(E_7)_2} = (A_1)_2 \otimes M(5,4), \quad \text{and} \quad \frac{(E_7)_1 \otimes (E_7)_1}{(E_7)_2} = M(5,4).$$

[2]The coset A is also found in an unpublished draft of Arakawa, Creutzig and Kawasetsu. The coset B has the same central charge as $M(13,10)$ was noticed in (Cheng-Gannon-Lockhart 20)

# Coset A

**Coset A**

$$\frac{(E_8)_2}{(E_{7+1/2})_2} = M_{\text{eff}}(13,4).$$

Denote $\mathfrak{g} = E_{7+1/2}$, we find the following precise character relations where we use the $M(13,4)$ primary labels:

$$\chi_0^{(E_8)_2} = \chi_{1,3}\chi_0^{(\mathfrak{g})_2} + \chi_{1,5}\chi_{\frac{10}{13}}^{(\mathfrak{g})_2} + \chi_{1,7}\chi_{\frac{12}{13}}^{(\mathfrak{g})_2} + \chi_{1,11}\chi_{\frac{18}{13}}^{(\mathfrak{g})_2} + \chi_{1,9}\chi_{\frac{19}{13}}^{(\mathfrak{g})_2} + \chi_{1,1}\chi_{\frac{21}{13}}^{(\mathfrak{g})_2},$$
$$\chi_{\frac{15}{16}}^{(E_8)_2} = \chi_{2,3}\chi_0^{(\mathfrak{g})_2} + \chi_{2,5}\chi_{\frac{10}{13}}^{(\mathfrak{g})_2} + \chi_{2,7}\chi_{\frac{12}{13}}^{(\mathfrak{g})_2} + \chi_{2,11}\chi_{\frac{18}{13}}^{(\mathfrak{g})_2} + \chi_{2,9}\chi_{\frac{19}{13}}^{(\mathfrak{g})_2} + \chi_{2,1}\chi_{\frac{21}{13}}^{(\mathfrak{g})_2},$$
$$\chi_{\frac{3}{2}}^{(E_8)_2} = \chi_{3,3}\chi_0^{(\mathfrak{g})_2} + \chi_{3,5}\chi_{\frac{10}{13}}^{(\mathfrak{g})_2} + \chi_{3,7}\chi_{\frac{12}{13}}^{(\mathfrak{g})_2} + \chi_{3,11}\chi_{\frac{18}{13}}^{(\mathfrak{g})_2} + \chi_{3,9}\chi_{\frac{19}{13}}^{(\mathfrak{g})_2} + \chi_{3,1}\chi_{\frac{21}{13}}^{(\mathfrak{g})_2}.$$

The three characters of $E_8$ correspond to $\mathbf{1}, \mathbf{248}, \mathbf{3875}$ reps respectively.

# Coset B

The second coset construction is more intricate. Consider the following block-diagonal modular invariant of $M(13,10)$. We choose

$$\begin{aligned} \chi_{1,i} &= \chi^{M(13,10)}_{(1,i)} + \chi^{M(13,10)}_{(1,13-i)}, \quad & i = 1,2,\dots,6, \\ \chi_{3,i} &= \chi^{M(13,10)}_{(3,i)} + \chi^{M(13,10)}_{(3,13-i)}, \quad & i = 1,2,\dots,6, \\ \chi_{5,i} &= \chi^{M(13,10)}_{(5,i)}, \quad & i = 1,2,\dots,6. \end{aligned}$$

They form the $(D_6, A_{12})$ modular invariant of $M(13,10)$:

$$Z_{(D_6,A_{12})} = \sum_{i=1}^{6} (|\chi_{1,i}|^2 + |\chi_{3,i}|^2 + 2|\chi_{5,i}|^2).$$

# Coset B

**Coset B**

$$\frac{(E_{7+1/2})_1 \otimes (E_{7+1/2})_1}{(E_{7+1/2})_2} = M_{(D_6,A_{12})}(13,10).$$

Denote $\mathfrak{g} = E_{7+1/2}$, we find the following precise character relations:

$$\chi_0^{(\mathfrak{g})_1} \otimes \chi_0^{(\mathfrak{g})_1} = \chi_{1,1}\chi_0^{(\mathfrak{g})_2} - \chi_{1,6}\chi_{\frac{10}{13}}^{(\mathfrak{g})_2} + \chi_{1,2}\chi_{\frac{12}{13}}^{(\mathfrak{g})_2} - \chi_{1,5}\chi_{\frac{18}{13}}^{(\mathfrak{g})_2} + \chi_{1,3}\chi_{\frac{19}{13}}^{(\mathfrak{g})_2} + \chi_{1,4}\chi_{\frac{21}{13}}^{(\mathfrak{g})_2},$$
$$\chi_0^{(\mathfrak{g})_1} \otimes \chi_{\frac{4}{5}}^{(\mathfrak{g})_1} = -\chi_{5,1}\chi_0^{(\mathfrak{g})_2} + \chi_{5,6}\chi_{\frac{10}{13}}^{(\mathfrak{g})_2} - \chi_{5,2}\chi_{\frac{12}{13}}^{(\mathfrak{g})_2} + \chi_{5,5}\chi_{\frac{18}{13}}^{(\mathfrak{g})_2} - \chi_{5,3}\chi_{\frac{19}{13}}^{(\mathfrak{g})_2} - \chi_{5,4}\chi_{\frac{21}{13}}^{(\mathfrak{g})_2},$$
$$\chi_{\frac{4}{5}}^{(\mathfrak{g})_1} \otimes \chi_{\frac{4}{5}}^{(\mathfrak{g})_1} = \chi_{3,1}\chi_0^{(\mathfrak{g})_2} - \chi_{3,6}\chi_{\frac{10}{13}}^{(\mathfrak{g})_2} + \chi_{3,2}\chi_{\frac{12}{13}}^{(\mathfrak{g})_2} - \chi_{3,5}\chi_{\frac{18}{13}}^{(\mathfrak{g})_2} + \chi_{3,3}\chi_{\frac{19}{13}}^{(\mathfrak{g})_2} + \chi_{3,4}\chi_{\frac{21}{13}}^{(\mathfrak{g})_2}.$$

- The degeneracy 2 is consistent on the two sides of the coset relation.
- The negative signs in the above character relations reflect the novelty of intermediate Lie algebra.

# $E_{7+1/2}$ at higher levels

Currently we do not know how to define VOA $(E_{7+1/2})_k$ for $k > 2$ or compute their characters, but we have some predictions inspired from the similarity between $(E_{7+1/2})_k$ and $(E_7)_k$ for $k = 1, 2$.

**Conjecture**

*Affine VOAs $(E_{7+1/2})_k$ and $(E_7)_k$ have the same number of characters $r(k)$ at least for $k \leq 5$:*

$$\sum r(t)x^t = \frac{1}{(1-x)^2(1-x^2)^3(1-x^3)^2(1-x^4)}$$
$$= 1 + 2x + 6x^2 + 12x^3 + 25x^4 + 44x^5 + O(x^6).$$

This indicates the irreducible representations of $E_{7+1/2}$ are marked by $E_7$ Dynkin labels.

# Dynkin diagram

Figure: Dynkin diagram of $E_7$ and irreducible representations associated with fundamental weights.

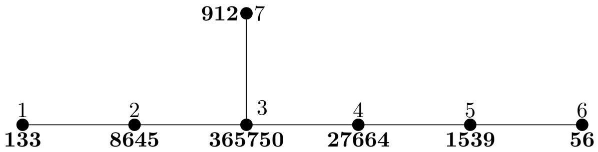
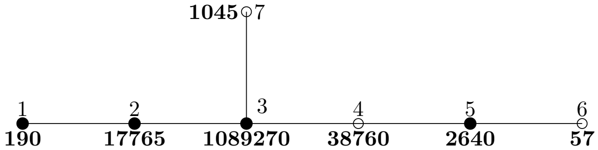



Figure: The analogy of Dynkin diagram for $E_{7+1/2}$ and irreducible representations associated with fundamental weights. We call the three circled nodes as *fermionic fundamental weights*. All representations here have appeared in (Cohen-de Man 96).



# Weyl vector

Define the Weyl vector for $E_{7+1/2}$ as

$$\rho = \rho_{E_7} + \rho_{\mathbf{56}} = \frac{1}{2}\Big(\Delta_+ + \frac{1}{2}\mathbf{56}\Big).$$

Explicitly, we have

$$\rho_{E_7} = (1,1,1,1,1,1,1), \qquad \rho_{\mathbf{56}} = (0,0,0,1,0,1,1).$$

The $w_4, w_6, w_7$ positions are like fermions, while rest are like bosons.

For $\mathfrak{g} = E_{7+1/2}$ and $h^\vee = 24$ we have

$$\langle \rho, \rho \rangle_{E_7} = \frac{h^\vee \dim(\mathfrak{g})}{12} = 380.$$

This means that $E_{7+1/2}$ satisfies the Freudenthal-de Vries strange formula just like simple Lie algebras.

# Quadratic Casimir invariants for $E_{7+1/2}$

Based on the quadratic Casimir invariants of level 1,2, we conjecture the following universal formula for irreducible representations of $E_{7+1/2}$:

$$C_2(R_\lambda) = \langle \lambda + 2\rho, \lambda \rangle_{E_7} + \langle \lambda, \lambda \rangle_{\text{odd}},$$

where the bilinear form $\langle , \rangle_{\text{odd}}$ is defined by the matrix

$$M_{ij} = \begin{cases} 1/2, & i, j \in S_{\text{odd}},\ i = j, \\ -1/2, & i, j \in S_{\text{odd}},\ i \neq j, \\ 0, & \text{otherwise}, \end{cases} .$$

where introduce the set of fermionic fundamental weights as $S_{\text{odd}} = \{w_4, w_6, w_7\}$.

# $E_{7+1/2}$ at higher levels

Consider a general degree $d$ MLDE

$$\Big[\sum_{i=0}^{d}\phi_i(\tau)D^i\Big]\chi(\tau)=0,\qquad \phi_i \text{ weight } 2d-2i+2l.$$

The highest order coefficient is a holomorphic modular form of weight $2l$. The integer index $l$ is related to degree $d$ and exponents $\alpha_i=-\frac{c}{24}+h_i$ of $\chi$ by

$$\frac{l}{6}=\frac{d(d-1)}{12}-\sum_{i=1}^{d}\alpha_i.$$

Using the conjectural $C_2(\lambda)$ formula, we find for both $E_7$ and $E_{7+1/2}$ with level $1\le k\le 5$,

| $k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $r(k)$ | 2 | 6 | 12 | 25 | 44 |
| $l(k)$ | 0 | 0 | 20 | 160 | 636 |

# $E_{7+1/2}$ at higher levels

However, for level $k > 5$,

| | $k$ | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|
| | $r(k)$ | 79 | 128 | 208 | 318 |
| $E_7$ | $l(k)$ | 2384 | 6804 | 19015 | 46056 |
| $E_{7+1/2}$ | $l(k)$ | $\frac{23837}{10}$ | $\frac{210906}{31}$ | $\frac{152105}{8}$ | $\frac{506574}{11}$ |

**Conjecture**

*VOA $(E_{7+1/2})_k$ is only rational for $1 \le k \le 5$. For these levels, the characters of $(E_{7+1/2})_k$ and $(E_7)_k$ satisfy the MLDEs with the same degree $r(k)$ and the same index $l$.*

# Rank-two instanton VOAs

- The VOAs associated to rank-two $H_G$ theories for the Deligne series was studied in (Beem-Meneghelli-Peelaers-Rastelli 19).
- The Schur indices also computed in (Gu-Klemm-KS-Wang 19), equivalently the VOA vacuum characters (turning off the $SU(2)_{\epsilon_-}$ symmetry), satisfy an uniform 4th order MLDE

$$\Big[D^4 + \tfrac{2-h^\vee}{12}\Theta_{0,1}D^3 - \Big(\tfrac{25+3h^\vee+8h^{\vee 2}}{288}\Theta_{0,2} + \tfrac{1-9h^\vee-4h^{\vee 2}}{288}\Theta_{1,1}\Big)D^2$$
$$+\Big(\tfrac{138+41h^\vee-36h^{\vee 2}+h^{\vee 3}}{6912}\Theta_{0,3} - \tfrac{38+15h^\vee+20h^{\vee 2}-5h^{\vee 3}}{2304}\Theta_{1,2}\Big)D$$
$$+\tfrac{(11+5h^\vee)(11-3h^\vee-11h^{\vee 2}+3h^{\vee 3})}{331776}\Theta_{0,4} + \tfrac{(11+5h^\vee)(11-51h^\vee+25h^{\vee 2}+9h^{\vee 3})}{82944}\Theta_{1,3}$$
$$-\tfrac{167-662h^\vee+120h^{\vee 2}+270h^{\vee 3}+65h^{\vee 4}}{110592}\Theta_{2,2}\Big]\chi_{vac} = 0,$$

where $\Theta_{r,s}(\tau) = \theta_2(\tau)^{4r}\theta_3(\tau)^{4s} + \theta_2(\tau)^{4s}\theta_3(\tau)^{4r}\ , \quad r \leqslant s.$

- Specializing to $h^\vee = 24$, we also find perfect solution

# Rank-two $E_{7+1/2}$ instanton VOA

- We find the vacuum character is

$$\chi_{vac} = q^{131/24}(1 + 193q + 380q^{3/2} + 18914q^2 + 68060q^{5/2} + 1299299q^3 + 6168280q^{7/2} + 70763062q^4 + 379716500q^{9/2} + \dots)$$

- We find the flavored vacuum character up to the overall factor is

$$\chi_{vac}(q,m) = 1 + (\chi_3 + \mathbf{190})q + \mathbf{190}\chi_2 q^{3/2} + (\chi_5 + (\mathbf{190}+1)\chi_3 + \mathbf{15504} + \mathbf{2640} + \mathbf{190} + 2)q^2 + \dots$$

Here $\chi_n$ is the character of the $n$-dimensional irreducible rep of $SU(2)$.

- We conjecture these exist a good VOA with the above vacuum character.
- It can be expected that the rank-$n$ instanton VOA associated with $E_{7+1/2}$ has symmetry

$$(E_{7+1/2})_{-5n} \times SU(2)_{-(5n+1)(n-1)/2}.$$

# A conjectural Weyl dimensional formula for $E_{7+1/2}$

Consider the following analogy of Weyl dimension formula

$$\dim(R_\lambda) = \frac{\prod_{\alpha\in\Delta_+}(\lambda+\rho,\alpha)\prod_{\alpha\in\frac{1}{2}\mathbf{56}}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Delta_+}(\rho,\alpha)\prod_{\alpha\in\frac{1}{2}\mathbf{56}}(\rho,\alpha)},$$

where $(,)$ is the $E_7$ bilinear form. We checked this formula for arbitrary purely bosonic weights, i.e. $\lambda=[n_1,n_2,n_3,0,n_5,0,0]$, in particular the dimensions are integers. The resulting dimensions are

$$1, 190, 2640, 15504, 17765, 392445, 749360, 1089270, 1770496, 2078505,$$
$$24732110, 26001690, 28139760, 64489040, 89109240, 111532869, 252065970\ldots$$

Many dimensions predicted in (Cohen-de Man 96) appear here.

# A conjectural Weyl dimensional formula for $E_{7+1/2}$

When $n_2 = n_3 = n_5 = 0$, our formula reduces to the dimension formula for $n\theta$ reps proved in (Landsberg-Manivel 06). We find

$$\dim(n\theta) = \frac{(2n+23)}{2^{39}3^{20}5^9 7^6 11^4 13^3 17^2 19^1 23^1} \prod_{j=1}^{22}(n+j) \prod_{j=5}^{18}(n+j) \prod_{j=8}^{15}(n+j)$$

It is easy to show this is equivalent to the Theorem 7.1 of (Landsberg-Manivel 06).

We also find the analogy of Weyl dimension formula for irreducible representations with only one fermionic fundamental weight.

# Data for $E_{7+1/2}$ irreducible representations

Table: All irreducible representations of $E_7$ and $E_{7+1/2}$ with level $k \leq 3$.

| $k$ | $\lambda$ | $E_7$ | $C_2$ | $I/6$ | $E_{7+1/2}$ | $C_2$ | $I/6$ | CdM | F/B |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0000010 | **56** | $\frac{57}{2}$ | 1 | **57** | 40 | 1 | $f$ | F |
| 2 | 1000000 | **133** | 36 | 3 | **190** | 48 | 4 | $\mathfrak{g}$ | B |
| 2 | 0000001 | **912** | $\frac{105}{2}$ | 30 | **1045** | 72 | 33 | $-Y_4^*$ | F |
| 2 | 0000100 | **1539** | 56 | 54 | **2640** | 76 | 88 | $Y_2^*$ | B |
| 2 | 0000020 | **1463** | 60 | 55 | **1520** | 84 | 56 | $Y_3^*$ | B |
| 3 | 1000010 | **6480** | $\frac{133}{2}$ | 270 | **9728** | 90 | 384 | $-$ | F |
| 3 | 0100000 | **8645** | 72 | 390 | **17765** | 96 | 748 | $X_2$ | B |
| 3 | 0001000 | **27664** | $\frac{165}{2}$ | 1430 | **38760** | 112 | 1904 | $-G^*$ | F |
| 3 | 0000011 | **40755** | 84 | 2145 | **87040** | 114 | 4352 | $C^*$ | B |
| 3 | 0000110 | **51072** | $\frac{177}{2}$ | 2832 | **102410** | 120 | 5390 | $-$ | F |
| 3 | 0000030 | **24320** | $\frac{189}{2}$ | 1440 | **25840** | 132 | 1496 | $-H^*$ | F |

# Data for $E_{7+1/2}$ irreducible representations

Table: All irreducible representations of $E_7$ and $E_{7+1/2}$ with level $k=4$.

| $k$ | $\lambda$ | $E_7$ | $C_2$ | $I/6$ | $E_{7+1/2}$ | $C_2$ | $I/6$ | CdM | F/B |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 2000000 | **7371** | 76 | 351 | **15504** | 100 | 680 | $Y_2$ | B |
| 4 | 1000001 | **86184** | $\frac{185}{2}$ | 4995 | **150480** | 124 | 8184 | – | F |
| 4 | 1000100 | **152152** | 96 | 9152 | **392445** | 128 | 22032 | $A$ | B |
| 4 | 1000020 | **150822** | 100 | 9450 | **237405** | 136 | 14161 | $D^*$ | B |
| 4 | 0100010 | **362880** | $\frac{209}{2}$ | 23760 | **812592** | 140 | 49896 | – | F |
| 4 | 0010000 | **365750** | 108 | 24750 | **1089270** | 144 | 68796 | $X_3$ | B |
| 4 | 0000002 | **253935** | 112 | 17820 | **347490** | 152 | 23166 | $I^*$ | B |
| 4 | 0000101 | **861840** | $\frac{229}{2}$ | 61830 | **1896960** | 154 | 128128 | – | F |
| 4 | 0001010 | **980343** | 116 | 71253 | **3023280** | 156 | 206856 | $F^*$ | B |
| 4 | 0000021 | **885248** | $\frac{237}{2}$ | 65728 | – | 160 | – | – | F |
| 4 | 0000200 | **617253** | 120 | 46410 | **2078505** | 160 | 145860 | $J$ | B |
| 4 | 0000120 | **915705** | 124 | 71145 | – | 168 | – | – | B |
| 4 | 0000040 | **293930** | 132 | 24310 | – | 184 | – | – | B |

# $E_{7+1/2}$ as a gauge algebra

If $E_{7+1/2}$ can be realized as a gauge algebra, then the 5d one $E_{7+1/2}$ instanton Nekrasov partition function – one $E_{7+1/2}$ instanton Hilbert series should be

$$Z_1^{\mathrm{Nek}}(v) = v^{h^\vee - 1}\sum_{n=0}^{\infty} v^{2n}\chi^{\mathfrak{g}}_{n\theta} \quad \text{(Benvenuti-Hanany-Mekareeya 10)}$$

$$= \frac{1}{(v-v^{-1})^{46}}(v^{\pm 23} + 144v^{\pm 21} + 7799v^{\pm 19} + 217646v^{\pm 17} + 3587175v^{\pm 15} + 37732006v^{\pm 13} + 266204829v^{\pm 11} + 1303208244v^{\pm 9} + 4533843651v^{\pm 7} + 11399199625v^{\pm 5} + 20952141111v^{\pm 3} + 28356500429v^{\pm 1}).$$

The rational expression of $v = e^{\epsilon_+}$ is palindromic as required.

**Question**

How to turn on the $E_{7+1/2}$ gauge fugacities?

# Thank you!

# Virasoro minimal models $M(p,p')$

Virasoro minimal models are marked by a pair of coprime integers $(p,p')$ with $p>1, p'>1$ and have central charge and conformal weights

$$c = 1 - 6\frac{(p-p')^2}{pp'}, \qquad h_{r,s} = \frac{(pr-p's)^2-(p-p')^2}{4pp'}.$$

There are in total $(p-1)(p'-1)/2$ characters which can be chosen from $1 \le r \le p'-1$, $1 \le r \le p'-1$ and $p's < pr$.

Typical examples of Virasoro minimal models include

- $M(5,2)$ Lee-Yang model
- $M(4,3)$ Ising model
- $M(5,4)$ tricritical Ising model

Minimal models are unitary when $|p-p'|=1$. For example, Lee-Yang model is non-unitary.

# Wess-Zumino-Witten models $(\mathfrak{g})_k$

WZW models are 2d RCFTs with additional conserved currents generating an affine Lie algebra. They are nonlinear sigma models with group manifold $G$ as target, depending on a positive integer $k$ called level.

Denote $C_2(R_\lambda) = \langle \lambda + 2\rho, \lambda \rangle$ as the quadratic Casimir invariant. The central charge and conformal weights are

$$\mathrm{WZW}\ (\mathfrak{g})_k: \quad c = \frac{\mathrm{dim}(\mathfrak{g})k}{h^\vee_\mathfrak{g} + k}, \quad h_i = \frac{C_2(R_\lambda)}{2(h^\vee_\mathfrak{g} + k)} \text{ with } a^\vee \cdot \lambda = k.$$

The characters of WZW model $(\mathfrak{g})_k$ are also the level $k$ characters of untwisted affine Lie algebra $\mathfrak{g}^{(1)}$ which have the following universal expansion

$$\widehat{\chi}^\mathfrak{g}_\lambda(\tau) = q^{-\frac{1}{24}\frac{\mathrm{dim}(\mathfrak{g})k}{h^\vee_\mathfrak{g}+k} + \frac{C_2(R_\lambda)}{2(h^\vee_\mathfrak{g}+k)}}(\mathrm{dim}(\chi^\mathfrak{g}_\lambda) + \mathcal{O}(q)).$$

# Modularity of characters of 2d RCFTs

1. Denote the least common denominator of $h_i - c/24$ as $N$, which is called the conductor of the theory.
2. $\{\chi_i\}$ form a vector-valued modular form of $SL(2,\mathbb{Z})$, i.e., there exists a $d$-dimensional representation $\rho : SL(2,\mathbb{Z}) \to GL(d,\mathbb{C})$ such that for any $\gamma \in SL(2,\mathbb{Z})$,
$$\chi_i(\gamma\tau) = \sum_j \rho(\gamma)_{ij}\chi_j(\tau).$$
3. Each character $\chi_i$ is invariant under $\tau \to \gamma\tau$ for any $\gamma \in \Gamma(N)$ defined as
$$\Gamma(N) = \left\{\gamma \in SL(2,\mathbb{Z}) \middle| \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\right\},$$
which is the kernel of the mod $N$ map $\mu_N : SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}_N)$.

# Hecke operator

The Hecke transformation for $SL(2,\mathbb{Z})$ modular forms is a well-known operation in number theory. It maps a weight $k$ modular form $f(\tau)$ to another weight $k$ modular form by

$$(\mathsf{T}_p f)(\tau) := p^{k-1} \sum_{a,d>0,ad=p} \frac{1}{d^k} \sum_{b \pmod d} f\left(\frac{a\tau+b}{d}\right).$$

The proper generalization to RCFT characters that keeps both the vector-valued nature and $\Gamma(N)$ modularity was given in (Harvey-Wu 18).

# Hecke operator

Suppose $p$ is coprime to $N$. Denote $\bar{p}$ as the multiplicative inverse of $p$ modulo $N$ and $\sigma_p$ as $\mu_N^{-1}\mathrm{diag}(\bar{p},p)$. Then Hecke operator $\mathsf{T}_p$ acts on $\chi_i(\tau)$ as (Harvey-Wu 18)

$$(\mathsf{T}_p\chi)_i(\tau) := \sum_j \rho_{ij}(\sigma_p)\chi_j(p\tau) + \sum_{b=0}^{p-1}\chi_i\Big(\frac{\tau+bN}{p}\Big).$$

The matrix $\rho(\sigma_p)$ can be computed by the $S$ and $T$ matrices as

$$\rho(\sigma_p) = \rho(T^{\bar{p}}S^{-1}T^{p}ST^{\bar{p}}S),$$

where the two generators of $SL(2,\mathbb{Z})$,

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hecke operation maps central charge $c \to pc$ and conformal weights $h_i \to ph_i$ mod $\mathbb{Z}$.