Characters of log VOAs and quantum invariants

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Motivation

- Curious relationships between various quantum invariants of knots/3-manifolds and characters of (log)VOAs have been discovered.
  - colored Jones polynomial of torus knots/links and characters of Virasoro/singlet/triplet VOAs.
  - homological block of 3 or 4-fibered Seifert manifolds and $(1,p)/(p,p')$-logVOAs.
- However, compared to the former, the latter theory is less well known. It is important to construct a large number of interesting logVOAs and to develop a unified methodology for study of them.

  In particular, correspondence between knots/3-manifolds and logVOAs.
- In this talk, I propose a clue to this problem by developing the geometric calculation method of characters of $(1,p)$-logVOA proposed by Feigin–Tipunin.

Setting

- $p \in \mathbb{Z}_{\geq 2}$, $\mathfrak{g}$: (ADE type) simple Lie algebra
- Let us consider the finite parameter set

$$\Lambda_p := \{ \sum_{i=1}^{\mathrm{rank}\mathfrak{g}} \tfrac{r_i - 1}{p} \varpi_i \mid 1 \leq r_i \leq p \} \simeq \tfrac{1}{p} P / P$$

  For $\lambda \in \frac{1}{p} P$, $[\lambda]$ denotes the representative in Λ_p.
- The Weyl group W acts on Λ_p by $[\lambda] \mapsto [\sigma * \lambda]$, where $\sigma * \lambda = \sigma(\lambda + \frac{1}{p}\rho) - \frac{1}{p}\rho$. Set

$$\epsilon_{[\lambda]}(\sigma) = \tfrac{1}{p}(\sigma * [\lambda] - [\sigma * \lambda]) \in P$$

  In other words, $\epsilon_{[\lambda]}(\sigma)$ is the “carry over” of the W-action.

Felder complex

Definition (Felder complex)

We call the data $(V_{[\lambda]})_{[\lambda]\in\Lambda_p}$ Felder complex if

- $V_{[\lambda]}$'s are weight B-modules with the grading $V_{[\lambda]} = \bigoplus_{\Delta\in\Delta_{[\lambda]}+\mathbb{Z}_{\geq 0}} (V_{[\lambda]})_\Delta$ compatible with the B-action.
- There exists linear operators $Q_i^{[\lambda]} \colon V_{[\lambda]} \to V_{[\sigma_i * \lambda]}$ such that
  - $\ker Q_i^{[\lambda]}$ admits the P_i-action.
  - For $[\lambda] \notin \Lambda_p^{\sigma_i}$, we have

$$0 \to \ker Q_i^{[\lambda]} \to V_{[\lambda]} \to \ker Q_i^{[\sigma_i * \lambda]}(\epsilon_{[\lambda]}(\sigma_i)) \to 0$$

    as B-modules.

(More precisely, we need more parameters as $V_{\hat{\lambda};[\lambda]}$, $V_{\pm,\hat{\lambda};[\lambda]}, \ldots$)

Felder complex is illustrated as follows.

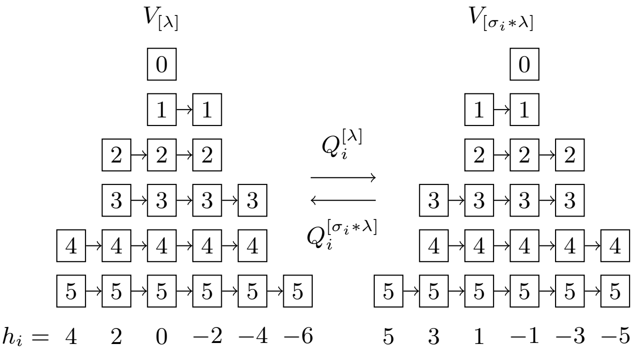

Figure: Felder complex.

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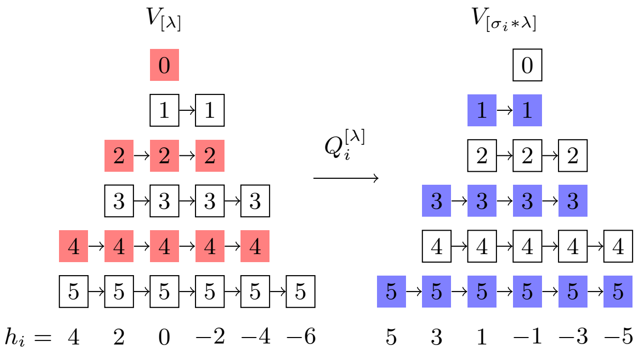
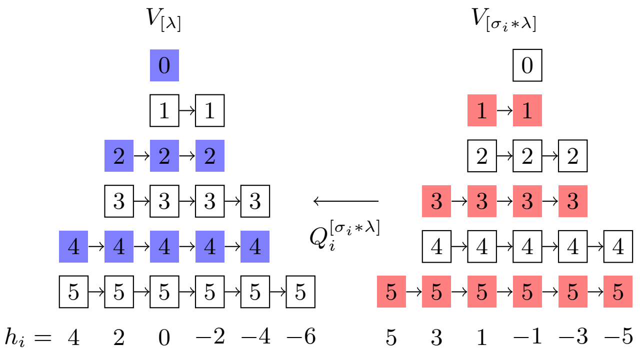


$$0 \to \underbrace{\ker Q_i^{[\lambda]}}_{\bigoplus_{k\geq 0} \mathbb{C}^{2k+1} \otimes 2k} \hookrightarrow \text{LHS} \twoheadrightarrow \underbrace{\frac{\text{LHS}}{\ker Q_i^{[\lambda]}}} \to 0.$$

$$\simeq \underbrace{\ker Q_i^{\sigma_i * \lambda}}_{\bigoplus_{k\geq 0} \mathbb{C}^{2k+2} \otimes 2k+1} \quad (\epsilon_\lambda(\sigma_i))$$



$$0 \to \underbrace{\ker Q_i^{[\sigma_i*\lambda]}}_{\bigoplus_{k\geq 0} \mathbb{C}^{2k+2} \otimes\, 2k+1} \hookrightarrow \text{RHS} \twoheadrightarrow \underbrace{\frac{\text{RHS}}{\ker Q_i^{[\sigma_i*\lambda]}}}_{\simeq\, \underbrace{\ker Q_i^{[\lambda]}}_{\bigoplus_{k\geq 0} \mathbb{C}^{2k+1} \otimes\, 2k} \quad (\epsilon_{\sigma_j * \lambda}(\sigma_j))} \to 0.$$

Feigin–Tipunin's construction $H^0(G \times_B V_{[\lambda]})$

We call $H^0(G \times_B V_{[\lambda]})$ Feigin–Tipunin's construction.

Theorem (S'21, S'22, Creutzig-Nakatsuka-S)

If $(V_{[\lambda]})_{[\lambda]\in\Lambda_p}$ is a Felder complex, then we have

- *The evaluation map at $\mathrm{id} \in G/B$ gives*

$$H^0(G \times_B V_{[\lambda]}) \hookrightarrow \bigcap_{i=1}^{\mathrm{rank}\mathfrak{g}} \ker Q_i^{[\lambda]} \subseteq V_{[\lambda]}.$$

  In particular, $H^0(G \times_B V_{[\lambda]})$ is isomorphic to the maximal G-submodule of $V_{[\lambda]}$ and $\hookrightarrow$ above is $\simeq$ iff $[\lambda]$ is near to 0.

- *For $\beta \in P_+$, we have $\mathrm{ch}_q\, V_{[\lambda]}^{h=\sigma\circ\beta} = \mathrm{ch}_q\, V_{[\sigma*\lambda]}^{h=\beta-\epsilon_{[\lambda]}(\sigma)}$*

- *(Borel–Weil–Bott type duality) If $(p[\lambda]+\rho,\theta) \le p$, then we have $H^n(G \times_B V_{[\lambda]}) \simeq H^{n+l(w_0)}(G \times_B V_{[w_0*\lambda]}(-\rho))$. In particular, $H^{n>0}(G \times_B V_{[\lambda]}) = 0$.*

$\mathrm{ch}_q\, V_{[\lambda]}^{h=\sigma\circ\beta} = \mathrm{ch}_q\, V_{[\sigma*\lambda]}^{h=\beta-\epsilon_{[\lambda]}(\sigma)}$

In the case $\mathfrak{g} = \mathfrak{sl}_2$, $\mathrm{ch}_q\, V_{[\lambda]}^{h=\sigma\circ\beta} = \mathrm{ch}_q\, V_{[\sigma*\lambda]}^{h=\beta-\epsilon_{[\lambda]}(\sigma)}$ is illustrated is follows.

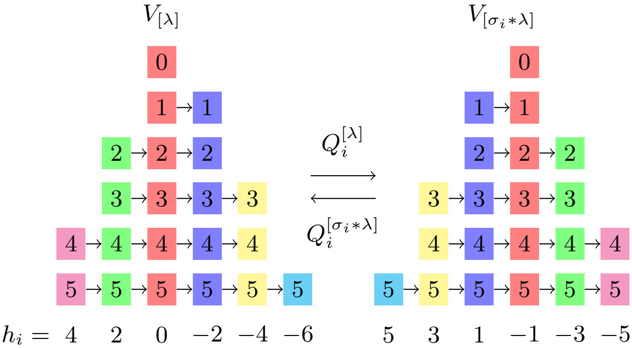


Figure: $\mathrm{ch}_q\, V_{[\lambda]}^{h=\sigma\circ\beta} = \mathrm{ch}_q\, V_{[\sigma*\lambda]}^{h=\beta-\epsilon_{[\lambda]}(\sigma)}$ for $\mathfrak{g} = \mathfrak{sl}_2$.

BWB duality

In the case $\mathfrak{g} = \mathfrak{sl}_2$, BWB duality is proven as follows.

- By applying the long exact sequence of $H^\bullet(\mathrm{SL}_2 \times_B -)$ to the short exact sequence
$$0 \to \ker Q^{[\lambda]} \to V_{[\lambda]} \to \ker Q^{[\sigma*\lambda]}(-\varpi) \to 0,$$
we have $H^n(\mathrm{SL}_2 \times_B V_{[\lambda]}) \simeq \delta_{n,0} \ker Q^{[\lambda]}$.
- On the other hand, by applying the long exact sequence of $H^\bullet(\mathrm{SL}_2 \times_B -)$ to the short exact sequence
$$0 \to \ker Q^{[\sigma*\lambda]}(-\varpi) \to V_{[\sigma*\lambda]}(-\varpi) \to \ker Q^{[\lambda]}(-2\varpi) \to 0,$$
we have $H^n(\mathrm{SL}_2 \times_B V_{[\sigma*\lambda]}(-\varpi)) \simeq \delta_{n,1} \ker Q^{[\lambda]}$.
- Therefore, $H^n(\mathrm{SL}_2 \times_B V_{[\lambda]}) \simeq H^{n+1}(\mathrm{SL}_2 \times_B V_{[\sigma*\lambda]}(-\varpi))$.

Felder complex and $\mathrm{char} > 0$

Remark

Borel–Weil–Bott type duality above implies that the theory of Felder complex and that of reductive algebraic group with $\mathrm{char} > 0$ (or quantum group at root of unity) are equivalent in some sense. In particular, it is expected that despite the BWB duality above holds only for the case $(p[\lambda] + \rho, \theta) \leq p$, we have

$$H^{n>0}(G \times_B V_{[\lambda]}) = 0$$

for any $[\lambda] \in \Lambda_p$ because of the Kempf vanishing theorem in another side. Moreover, by studying the counterparts of the results by Bezrukavnikov et al., we might be able to prove the log-Kazhdan-Lusztig corrspondence at the level of abelian categories.

Character formula by Atiyah–Bott formula

Let $\mathrm{ch}_q V$ be the character of V defined by

$$\mathrm{ch}_q V = \sum_{\Delta} \dim V_{\Delta} q^{\Delta}.$$

Then we have

$$\begin{aligned}
\mathrm{ch}_q H^0(G \times_B V_{[\lambda]}) &= \sum_{n \geq 0} (-1)^n \, \mathrm{ch}_q H^n(G \times_B V_{[\lambda]}) \\
&= \sum_{\beta \in P_+} \dim L(\beta) \sum_{\sigma \in W} (-1)^{l(\sigma)} \, \mathrm{ch}_q V_{[\lambda]}^{h=\sigma \circ \beta} \\
&= \sum_{\beta \in P_+} \dim L(\beta) \sum_{\sigma \in W} (-1)^{l(\sigma)} \, \mathrm{ch}_q V_{[\sigma * \lambda]}^{h=\beta - \epsilon_{[\lambda]}(\sigma)},
\end{aligned}$$

i.e. $\mathrm{ch}_q H^0(G \times_B V_{[\lambda]})$ is reduced to $\mathrm{ch}_q V_{[\sigma * \lambda]}^{h=\beta - \epsilon_{[\lambda]}(\sigma)}$.

Example: $(1,p)$-logVOA for $(\mathfrak{g}, f_{\mathrm{prin}})$

- $V_{\sqrt{p}Q}$: lattice VOA assoc to the rescaled root lattice $\sqrt{p}Q$
- The conformal vector is given by

$$\omega := \frac{1}{2} \sum_{1 \le i,j \le \mathrm{rank}\mathfrak{g}} c^{ij} \alpha_{i(-1)} \alpha_j + \sqrt{p}(1 - \tfrac{1}{p}) \rho_{(-2)} \mathbf{1}.$$

- $V_{[\lambda]} = V_{\sqrt{p}(Q-\hat{\lambda})+[\lambda]}$, $\quad \hat{\lambda}$: minuscle weight
- The B-module structure on $V_{[\lambda]}$ is given by

$$f_i = \int e^{\sqrt{p}\alpha_i} dz, \quad h_i = \lceil -\tfrac{1}{p} \alpha_{i(0)} \rceil$$

- The linear operator $Q_i^{[\lambda]}$ is the short screening operator

$$Q_i^{[\lambda]} = \int e^{-\frac{1}{\sqrt{p}}\alpha_i}(z_1) \cdots e^{-\frac{1}{\sqrt{p}}\alpha_i}(z_{(p[\lambda]+\rho, \alpha_i)}) d\vec{z}$$

- We call $H^0(G \times_B V_{\sqrt{p}Q})$ the $(1,p)$-logVOA for $(\mathfrak{g}, f_{\mathrm{prin}})$.

Example: $(1,p)$-logVOA for $(\mathfrak{g}, f_{\mathrm{prin}})$

Then $(V_{[\lambda]})_{[\lambda]\in\Lambda_p}$ consists a Felder complex and we have the following.

Theorem (S'21, S'22)

- $H^0(G\times_B V_{[\lambda]}) \hookrightarrow \bigcap_{i=1}^{\mathrm{rank}\,\mathfrak{g}} \ker Q_i^{[\lambda]}$ *and it is isomorphic iff* $[\lambda]$ *is near to* 0. *In particular, two definitions of* $(1,p)$*-logVOA coincides.*
- $H^0(G\times_B V_{[\lambda]}) \simeq \bigoplus_{\beta\in P_+} L(\beta)\otimes \mathcal{W}_{\beta+[\lambda]}$, *where* $\mathcal{W}_{\beta+[\lambda]}$ *is a* $\mathcal{W}_0$*-module with l.w.* $\Delta_{\beta+[\lambda]}$. *Note that* $\mathcal{W}^{p-h}(\mathfrak{g})$ *is a sub VOA of* $\mathcal{W}_0$.
- *(For* $(p[\lambda]+\rho,\theta)\leq p$*) we have*

$$\mathrm{ch}_q\, H^0(G\times_B V_{[\lambda]}) = \frac{1}{\eta(q)^{\mathrm{rank}\,\mathfrak{g}}}\sum_{\beta\in P_+}\dim L(\beta)\sum_{\sigma\in W}(-1)^{l(\sigma)}q^{\Delta_{-\sqrt{p}\beta+\sigma*[\lambda]}}.$$

- *For* $(p[\lambda]+\rho,\theta)\leq p$, $H^0(G\times_B V_{[\lambda]})$ *is simple as* $(1,p)$*-logVOA module and each* $\mathcal{W}_{\beta+[\lambda]}$ *so is as* $\mathcal{W}^{p-h}(\mathfrak{g})$*-modules.*

Example: $(1,p)$-logVOA for $(\mathfrak{sl}_2, 0)$

- $V^k(\mathfrak{sl}_2) \hookrightarrow \beta\gamma \otimes V_{\sqrt{p}\mathrm{A}_1} \hookrightarrow \Pi[0] \otimes V_{\sqrt{p}\mathrm{A}_1}$
- $V_{[r]} \in \{\beta\gamma \otimes V_{r,s},\ \tau(\Pi[\frac{r}{p}] \otimes V_{r,s}),\ \Pi[b] \otimes V_{r,s} \mid [b] \neq [0], [\frac{r}{p}]\}$
- The B-module structure on $V_{[r]}$ is given by

$$f = \int \beta \otimes e^{\sqrt{p}\alpha} dz,\ h = \lceil -\tfrac{1}{\sqrt{p}}\alpha_{(0)} + \tfrac{1}{p}(u+v)_{(0)}) \rceil$$

  and the grading is the conformal one.
- The linear operator is the short screening operator

$$Q^{[r]} = \int \Pi_{i=1}^{r} e^{-\frac{1}{\sqrt{p}}\alpha + \frac{1}{p}(u+v)}(z_i) d\vec{z}$$

- We call $H^0(\mathrm{SL}_2 \times_B \beta\gamma \otimes V_{\sqrt{p}\mathrm{A}_1})$ the $(1,p)$-logVOA for $(\mathfrak{sl}_2, 0)$.

Example: $(1,p)$-logVOA for $(\mathfrak{sl}_2, 0)$

Then $(V_{[r]})_{1\leq r\leq p}$ consists a Felder complex and we have the following:

Theorem (Creutzig-Nakatsuka-S)

- $H^0(G\times_B V_{[r]})\simeq \ker Q^{[r]}$.
- $(\mathrm{SL}_2, V^k(\mathfrak{sl}_2))$*-module structure on* $H^0(G\times_B V_{[r]})$.
- *BWB duality and character formula (two variables).*
- *simplicity theorem.*

The same type results would be hold for general $(1,p)$-logVOA for $(\mathfrak{g}, f)$.

Beyond $(1, p)$-logVOAs?

From the results above, certain aspects of the representation theory and structure of $(1, p)$-logVOA are controlled by the theory of Felder complex, which is essentially not a issue on VOA, but simple Lie algebra/group (put more simply, $\mathfrak{sl}_2$). In other words, regardless of the complexity of the specific form of the VOA-modules, B-action, etc., its representation theory can be studied.

Question

Is the theory of Felder complex used to study other logVOAs?
In other words, how fundamental a position does the theory of $(1, p)$-logVOA occupy in the that of logVOAs?

Example: (p_1, p_2)-logVOA for $(\mathfrak{sl}_2, f_{\mathrm{prin}})$

- $p_1, p_2 \in \mathbb{Z}_{\geq 2}$: coprime, $\quad p := p_1 p_2$
- $V_{\sqrt{p}\mathrm{A}_1}$ is the lattice VOA with the conformal vector

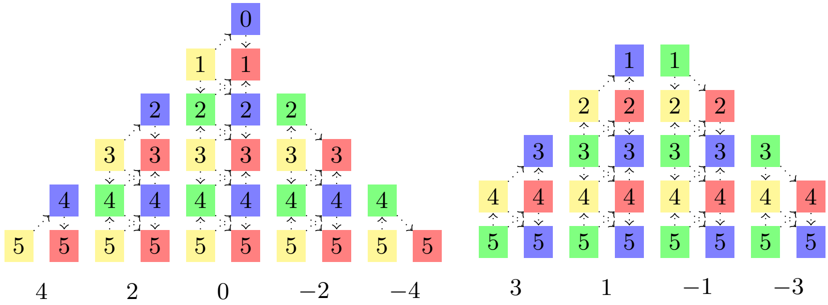
$$\frac{1}{4}\alpha_{(-1)}\alpha + \sqrt{p}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\rho_{(-2)}\mathbf{1}$$

- $V_{[r_1],[r_2]} := V_{\sqrt{p}(\mathrm{A}_1 - \hat{\lambda}) - [r_1] + [r_2]}$, where $[r_1] \in \Lambda_{p_1}$ and $[r_1] \in \Lambda_{p_2}$.
- The linear operators

$$Q^{[r_1]} \colon V_{[r_1],[r_2]} \to V_{[p_1 - r_1],[r_2]}, \quad Q^{[r_2]} \colon V_{[r_1],[r_2]} \to V_{[r_1],[p_2 - r_2]}$$

  are short screening operators.
- $V_{[r_1],[r_2]}$ does not consists a Felder complex, but $\mathrm{Im}\, Q^{[p_1 - r_1]} \subsetneq V_{[r_1],[r_2]}$ consists a Felder complex with the B-action definined by Frobenius homomorphism.

The socle sequence of $V_{[r_1],[r_2]}$ is illustrated as



- k, k, k, k : simple $U(L)$-modules $L_{p_1-r_1,r_2,-k}$, $L_{r_1,r_2,-k}$, $L_{p_1-r_1,p_2-r_2,-k}$, $L_{r_1,p_2-r_2,-k}$, respectively.
- $M_0 \supsetneq M_1 \supsetneq M_2$, where $M_0/M_1 = k$, $M_1/M_2 = k \oplus k$, $M_3 = k$.
- If we exchange r_1 with $p_1 - r_1$ (resp.r_2 with $p_2 - r_2$), then the colors also exchange.

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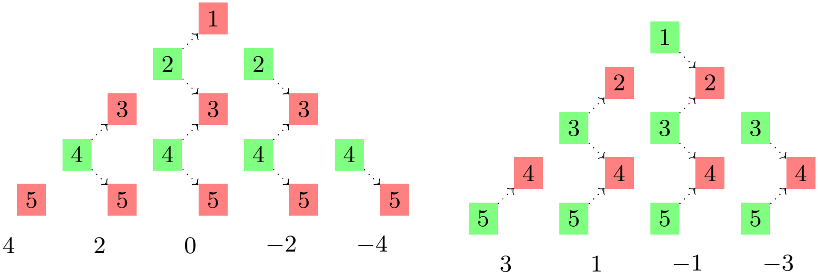
The socle sequence of $\operatorname{Im} Q^{[p_1 - r_1]}$ is illustrated as

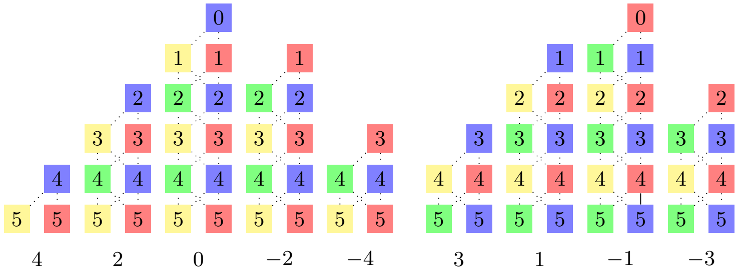


- They consist a Felder complex by the B-action by the Frobenius homomorphism and the remaining short screening operator $Q^{[r_2]}$.
- In particular, by taking $H^0(\mathrm{SL}_2 \times_B -)$, only k remains, which consists the simple module $\mathcal{X}_{[r_1],[r_2]}$ of the (p_1, p_2)-logVOA.
  In particular, we can reduce $\mathrm{ch}_q\, \mathcal{X}_{[r_1],[r_2]}$ to $\mathrm{ch}_q \operatorname{Im} Q^{[r_1] \geq 0}$.

Question

Can we calculate $\mathrm{ch}_q \operatorname{Im} Q^{[r_1]\geq 0}$ *using Felder complex (or Atiyah–Bott)?*

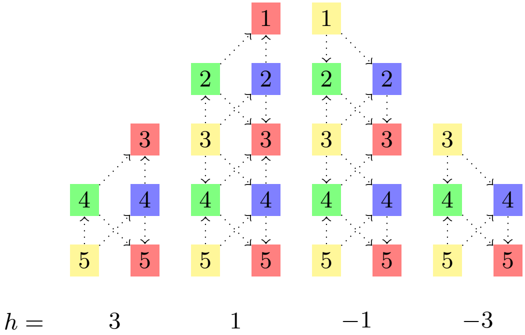
Let us consider a $L(c_{p_1,p_2},0)$-module $\tilde{V}_{[r_1],[r_2]}$ illustrated as



- I don't know the socle sequence, but the composition factors are supposed to be given as above. i.e. add k and $k-1$ to $V^{h=-k<0}_{[r_1],[r_2]}$. For a makeshift definition of $\tilde{V}_{[r_1],[r_2]}$, see [Hikami-S].
- Then they has the shape of Felder complex by regarding the pairs (k , $k+1$) and (k , $k+1$) as weight vectors.

Calculation on $\mathrm{ch}_q\,\mathcal{X}_{[r_1],[r_2]}$

Using this $\tilde{V}_{[r_1],[r_2]}$, let us calculate $\mathrm{ch}_q\,\mathcal{X}_{[r_1],[r_2]}$ using Felder complex (or Atiyah–Bott formula).
Let us recall that the socle sequence of $V_{[p_1-r_1],[r_2]}$ is given as follows.



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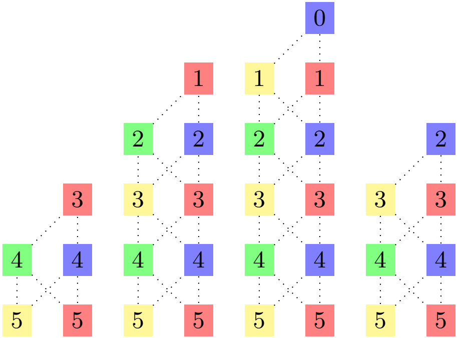
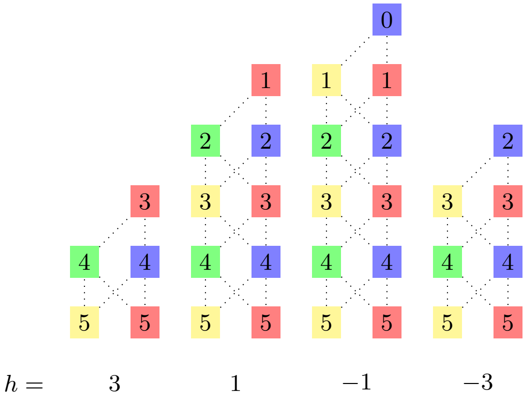
Then $\tilde{V}_{[p_1 - r_1],[r_2]}$ has the shape as follows.



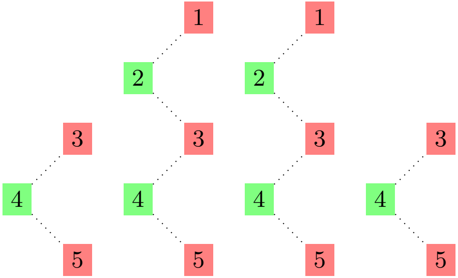
$h =$ 3 1 -1 -3

$\tilde{V}_{[p_1-r_1],[r_2]}$



$H^0(G \times_B \tilde{V}_{[p_1 - r_1],[r_2]})$

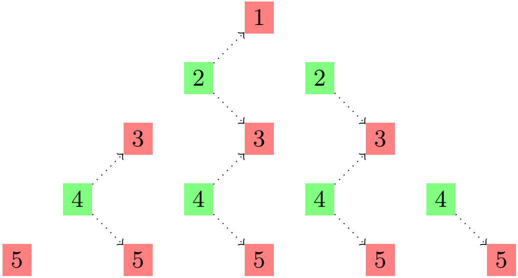
By taking $H^0(\mathrm{SL}_2 \times_B -)$, we obtain the following.



$h = \quad 3 \quad 1 \quad -1 \quad -3$

$\tilde{H}^0(G \times_B \tilde{V}_{[r_1],[r_2]}) := \operatorname{Im} Q^{[p_1-r_1]}$

Set $\tilde{H}^0(\mathrm{SL}_2 \times_B \tilde{V}_{[r_1],[r_2]}) := \operatorname{Im} Q^{[p_1-r_1]}$, which is illustrated as



$h = \quad 4 \quad 2 \quad 0 \quad -2 \quad -4$

Then we have

$$\mathrm{ch}_q\, \tilde{H}^0(\mathrm{SL}_2 \times_B \tilde{V}_{[r_1],[r_2]})^{h=k\varpi} = \mathrm{ch}_q\, H^0(\mathrm{SL}_2 \times_B \tilde{V}_{[p_1-r_1],[r_2]})^{h=(k+1)\varpi}$$

$H^0(G \times_B \tilde{H}^0(G \times_B \tilde{V}_{[r_1],[r_2]}))$

Since $\tilde{H}^0(\mathrm{SL}_2 \times_B V_{[r_1],[r_2]})$ consists a Felder complex again, we take $H^0(\mathrm{SL}_2 \times_B -)$ and obtain $\mathcal{X}_{[r_1],[r_2]} = H^0(\mathrm{SL}_2 \times_B \tilde{H}^0(\mathrm{SL}_2 \times_B V_{[r_1],[r_2]}))$.

			1		
		3	3	3	
	5	5	5	5	5
$h =$	4	2	0	-2	-4

Indeed, we have

$$
\begin{aligned}
&\eta(q)\,\mathrm{ch}_q\, H^0(\mathrm{SL}_2 \times_B \tilde{H}^0(\mathrm{SL}_2 \times \tilde{V}_{[r_1],[r_2]}))^{h=0} \\
&= \sum_{n_2 \geq 0} (\mathrm{ch}_q\, \tilde{H}^0(\mathrm{SL}_2 \times_B \tilde{V}_{[r_1],[r_2]})^{h=2n_2} - \mathrm{ch}_q\, \tilde{H}^0(\mathrm{SL}_2 \times_B \tilde{V}_{[r_1],[p_2-r_2]})^{h=2n_2+1}) \\
&= \sum_{n_2 \geq 0} (\mathrm{ch}_q\, \tilde{H}^0(\mathrm{SL}_2 \times_B \tilde{V}_{[p_1-r_1],[r_2]})^{h=2n_2+1} - \mathrm{ch}_q\, \tilde{H}^0(\mathrm{SL}_2 \times_B \tilde{V}_{[p_1-r_1],[p_2-r_2]})^{h=2n_2} \\
&= \sum_{n_1, n_2 \geq 0} (\mathrm{ch}_q\, \tilde{V}^{h=2n_1+2n_2+1}_{[p_1-r_1],[r_2]} - \mathrm{ch}_q\, \tilde{V}^{h=2n_1+2n_2+2}_{[p_1-r_1],[p_2-r_2]}) \\
&\quad - (\mathrm{ch}_q\, \tilde{V}^{h=2n_1+2n_2+2}_{[r_1],[r_2]} - \mathrm{ch}_q\, \tilde{V}^{h=2n_1+2n_2+3}_{[r_1],[p_2-r_2]}) \\
&= \sum_{n \geq 0} n(q^{\Delta_{p_1-r_1,r_2,2n+1}} - q^{\Delta_{p_1-r_1,p_2-r_2,2n+2}} - q^{\Delta_{r_1,r_2,2n+2}} + q^{\Delta_{r_1,p_2-r_2,2n+3}})
\end{aligned}
$$

and it coinsides with $\mathrm{ch}_q\, \mathcal{X}^{h=0}_{[r_1],[r_2]}$.

$(1, p_1p_2)$-logVOA v.s. (p_1, p_2)-logVOA

- In the case of $(1, p_1p_2)$-logVOA, there is only one parameter $[\lambda] \in \Lambda_{p_1p_2}$, and we consider the Felder complex w.r.t this $[\lambda]$.
- On the other hand, in the case of (p_1, p_2)-logVOA, there are two parameters $[\lambda_1] \in \Lambda_{p_1}$ and $[\lambda_2] \in \Lambda_{p_2}$, and we apply the theory of Felder complex (or Atiyah–Bott character formula) to $[\lambda_1]$ and $[\lambda_2]$ separately.
- These two Felder complexes are related by the "symmetrization of Felder complex"
$$\mathrm{ch}_q\, \tilde{H}^0(G \times_B \tilde{V})^{h=k\geq 0} = \mathrm{ch}_q\, H^0(G \times_B \tilde{V})^{h=k+1}$$
above, and it enables us to the nested-use of Atiyah–Bott character formulae.
- These discussions also holds for
$$([\lambda_1], \dots, [\lambda_N]) \in \Lambda_{p_1} \times \cdots \times \Lambda_{p_N}$$

Nested Felder complex/Feigin–Tipunin's construction

From the above discussion, it is natural to consider the following.

- $p_1, \ldots, p_N \in \mathbb{Z}_{\geq 2}$: coprime, $p := p_1 \cdots p_N$.
- $Q_0 := \frac{1}{p_1} - \frac{1}{p_2} - \cdots - \frac{1}{p_N}$.
- $V_{\sqrt{p}Q}$: lattice VOA with conformal vector

$$\omega := \frac{1}{2} \sum_{1 \leq i,j \leq \mathrm{rank}\mathfrak{g}} c^{ij} \alpha_{i(-1)} \alpha_j + \sqrt{p} Q_0 \rho_{(-2)} \mathbf{1}.$$

- For $[\lambda_i] \in \Lambda_{p_i}$, set

$$\vec{\lambda} = ([\lambda_1], \ldots, [\lambda_N]) \in \Lambda_{p_1} \times \cdots \times \Lambda_{p_N}$$

- $V_{\vec{\lambda}} := V_{\sqrt{p}(Q - \hat{\lambda}) + \sqrt{p}(-\lambda_1 + \lambda_2 \cdots + \lambda_N)}$: simple $V_{\sqrt{p}Q}$-module ($\hat{\lambda}$; minuscle weight).

Definition (Nested Felder complex/Feigin–Tipunin's construction)

We call $(\underbrace{\tilde{H}^0(G\times_B\cdots\tilde{H}^0(G\times_B}_{0\leq m\leq N-1}\tilde{V}_{\vec{\lambda}}\cdots))_{[\vec{\lambda}]\in\Lambda_p}$ nested Felder complex if

- For each $0\leq m\leq N-1$, $\underbrace{\tilde{H}^0(G\times_B\cdots\tilde{H}^0(G\times_B}_{m}\tilde{V}_{\dots,[\lambda_{N-m}],\dots}\cdots)$

  consists a Felder complex.
- For $\beta\in P_+$ and $\sigma\in W$, we have

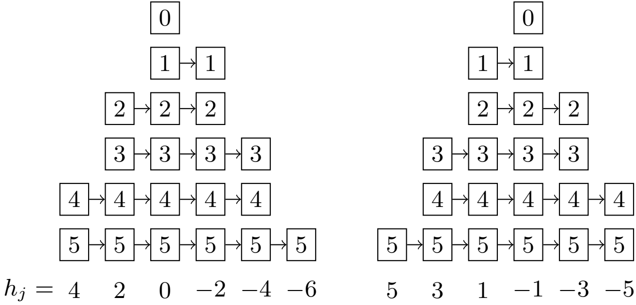
$$\bullet\,\mathrm{ch}_q\,\tilde{V}_{\vec{\lambda}}^{h=\beta-\epsilon_{[\lambda_N]}(\sigma)}=\mathrm{ch}_q\,V_{\vec{\lambda}}^{h=\beta},$$

$$\bullet\,\mathrm{ch}_q\,\tilde{H}^0(G\times_B\tilde{V}_{[\lambda_1],\dots})^{h=\beta}=\mathrm{ch}_q\,\tilde{H}^0(G\times_B\tilde{V}_{[w_0*\lambda_1],\dots})^{h=\beta+\rho},$$

$$\bullet\,\mathrm{ch}_q\,\underbrace{\tilde{H}^0(G\times_B\cdots\tilde{H}^0(G\times_B}_{m\geq 2}\tilde{V}_{\vec{\lambda}}\cdots)^{h=\beta-\epsilon_{[\lambda_{N-m}]}(\sigma)}$$

$$=\mathrm{ch}_q\,H^0(G\times_B\underbrace{\tilde{H}^0(G\times_B\cdots\tilde{H}^0(G\times_B}_{m-1}\tilde{V}_{\vec{\lambda}}\cdots)^{h=\beta-\epsilon_{[\lambda_{N-m}]}(\sigma)+\rho}$$

The second condition above is "symmetrization of Felder complex" or "connection of Felder complex w.r.t $[\lambda_m]$ and that w.r.t $[\lambda_{m-1}]$". Let us recall that a Felder complex is illustrated as



[redacted]

Then the second condition is illustrated is follows.

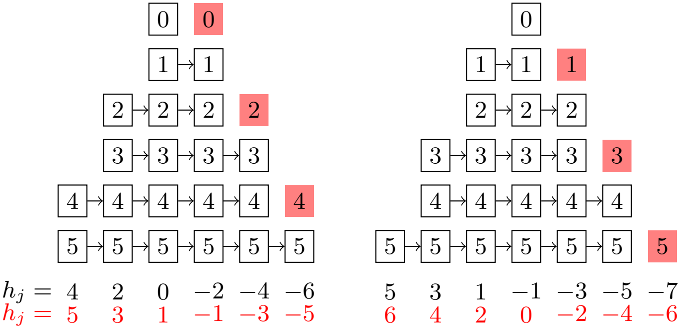
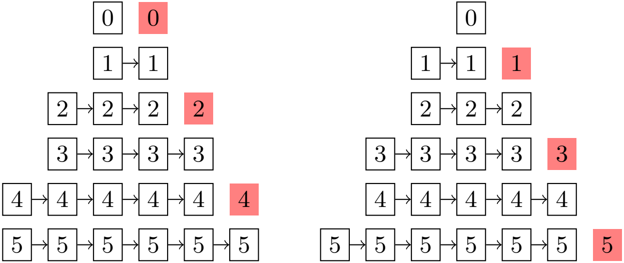


Figure: $\tilde{H}^0(G \times_B V)$ and $H^0(G \times_B V)$.

As we see in the case of $N = 2$ (i.e. (p_1, p_2)-logVOA), if we regard the pair ($\boxed{k}$, $\boxed{k+1}$) (or (k [highlighted], $\boxed{k+1}$)) as weight vectors, then these pictures are regarded as the maximal G-submodule of a (larger) Felder complex.



$h_j = 4 \quad 2 \quad 0 \quad -2 \quad -4 \quad -6$ $\qquad$ $5 \quad 3 \quad 1 \quad -1 \quad -3 \quad -5 \quad -7$

$h_j = 5 \quad 3 \quad 1 \quad -1 \quad -3 \quad -5$ $\qquad$ $6 \quad 4 \quad 2 \quad 0 \quad -2 \quad -4 \quad -6$

In the same manner as $N=2$, we obtain the character formula.

Theorem (S)

We have the character formula

$$\eta(q)^{\operatorname{rank}\mathfrak{g}} \operatorname{ch}_q H^0(G\times_B \underbrace{\tilde{H}^0(G\times_B \cdots \tilde{H}^0(G\times_B}_{N-1} \tilde{V}_{\hat{\lambda};\lambda_1,\dots,\lambda_N})\cdots)^{h=\hat{\lambda}}$$

$$= \sum_{\gamma_1,\dots,\gamma_N\geq 0} p(\gamma_1)\cdots p(\gamma_N) \sum_{w_1,\dots,w_N\in W} (-1)^{l(w_1)+\cdots+l(w_N)}$$

$$\operatorname{ch}_q V_{w_1*w_0*\lambda_1,w_2*\lambda_2,\dots,w_N*\lambda_N}^{h=\gamma_1+\cdots+\gamma_N+\hat{\lambda}+(N-1)\rho-\epsilon_{w_0*\lambda_1}(w_1)-\sum_{i=2}^N \epsilon_{\lambda_i}(w_i)},$$

where $p(-)$ is the Kostant's partition function.

In particular, when $\mathfrak{g}=\mathfrak{sl}_2$ and $\hat{\lambda}=0$, the character is given by

$$\sum_{n\geq 0}\binom{n+N-1}{N-1}\sum_{\epsilon_1,\dots,\epsilon_N\in\{\pm 1\}} \epsilon_1\cdots\epsilon_N q^{\frac{p}{4}(2n+N-\sum_{i=1}^N \frac{\epsilon_i r_i}{p_i})},$$

which coincides with the homological block of $(N+2)$-fibered Seifert manifold!

Conclusion

When $\mathfrak{g} = \mathfrak{sl}_2$, $\mathrm{ch}_q\, H^0(G \times_B \tilde{H}^0(G \times_B \cdots))$ is calculated as

- First, we decompose the Fock spaces to 2^N types of colors/components.
- We can decompose the colors as $2^N = 2^{N-1} + 2^{N-1}$, where the first 2^{N-1} colors appear symmetric w.r.t. Cartan weight h, but the latter is shifted by -1. In other words, we obtain a Felder complex w.r.t $[\lambda_N]$.
- By applying $H^0(G \times_B -)$, only 2^{N-1} colors that are symmetric w.r.t the Cartan weight h are taken out.
- In $h \geq 0$, it has the same character as a Felder complex w.r.t $[\lambda_{N-1}]$ consisting of $2^{N-1} = 2^{N-2} + 2^{N-2}$ colors. So we can compute the character by applying $H^0(G \times_B -)$ again (in other words, by taking 2^{N-2} colors that are syemmetric w.r.t the Cartan weight h, again).
- By repeating this procedure, we can take out only one color, which is the desired logVOA(-module) of the form

$$\text{“}H^0(G \times_B H^0(G \times_B \cdots H^0(G \times_B \tilde{V}_{\vec{\lambda}} \cdots)\text{”}$$

Future work (1)

Let us recall the question above.

Question

Is the theory of Felder complex used to study other logVOAs?
In other words, how fundamental a position does the theory of $(1, p)$-logVOA occupy in the that of logVOAs?

The following conjecture claims that the theory of Felder complex/$(1, p)$-logVOA is fundamental in that of conjectural logVOAs corresponding to (Seifert/plumbed) 3-manifolds.

Conjecture

There exists logVOA(-modules)

$$H^0(G \times_B \tilde{H}^0(G \times_B \cdots \tilde{H}^0(G \times_B \tilde{V}_{\vec{\lambda}} \cdots)$$

(i.e. given by nested Feigin–Tipunin construction) such that the character coincides with the homological block of corresponding Seifert manifold.

Future work (2)

- For $c \leq r$, the limit of $\mathfrak{sl}_r$-colored Jones polynomial of the torus link $T(c, cp)$ gives the character of $(1, p)$-logVOA for $(\mathfrak{sl}_c, f_{\text{prin}})$.
- On the other hand, the limits of $\mathfrak{sl}_2$-colored Jones polynomial of the torus link $T(2p, 2p')$ (minus certain modular form) gives the character of (p, p')-logVOA for $(\mathfrak{sl}_2, f_{\text{prin}})$.
- So it is expected that the limits of $\mathfrak{sl}_r$-colored Jones polynomial of the torus link $T(cp, cp')$ (minus certain modular form) gives the character of (p, p')-logVOA for $(\mathfrak{sl}_c, f_{\text{prin}})$, but (p, p')-logVOA is constructed only for the case $r = 2$.
- However, we can expect the character of (irreducible modules of) (p, p')-logVOA for $(\mathfrak{g}, f_{\text{prin}})$ is given by

$$\sum_{\gamma_1, \gamma_2 \geq 0} p(\gamma_1) p(\gamma_2) \sum_{w_1, w_2 \in W} (-1)^{l(w_1)+l(w_2)} \operatorname{ch}_q V_{w_1 * w_0 * \lambda_1, w_2 * \lambda_2}^{h = \gamma_1 + \gamma_2 + \hat{\lambda} + \rho - \epsilon_{[w_0 * \lambda_1]}(w_1) - \epsilon_{[\lambda_2]}(w_2)},$$

and thus we can check the expectation above.

Thank you!