Combinatorial relations among relations for standard $C_n^{(1)}$-modules of level 2,3,...?

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Representation Theory XVIII
June 26-30, 2023, IUC Dubrovnik, Croatia

Joint work with Mirko Primc

Znanstveni centar izvrsnosti
za kvantne i kompleksne sustave te
reprezentacije Liejevih algebri

Projekt KK.01.1.1.01.0004

Projekt je sufinancirala Europska unija iz Europskog fonda za regionalni razvoj. Sadržaj ovog seminara isključiva je odgovornost Prirodoslovno-matematičkog fakulteta Sveučilišta u Zagrebu te ne predstavlja nužno stajalište Europske unije.

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PROVEDBA VRHUNSKIH ISTRAŽIVANJA U SKLOPU ZNANSTVENOG CENTRA IZVRSNOSTI ZA KVANTNE I KOMPLEKSNE SUSTAVE TE REPREZENTACIJE LIEJEVIH ALGEBRI

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Affine Lie algebras

- Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $\langle\ ,\ \rangle$ a symmetric invariant bilinear form on $\mathfrak{g}$ and we assume that $\langle\theta,\theta\rangle = 2$ for the maximal root θ
- Denote by $\Delta(= \Delta_+ \cup \Delta_-)$ roots (positive and negative roots)
- Triangular decomposition $\mathfrak{g} = \mathfrak{N}_+ + \mathfrak{h} + \mathfrak{N}_-$
- Fix root vectors X_α
- $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] + \mathbb{C}c \quad , \quad \tilde{\mathfrak{g}} = \hat{\mathfrak{g}} + \mathbb{C}d$ is the associated untwisted affine Kac-Moody Lie algebra
- $x(m) = x \otimes t^m$ for $x \in \mathfrak{g}$ and $i \in \mathbb{Z}$, c is the canonical central element, and $[d, x(m)] = mx(m)$
- $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{<0} + (\mathfrak{g} + \mathbb{C}c) + \hat{\mathfrak{g}}_{>0} \quad , \quad \hat{\mathfrak{g}}_{<0} = \sum_{m<0} \mathfrak{g}(m)$

Highest weight modules and VOA

- Λ highest weight, v_Λ highest weight vector
- Verma modul $M(\Lambda)$, $L(\Lambda)$ irr. modul
- level of representation $k = \Lambda(c)$ (for us $k = 1, 2, \cdots$)
- we can form the induced $\tilde{\mathfrak{g}}$-module (a generalized Verma modul)

$$N(k\Lambda_0)(or\ V^k(\mathfrak{g})) = \mathcal{U}(\tilde{\mathfrak{g}}) \otimes_{\mathcal{U}(\tilde{\mathfrak{g}})_{\geq 0}} \mathbb{C}v_{k\Lambda_0}$$

- $N(k\Lambda_0) \cong \mathcal{U}(\tilde{\mathfrak{g}})_{<0}$ (as vector space)

Relations for level k standard (vacuum) $\hat{\mathfrak{g}}$-modules

Let R be the finite dimensional $\mathfrak{g}$-module generated by the singular vector in $N(k\Lambda_0)$, i.e.

$$R = U(\mathfrak{g}) \cdot x_\theta(-1)^{k+1}\mathbf{1} \cong L_{\mathfrak{g}}((k+1)\theta),$$

where x_θ is a root vector for the maximal root θ with respect to a chosen Cartan decomposition of $\mathfrak{g}$. Then the coefficients $r(m)$, $r \in R$, $m \in \mathbb{Z}$, of vertex operators

$$Y(r,z) = \sum_{m\in\mathbb{Z}} r(m)z^{-m-k-1}$$

span a loop $\hat{\mathfrak{g}}$-module $\bar{R}$. Since $\bar{R}N(k\Lambda_0) \subset N(k\Lambda_0)$ is the maximal submodule of the generalized Verma module we have

$$L(k\Lambda_0) = \mathcal{THM} = N(k\Lambda_0)/\bar{R}N(k\Lambda_0) \quad \text{and} \quad \bar{R}\,|_{L(k\Lambda_0)} = 0, \quad (1)$$

Annihilating fields

- $L(k\Lambda_0) = N(k\Lambda_0)/N^1(k\Lambda_0)$
- $N^1(k\Lambda_0) = \bar{R}N(k\Lambda_0) = \mathcal{U}(\tilde{\mathfrak{g}})\bar{R}v_\Lambda \rightsquigarrow \bar{R}$ *Relations*
- Field $Y(x_\theta(-1)^{k+1}, z) = x_\theta(z)^{k+1}$ generates all annihilating fields of $L(k\Lambda_0)$
- $x_\theta(z)^{k+1} = \sum_{m\in\mathbb{Z}} r_{(k+1)\theta}(m)z^{-m-k-1}$

$$= \sum_{m\in\mathbb{Z}} [\sum_{i_1+\cdots+i_{k+1}=m} x_\theta(i_1)\cdots x_\theta(i_{k+1})]z^{-m-k-1}$$

We can use the relations $\bar{R}$ to construct a combinatorial bases of $L(k\Lambda_0)$—the basic idea is to reduce the PBW spanning set of $L(k\Lambda_0)$ to a basis $\mathcal{B}$ by using relations $r|_{L(k\Lambda_0)} = 0$, and to parameterize the monomial vectors

$$u(\pi)\mathbf{1} \in \mathcal{B} \subset L(k\Lambda_0) = U(\hat{\mathfrak{g}})\mathbf{1}$$

with monomials π in the symmetric algebra $S(\hat{\mathfrak{g}})$.

Combinatorial and Gröbner bases

Problem:
Find a combinatorial basis of $L(k\Lambda_0)$ $\Leftrightarrow$ Find a "Gröbner basis" of $\bar{R}N(k\Lambda_0)$

- solved for all $\tilde{\mathfrak{sl}}_2$-modules $L(\Lambda)$
  [Meurman - Primc: *Annihilating Fields of Standard Modules of $\tilde{\mathfrak{sl}}_2$ and Combinatorial Identities*; Memoirs of AMS 1999]
- solved for basic modules $L(\Lambda_0)$ for all affine symplectic Lie algebras $C_n^{(1)}$
  [Primc-Š: *Combinatorial bases of basic modules for affine Lie algebras $C_n^{(1)}$*;J. Math. Phys. 2016]
- conjectured for standard modules $L(k\Lambda_0)$ for affine symplectic Lie algebras $C_n^{(1)}$
  [Primc-Š: *Leading terms of relations for standard modules of affine Lie algebras $C_n^{(1)}$*; Ramanujan J. 2019]

The methodology by steps

- Choose an (appropriately) **totally ordered basis** B of $\mathfrak{g}$ and extend the strict order $\prec$ on B to $\bar{B}$ so that $m < m'$ implies $b(m) \prec b'(m')$. Since $b(m)\mathbf{1} = 0$ for $m \geq 0$, in some arguments it is enough to consider basis elements in

$$\bar{B}_{<0} = \{b(m) \mid b \in B, m < 0\} = \bar{B} \cap \hat{\mathfrak{g}}_{<0}.$$

- Denote by $\mathcal{P}$ the set of monomials

$$\pi = \prod_{b(j) \in \bar{B}} b(j)^{n_{b(j)}} \in S(\hat{\mathfrak{g}})$$

and by $\mathcal{P}_{<0} = \mathcal{P} \cap S(\hat{\mathfrak{g}}_{<0})$ and interpret as a **colored partition** of length $\ell(\pi)$, degree $|\pi|$ and support $\operatorname{supp} \pi$

The methodology by steps

- Determine the set of **leading terms** $\ell t(\bar{R}) \subset \mathcal{P}$ of relations $r(m) \in \bar{R} \setminus \{0\}$:

$$\rho = \ell t(r(m)) \quad \text{if} \quad r(m) = c_\rho u(\rho) + \sum_{\rho \prec \kappa} c_\kappa u(\kappa), \quad c_\rho \neq 0.$$

  Then we can parameterize a basis of the loop module $\bar{R}$ by its leading terms,

- By using relations $r(\rho)|_{L(k\Lambda_0)} = 0$ reduce the spanning set $\{u(\pi)\mathbf{1} \mid \pi \in \mathcal{P}_{<0}\}$ of $L(k\Lambda_0)$ to a basis

$$\mathcal{B} = \{u(\pi) \cdot \mathbf{1} \mid \pi \in \mathcal{P}_{<0} \setminus (\ell t(\bar{R}))\}.$$

  Here $\pi \in \mathcal{P}_{<0} \setminus (\ell t(\bar{R}))$ denotes monomials π which are not in the ideal $(\bar{R})$ generated by relations $\bar{R}$.

The methodology by steps

- By using relations $r(\rho)|_{L(k\Lambda_0)} = 0$ reduce the spanning set $\{u(\pi)\mathbf{1} \mid \pi \in \mathcal{P}_{<0}\}$ of $L(k\Lambda_0)$ to a basis

$$\mathcal{B} = \{u(\pi) \cdot \mathbf{1} \mid \pi \in \mathcal{P}_{<0} \setminus (\ell t(\bar{R}))\}.$$

- Remark: If we think of monomials π as colored partitions, then the spanning set of monomial vectors $\mathcal{B} \subset L(k\Lambda_0)$ is parameterized by partitions which do not contain any subpartition $\rho \in \ell t(\bar{R})$—this is some sort of combinatorial "difference conditions" on parts of the partition π.

Relation among relations

In order to describe a combinatorial basis of $\bar{R}N(k\Lambda_0)$ for an embedding $\rho \subset \pi$ such that $\pi = \rho\kappa$, $\rho \in \ell t(\bar{R})$ we have $r(\rho)u(\kappa) = u(\pi) + \sum_{\pi \prec \tau} c_\tau u(\tau)$, and $\ell t(u(\rho \subset \pi)) = \pi$. With this notation we can write the spanning set of $\bar{R}N(k\Lambda_0)$ as

$$u(\rho \subset \pi)\mathbf{1}, \quad \rho \in \ell t(\bar{R}), \pi \in (\ell t(\bar{R})) \cap \mathcal{P}_{<0}. \tag{2}$$

If for any two embeddings $\rho_1 \subset \pi$ and $\rho_2 \subset \pi$ we have **a relation among relations**

$$u(\rho_1 \subset \pi)\mathbf{1} - u(\rho_2 \subset \pi)\mathbf{1} = \sum_{\pi \prec \pi',\ \rho' \subset \pi'} c_{\rho' \subset \pi'}\, u(\rho' \subset \pi')\mathbf{1}, \tag{3}$$

then we can reduce the spanning set (2) by using (3), and for each π we may take **just one embedding** $\rho(\pi) \subset \pi$, $\rho(\pi) \in \ell t(\bar{R})$ and the corresponding vector for the reduced spanning set of $\bar{R}N(k\Lambda_0)$

$$u(\rho(\pi) \subset \pi)\mathbf{1}, \quad \pi \in (\ell t(\bar{R})) \cap \mathcal{P}_{<0}. \tag{4}$$

Remark

If $\pi = \rho_1\rho_2\kappa$, $\rho_1, \rho_2 \in \ell t(\bar{R})$, then we have two embeddings $\rho_1 \subset \pi$ and $\rho_2 \subset \pi$ and $\ell(\pi) \geq 2k+2$ and (3) easily follows. Hence the problem is to check (3) "only" for

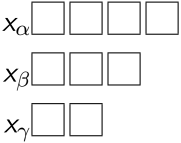
$$k+2 \leq \ell(\pi) \leq 2k+1.$$

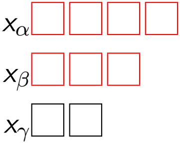
For $k=1$ we have $k+2 = 2k+1 = 3$, i.e. we have to check (3) only for $\ell(\pi) = 3$, and this was done in [PŠ; 2016].
On the other hand, for $k = 2 (= 3 = \cdots)$ we have to check (3) for $4 \leq \ell(\pi) \leq 5$ $(5 \leq \ell(\pi) \leq 7; \cdots)$. The main result are relations among relations (3) for $\ell(\pi) = 4$.
p.s.

- $k+2$ maximal intersection of leading terms
- $2k+1$ minimal intersection of leading terms

Maximal and minimal intersection k=1

$$k = 1 \Rightarrow k + 2 = 3 \quad \rightsquigarrow$$





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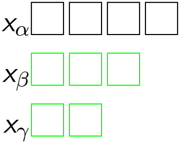
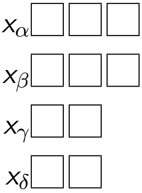
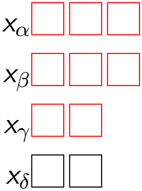


Maximal intersection k=2

$k = 2 \Rightarrow k + 2 = 4$





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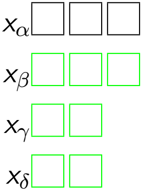
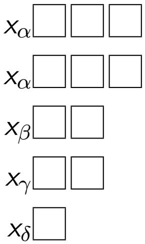
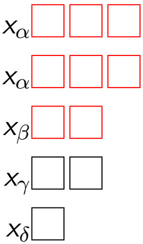


Minimal intersection k=2

$$k = 2 \Rightarrow 2k + 1 = 5 \quad \rightsquigarrow$$





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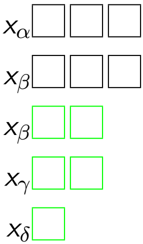


Ordered basis B and $\bar{B}$

- let B be the ordered basis of $\mathfrak{g}$
- We fix the basis $\bar{B}$ of $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$,

$$\bar{B} = \bigcup_{j \in \mathbb{Z}} B \otimes t^j,$$

- Let $\preceq$ be a linear order on $\bar{B}$ such that

$$i < j \quad \text{implies} \quad b(i) \prec b'(j).$$

- degree $|b(i)| = i$

Ordered basis B and $\bar{B}$

$$\pi = \prod_{i=1}^{\ell} b_i(j_i), \quad b_i(j_i) \in \bar{B},$$

- π is a colored partition of degree $|\pi| = \sum_{i=1}^{\ell} j_i \in \mathbb{Z}$ and length $\ell(\pi) = \ell$, with parts $b_i(j_i)$ of degree j_i and color b_i
- we shall usually assume that parts of π are indexed so that

$$b_1(j_1) \preceq b_2(j_2) \preceq \cdots \preceq b_\ell(j_\ell).$$

- we associate with a colored partition π its shape $\operatorname{sh}\pi$,

$$j_1 \leq j_2 \leq \cdots \leq j_\ell \quad (\text{"plain" partition}).$$

- the set of all colored partitions with parts $b_i(j_i)$ of degree $j_i (j_i < 0)$ is denoted as $\mathcal{P}(\mathcal{P}_{<0})$

Colored partitions

- $N(k\Lambda_0) \cong \mathcal{U}(\hat{\mathfrak{g}}_{<0}) \cong \mathcal{S}(\mathfrak{g}_{<0})$
  (Thx to PBW Thm ; like vec.space)
-
  $$(\prod_{b\in\bar{B}} b^{mult(b)})\cdot v_{k\Lambda_0} \cong \prod_{b\in\bar{B}} b^{mult(b)} \quad \text{ordered monomials as in } \mathcal{P}_{<0}$$

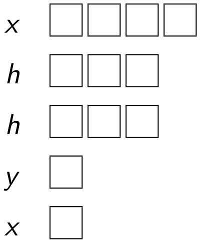
Colored partitions - example

Case: $\hat{\mathfrak{sl}}_2$; $B = \{x, h, y\}$; $y \prec h \prec x$

$$\textit{ordered monomial } u(\pi) = x(-4)h(-3)^2 y(-1)x(-1)v_{k\Lambda_0}$$

colored partitions

$$u(\pi) = x(-4)h(-3)^2 y(-1)x(-1)v_{k\Lambda_0} \quad \rightsquigarrow$$



$$\ell(\pi) = \sum_{b \in \bar{B}} mult(b) = 5 \quad |\pi| = \sum_{i=1}^{\ell} j_i = 12$$

Leading terms of the relation

On level k standard module $L(\Lambda)$ we have vertex operator relations

$$x_\theta(z)^{k+1} = \sum_{m\in\mathbb{Z}} r_{(k+1)\theta}(m) z^{-m-k-1} = 0$$

i.e. the coefficient (relations) of above annihilating fields are

$$r_{(k+1)\theta}(m) = \sum_{j_1+\cdots+j_{k+1}=m} x_\theta(j_1)\cdots x_\theta(j_{k+1}) \ .$$

The smallest summand in this sum is proportional to

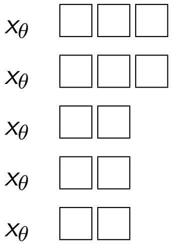
$$x_\theta(-j-1)^b x_\theta(-j)^a$$

for $a+b=k+1$ and $(-j-1)b+(-j)a=m$. Moreover, the shape of every other term Φ which appears in the sum is greater than the shape $(-j-1)^b(-j)^a$, so we can write

$$r_{(k+1)\theta}(m) = c\, x_\theta(-j-1)^b x_\theta(-j)^a + \sum_{\mathrm{sh}\,\Phi \succ (-j-1)^b(-j)^a} c_\Phi X(\Phi)$$

Leading terms of the relation - example

$$\ell t\,(r_{5\theta}(-12)) = x_\theta(-3)^2 x_\theta(-2)^3 \quad \rightsquigarrow$$



Remark:
For $a+b=k+1$ and $(-j-1)b+(-j)a=m$ we have only one possible shape. $b=|m|-(k+1)j$ *i.e.* $b \equiv |m|(k+1)$.

$$k=4\ ,\ \ m=-12 \Rightarrow b=2 \Rightarrow a=3 \Rightarrow j=-2$$

Leading terms of relation $r(m)$

The adjoint action of $U(\mathfrak{g})$ on $r_{(k+1)\theta}(m)$, $m \in \mathbb{Z}$, gives all other relations in $\bar{R}$. For $u \in U(\mathfrak{g})$ the relation $r(m) = u \cdot r_{(k+1)\theta}(m)$ can be written as

$$r(m) = \sum_{\text{sh } \Psi = (-j-1)^b(-j)^a} c_\Psi X(\Psi) + \sum_{\text{sh } \Psi \succ (-j-1)^b(-j)^a} c_\Psi X(\Psi) \ .$$

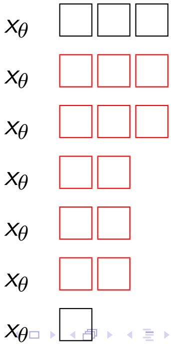
The actions of $u \in U(\mathfrak{g})$ in $\mathfrak{g}$-modules $\mathcal{U}$ and $\mathcal{S}$ are different, but we have $u\Big(c\, x_\theta(-j-1)^b x_\theta(-j)^a\Big) = \sum_{\text{sh } \Psi = (-j-1)^b(-j)^a} c_\Psi \Psi$ with the same coefficients c_Ψ as in the first summand in above equation. The smallest $\Psi \in \mathcal{P}^{k+1}(m)$ which appears in the first sum we call **the leading term of relation** $r(m)$ and we denote it as $\ell t\, r(m)$. Hence we can rewrite above equation as

$$r(m) = c_\Phi X(\Phi) + \sum_{\Psi \succ \Phi} c_\Psi X(\Psi), \qquad \Phi = \ell t\, r(m).$$

Embeddings of leading terms

- k=4...$\ell t \bar{R} = \{\ell t\, r(m)\}$ parametrize a basis $\{r(\rho) \mid \rho \in \ell t \bar{R}\}$ of $\bar{R}$
- for $\kappa \in \mathcal{P}$, $\rho \in \ell t \bar{R}$ and $\pi = \kappa\rho$ we say that ρ is embedded in π (we write $\rho \subset \pi$)
- $u(\rho \subset \pi) = u(\kappa) r(\rho)$
- $\ell t(u(\rho \subset \pi)) = \pi$

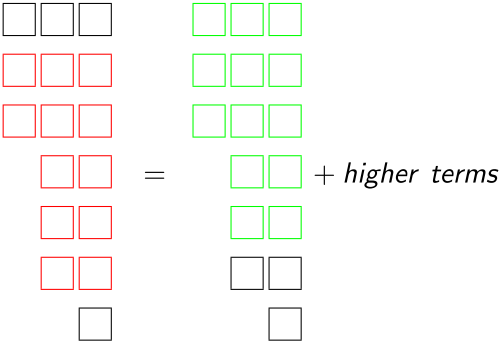
$$x_\theta(-3)\ \color{red}{r_{5\theta}(-12)}\ x_\theta(-1) \quad \rightsquigarrow$$



Relation among relations

For any π with two embeddings $\rho_1 \subset \pi$ and $\rho_2 \subset \pi$ we have a relation among relations

$$u(\rho_2 \subset \pi)\mathbb{1} = u(\rho_1 \subset \pi)\mathbb{1} + \text{higher terms???}$$



Simple Lie algebra of type C_n ($\mathfrak{sp}_{2n}$)

These vectors form a basis B of $\mathfrak{g}$ which we shall write in a triangular scheme, e.g. for $n = 3$ the basis B is

$$\begin{array}{llllll}
11 & & & & & \\
12 & 22 & & & & \\
13 & 23 & 33 & & & \\
1\underline{3} & 2\underline{3} & 3\underline{3} & \underline{33} & & \\
1\underline{2} & 2\underline{2} & 3\underline{2} & \underline{32} & \underline{22} & \\
1\underline{1} & 2\underline{1} & 3\underline{1} & \underline{31} & \underline{21} & \underline{11}
\end{array}$$

Case $C_n^{(1)}$

For general rank we may visualize admissible pair of cascades (=leading term) as figure below

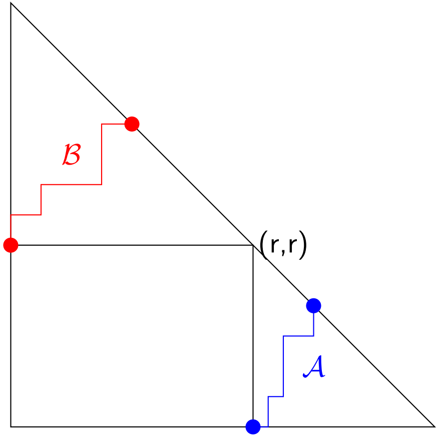


Figure 1

Case $C_n^{(1)}$

Theorem (Primc-Š 2019)

Let $(-j-1)^b(-j)^a, \quad j \in \mathbb{Z}, \quad a+b=k+1, \quad b \geq 0,$ *be a fixed shape and let* $\mathcal{B}$ *and* $\mathcal{A}$ *be two cascades in degree* $-j-1$ *and* $-j$*, with multiplicities* $(m_{\beta,j+1}, \beta \in \mathcal{B})$ *and* $(m_{\alpha,j}, \alpha \in \mathcal{A})$*, such that* $\sum_{\beta\in\mathcal{B}} m_{\beta,j+1} = b, \quad \sum_{\alpha\in\mathcal{A}} m_{\alpha,j} = a$. *Let* $r \in \{1, \cdots, n, \underline{n}, \cdots, \underline{1}\}$. *If the points of cascade* $\mathcal{B}$ *lie in the upper triangle* $\triangle_r$ *and the points of cascade* $\mathcal{A}$ *lie in the lower triangle* ${}^r\triangle$*, then*

$$\prod_{\beta\in\mathcal{B}} X_\beta(-j-1)^{m_{\beta,j+1}} \prod_{\alpha\in\mathcal{A}} X_\alpha(-j)^{m_{\beta,j}}$$

is the leading term of a relation for level k *standard module for affine Lie algebra of the type* $C_n^{(1)}$.

Conjecture

Let $n \geq 2$ and $k \geq 2$. We consider the standard module $L(k\Lambda_0)$ for the *aff* $\mathcal{LA}$ of type $C_n^{(1)}$ ($\{X_{ab}(j) \mid ab \in B, j \in \mathbb{Z}\} \cup \{c, d\}$ base).
We conjecture that the set of monomial vectors

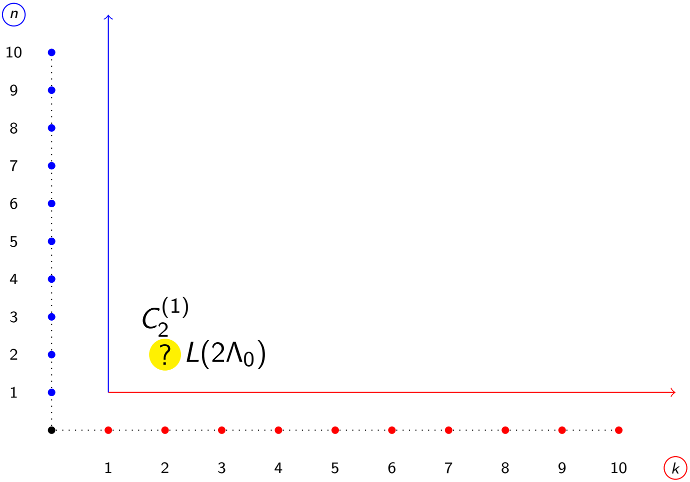
$$\prod_{ab \in B,\, j>0} X_{ab}(-j)^{m_{ab;j}}\, v_0,$$

satisfying difference conditions $\sum_{ab \in \mathcal{B}} m_{ab;j+1} + \sum_{ab \in \mathcal{A}} m_{ab;j} \leq k$ *for any admissible pair of cascades* $(\mathcal{B}, \mathcal{A})$*, is a basis of* $L(k\Lambda_0)$*.*

The conjecture is true for

- $n = 1$ and all $k \geq 1$ [Meurman-Primc]
- $k = 1$ for all $n \geq 2$ [Primc-Š 2016]
  [J. Dousse-I. Konan preprint][M. C. Russel preprint]

Recent situation...



New frontiers

- Case $C_n^{(1)}$ for $k = 2$
- Case $C_2^{(1)}$ for $k \geq 2$
- Case $C_n^{(1)}$ for $n \geq 2$ and $k \geq 2$

Main theorem [(arXiv:2301.11222)]

Theorem

For any two embeddings $\rho_1 \subset \pi$ *and* $\rho_2 \subset \pi$ *in* $\pi \in \mathcal{P}^4(m)$*, where* $\rho_1, \rho_2 \in \ell t(\bar{R})$*, we have a level 2 relation for* $C_n^{(1)}$

$$u(\rho_1 \subset \pi) - u(\rho_2 \subset \pi) = \sum_{\pi \prec \pi', \ \rho \subset \pi'} c_{\rho \subset \pi'} \, u(\rho \subset \pi'). \qquad (5)$$

Before the proof: a much more appropriate notation

We will reinterpret the term of cascade with two identical triangles from Figure 1, but one is rotated and mirrored and then both are rotated.

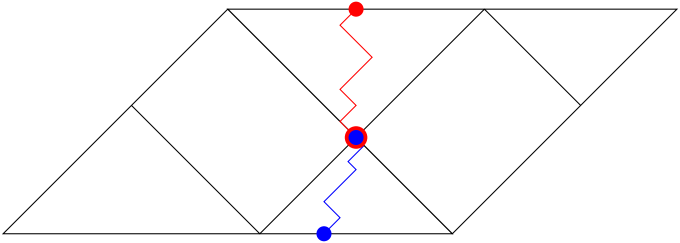


Figure 2

From Figure 2, it is already obvious that the pair of admissible cascades has become a zig-zag line.

A much more appropriate notation

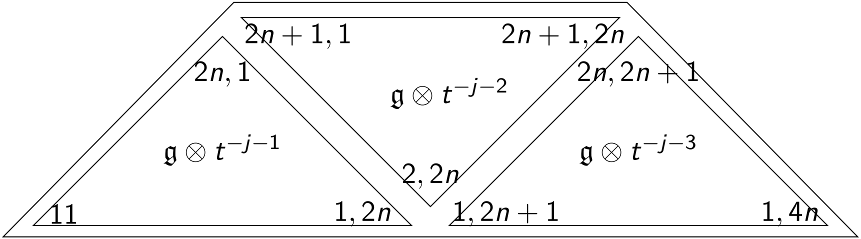


Figure 7

Proof

To prove the theorem, for a fixed ordinary partition p of length 4 we need to count the number of two-embeddings for $\operatorname{sh}\pi = p$

$$\sum_{\operatorname{sh}\pi=p,\ \pi\in\mathcal{P}^4(m)} N(\pi).$$

where is

$$N(\pi) = \max\{\#\mathcal{E}(\pi) - 1, 0\} \quad \mathcal{E}(\pi) = \{\rho \in \mathcal{lt}(\bar{R}) \mid \rho \subset \pi\}.$$

It turns out it is enough, but much easier, to count for a trapezoid T the number

$$N_T = \sum_{\pi,\ \ell(\pi)=4,\ \operatorname{supp}\pi\subset T} N(\pi).$$

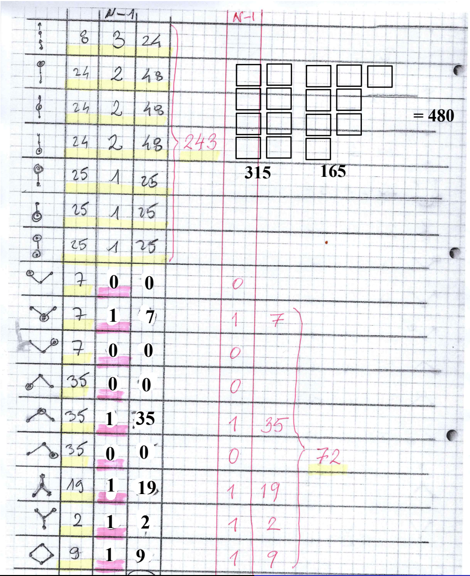
Proof

By 14 technical Lemmas were proved for three successive triangles that the number of two-embeddings for 16 admissible supports of 4 different types (A_r, B_r, C_r, D_r) of π is

$$N_T = \sum_{r=2}^{4} N_T(A_r) + \sum_{r=1}^{2} \sum_{\delta = |,||} (N_T(B_{r\,\delta}) + N_T(C_{\delta\,r}) + N_T(D_{1\,\delta\,1}))$$

$$= \frac{7(10n-1)}{4} \binom{2n+6}{7} .$$

Case $C_2^{(1)}$ for $k = 2$ and $n = 2$



Proof

By using the relation

$$x_\theta(z)\frac{d}{dz}(x_\theta(z)^{k+1}) = (k+1)x_\theta(z)^{k+1}\frac{d}{dz}x_\theta(z)$$

for each n we can construct

$$dimL((k+2)\theta) + dimL((k+2)\theta - \alpha^*) + dimL((k+1)\theta)$$

linearly independent relation among relations of length $k+2$.
For level 2 $C_n^{(1)}$-standard modul above number of relations among relation is equal* to

$$2n\binom{2n+6}{7}.$$

$*$ Weyl dimension formula

Proof

Moreover, the following inequality holds

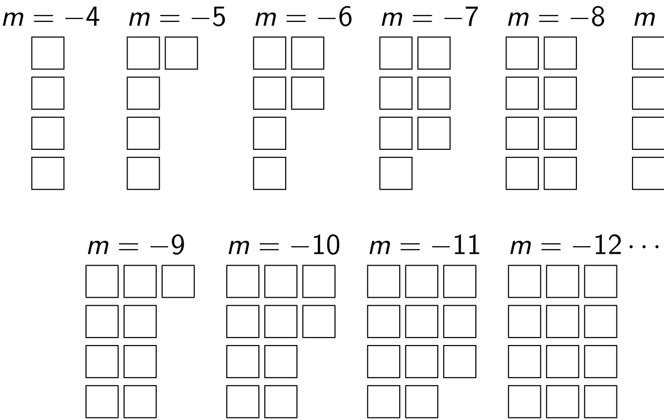
$$(\star) \qquad 2n\binom{2n+2k+2}{2k+3} \leq \sum_{|\pi|=m;\ell(\pi)=k+2} N(\pi) \quad \text{where}$$

$N(\pi) = max\{card(\varepsilon(\pi)) - 1, 0\}$, $\varepsilon(\pi) = \{\rho \in \ell t(\bar{R}) \mid \rho \subset \pi\}$.

If in $(\star)$ equality holds for all m than we have the proof of theorem (for all π of lenght $\ell(\pi) = k+2$).

Proof

List of Young tableaux for three successive triangles



Proof

For three successive triangles above number of relation among relations can be replaced with the equivalent one for all 11 listed Young diagrams

$$\sum_{m=4}^{12} \sum_{\pi \in \mathcal{P}^4(m)} N(\pi) = 9 \times 2n \binom{2n+6}{7} - 2 \times \dim L(4\theta) \qquad (6)$$

$$= \frac{7(10n-1)}{4} \binom{2n+6}{7} \qquad \square$$

Standard $C_n^{(1)}$-modules of level 2,3,. . . ?

C_2 k=3 za članak

C_2 #TrapezSve4.py

In[1]:= $n = 2$

Out[1]= 2

In[2]:= $\mathrm{SigmaTA5} = \sum_{i1=5}^{2n+1} \sum_{j1=1}^{4n+1-i1} \sum_{i2=4}^{i1-1} \sum_{j2=j1}^{i1+j1-i2} \sum_{i3=3}^{i2-1} \sum_{j3=j2}^{i2+j2-i3} \sum_{i4=1}^{i3-1} \sum_{j4=j3}^{i3+j3-i4} \sum_{i5=1}^{i4-1} \sum_{j5=j4}^{i4+j4-i5} 1$

Out[2]= 64

In[3]:= $\mathrm{SigmaTA4} = \sum_{i1=4}^{2n+1} \sum_{j1=1}^{4n+1-i1} \sum_{i2=3}^{i1-1} \sum_{j2=j1}^{i1+j1-i2} \sum_{i3=2}^{i2-1} \sum_{j3=j2}^{i2+j2-i3} \sum_{i4=1}^{i3-1} \sum_{j4=j3}^{i3+j3-i4} 1$

Out[3]= 216

In[4]:= $\mathrm{SigmaTA3} = \sum_{i1=3}^{2n+1} \sum_{j1=1}^{4n+1-i1} \sum_{i2=2}^{i1-1} \sum_{j2=j1}^{i1+j1-i2} \sum_{i3=1}^{i2-1} \sum_{j3=j2}^{i2+j2-i3} 1$

Out[4]= 268

In[5]:= $\mathrm{SigmaTA2} = \sum_{i1=2}^{2n+1} \sum_{j1=1}^{4n+1-i1} \sum_{i2=1}^{i1-1} \sum_{j2=j1}^{j1+i1-i2} 1$

Out[5]= 145

In[6]:= $\mathrm{SigmaTB3i} =$

$$\sum_{d=1}^{2n+1-3} \sum_{l=1}^{2n+2-d-3} \sum_{i1=l+d+3-1}^{2n+1} \sum_{i2=l+d+3-2}^{i1-1} \sum_{i3=l+d}^{i2-1} (4n+1-i1)(i1-i2+1)(i2-i3+1)(i3-l-d+1)$$

Out[6]= 132

In[7]:= $\mathrm{SigmaTB2i} = \sum_{d=1}^{2n+1-2} \sum_{l=1}^{2n+2-d-2} \sum_{i1=l+d+2-1}^{2n+1} \sum_{i2=l+d+2-2}^{i1-1} (4n+1-i1)(i1-i2+1)(i2-l-d+1)$

Out[7]= 211

THE END

THANK YOU!

Case $C_2^{(1)}$ for $k = 2$

$$(\star) \qquad 2n\binom{2n+2k+2}{2k+3} =?= \sum_{|\pi|=m;\ell(\pi)=k+2} N(\pi) \quad \text{where}$$

$$2n\binom{2n+2k+2}{2k+3} = 4\binom{10}{7} = 480$$

Case $C_n^{(1)}$ for $k = 2$ The proof of the main result

The Weyl dimension formula in the case of symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2n}$
(with the corresponding $\rho = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + 2\varepsilon_{n-1} + \varepsilon_n$) gives

$$\dim L(s\theta) = \binom{2n+2s-1}{2s}, \tag{7}$$

$$\dim L(4\theta - \alpha^\star) = \frac{(2n+7)(n-1)}{4}\binom{2n+5}{6}. \tag{8}$$

Hence from (14) and (15) we have

$$\dim Q_4(m) = \dim L(3\theta) + \dim L(4\theta) + \dim L(4\theta - \alpha^\star) = 2n\binom{2n+6}{7} \tag{9}$$

Case $C_n^{(1)}$ for $k = 2$

Let $\ell(\pi) = k + 2$ and assume that π allows two embeddings of leading terms of relations for level k standard modules. Then $\operatorname{supp}\pi$ is one of the following types (for $r, s \in \mathbb{N}$ and δ in the set of two symbols $|$ and $||$):

(A_r) $\operatorname{supp}\pi = \{a_1, \ldots, a_r\}$, $r \geq 2$, $a_1 \rhd \cdots \rhd a_r$.

($B_{r\,\delta}$) $\operatorname{supp}\pi = \{a_1, \ldots, a_r, b, c\}$, $r \geq 1$, $a_1 \rhd \cdots \rhd a_r$, $a_r \rhd b$, $a_r \rhd c$ and b and c are not comparable. We set δ to be $|$ if b and c are in the same row, and $||$ otherwise.

($C_{\delta\,r}$) $\operatorname{supp}\pi = \{b, c, a_1, \ldots, a_r\}$, $r \geq 1$, $a_1 \rhd \cdots \rhd a_r$, $b \rhd a_1$, $c \rhd a_1$ and b and c are not comparable. We set δ to be $|$ if b and c are in the same row, and $||$ otherwise.

($D_{r\delta\,s}$) $\operatorname{supp}\pi = \{a_1, \ldots, a_r, b, c, d_1, \ldots, d_s\}$, $r, s \geq 1$, $a_1 \rhd \cdots \rhd a_r$, $a_r \rhd b \rhd d_1$, $a_r \rhd c \rhd d_1$, $d_1 \rhd \cdots \rhd d_s$, and b and c are not comparable. We set δ to be $|$ if b and c are in the same row, and $||$ otherwise.

Case $C_n^{(1)}$ for $k = 2$

1. $N_T(A_r) = (r-1)\binom{k+1}{r-1}\,\Sigma_T(A_r)$,
2. $N_T(B_{r\,\delta}) = \binom{k-1}{r-1}\,\Sigma_T(B_{r\,\delta})$,
3. $N_T(C_{\delta\,r}) = \binom{k-1}{r-1}\,\Sigma_T(C_{\delta\,r})$,
4. $N_T(D_{r\,\delta\,s}) = \binom{k-1}{s+r-1}\,\Sigma_T(D_{r\,\delta\,s})$.

Case $C_n^{(1)}$ for $k=2$ The main result

Theorem

For any two embeddings $\rho_1 \subset \pi$ and $\rho_2 \subset \pi$ in $\pi \in \mathcal{P}^4(m)$, where $\rho_1, \rho_2 \in \ell t(\bar{R})$, we have a level 2 relation for $C_n^{(1)}$

$$u(\rho_1 \subset \pi) - u(\rho_2 \subset \pi) = \sum_{\pi \prec \pi',\ \rho \subset \pi'} c_{\rho \subset \pi'}\, u(\rho \subset \pi'). \tag{10}$$

Case $C_n^{(1)}$ for $k=2$ The proof of the main result

The Weyl dimension formula in the case of symplectic Lie algebra $\mathfrak{g}=\mathfrak{sp}_{2n}$
(with the corresponding $\rho = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + 2\varepsilon_{n-1} + \varepsilon_n$)
gives

$$\dim L(s\theta) = \binom{2n+2s-1}{2s}, \tag{11}$$

$$\dim L(4\theta-\alpha^\star) = \frac{(2n+7)(n-1)}{4}\binom{2n+5}{6}. \tag{12}$$

Hence from (14) and (15) we have

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Case $C_n^{(1)}$ for $k=2$ The proof of the main result

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Hence from (14) and (15) we have

$$\dim Q_4(m) = \dim L(3\theta) + \dim L(4\theta) + \dim L(4\theta - \alpha^\star) = 2n\binom{2n+6}{7} \tag{16}$$