On Feigin–Tipunin type extension of $\mathcal{W}$-algebras

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based on a joint work with T. Creutzig and S. Sugimoto

Representation theory XVIII

Object of this talk

$\mathfrak{g}$: reductive Lie alg., $f \in \mathcal{N}$, $\mathcal{W}^{\kappa}(\mathfrak{g}, f) = H_f(V^{\kappa}(\mathfrak{g}))$: affine $\mathcal{W}$-algebra.

Vertex algebra extensions

of $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$

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($\mathfrak{g}$: abelian, $f = 0$)

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Logarithmic VOA

($\mathfrak{g}$: ADE, $f = f_{\mathrm{prin}}$)

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- $\mathcal{W}^k(\mathfrak{g})$ has a free field realization by Miura map [Feigin–Frenkel].

$$0 \to \mathcal{W}^k(\mathfrak{g}) \to \pi_Q^{k+h^\vee} \xrightarrow{\bigoplus \mathcal{Q}_i} \bigoplus_{i \in I} \pi_{Q,\alpha_i}^{k+h^\vee} \quad (\mathcal{Q}_i = \int e^{\frac{1}{k+h^\vee}\alpha_i}(z)dz).$$

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$$\rightsquigarrow \quad B \curvearrowright V_{\sqrt{p}Q}, \quad f_i \to \mathcal{Q}_i, \quad h_i \to -\frac{1}{\sqrt{p}} h_{i(0)}.$$

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$$\rightsquigarrow \quad \mathcal{W}_\mathfrak{g}(p) \simeq \bigoplus_{\lambda \in P_+ \cap Q} L_{\lambda^\dagger} \otimes H_f(\mathbb{V}_\lambda^k) \quad \text{as } (G, \mathcal{W}^k(\mathfrak{g}))\text{-bimodules [FT, S]}$$

- For $\mathfrak{g} = \mathfrak{sl}_2$, $\mathcal{W}_{\mathfrak{g}}(p)$ is known to be the triplet algebra $\mathcal{W}(p)$ studied by [Adamović-Milas].

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$\mathcal{W}(p)$-mod is nonsemisimple [AM] and aftrer many important works [1], the logarithmic Kazhdan–Lusztig correspondence

$$\mathcal{W}(p)\text{-mod} \underset{\mathrm{BTC}}{\simeq} u_q(\mathfrak{sl}_2)\text{-mod}, \quad (q = e^{\pi i/p})$$

is established [Gannon-Negron, Creutzig-Lentner-Rupert].

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- For general $\mathfrak{g}$, not much about $\mathcal{W}_{\mathfrak{g}}(p)$ is known so far: some simple modules [FT,S] and logarithmic modules [AM]. Conjectually,

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Especially, the C_2-cofiniteness ($\Rightarrow$ finiteness of module category) remains a conjecture.

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C_2-cofiniteness of $\mathcal{W}$-algebras

Studying properties of $\mathcal{W}$-algebras by hand is usually hard in higher rank cases. A natural path is to use the BRST reduction functor

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The C_2-cofiniteness of (simple exceptional) $\mathcal{W}$-algebras is proven by Arakawa in this way:

- $X_{L_k(\mathfrak{g})} \simeq \overline{\mathbb{O}}_q \subset \mathcal{N}$ $\quad$ ($k = -h^\vee + \frac{p}{q}$: admissible)
- $X_{H_f(V)} \simeq X_V \times_{\mathfrak{g}} S_f$ $\quad$ ($S_f = f + \mathfrak{g}^e$: the Slodowy slice).

$\Rightarrow X_{\mathcal{W}_k(\mathfrak{g},f)} = X_{H_f(L_k(\mathfrak{g}))} = \overline{\mathbb{O}}_q \times_{\mathfrak{g}} S_f = \{\mathrm{pt}\}.$

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▶ Why not in our case?

$$\begin{array}{ccc}
V^k(\mathfrak{g}) & \longrightarrow & ??? \simeq \bigoplus L_{\lambda^\dagger} \otimes \mathbb{V}^k_\lambda \\
H_f \downarrow & & H_f \downarrow \\
\mathcal{W}^k(\mathfrak{g}) & \longrightarrow & \mathcal{W}_{\mathfrak{g}}(p) \simeq \bigoplus L_{\lambda^\dagger} \otimes H_f(\mathbb{V}^k_\lambda)
\end{array}$$

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Chiral Hecke algebra

($\mathfrak{g}$: simple or $\mathbb{C}$, $f = 0$)

$$A_k(\mathfrak{g}) \simeq \bigoplus_{\lambda \in P^\vee_+} L_{\lambda^\dagger} \otimes \mathbb{V}^k_\lambda \quad (k \in \mathbb{Z}_{<-h^\vee})$$

Vertex algebra ext's of $\mathcal{W}^\kappa(\mathfrak{g}, f)$

$$\mathfrak{A}^\kappa[\mathfrak{g}, f] = \bigoplus_{\lambda \in R} L_{\lambda^\dagger} \otimes H_f(\mathbb{V}^\kappa_\lambda)$$

4d VOA at level=∞?

($\mathfrak{g}$: $ABCD$ & some variants)

$$\lim_{k \to \infty} \mathscr{D}^{\mathrm{ch}}_{G,k}[n]/\mathcal{Z} \simeq \bigoplus L_{\lambda^\dagger} \otimes \mathbb{V}^{k^R}_\lambda$$

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$$V_L \simeq \bigoplus_{\lambda \in L} \mathbb{C}_{-\lambda} \otimes \pi^L_\lambda$$

Peter–Weyl theorem?

- The branching rule $\mathfrak{A}^\kappa[\mathfrak{g}] \simeq \bigoplus L_{\lambda^\dagger} \otimes \mathbb{V}_\lambda^k$ has a flavor of Peter–Weyl theorem for G (of Adjoint type)

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- In the vertex algebra setting, we have the chiral differential operators
$$\mathscr{D}^{\mathrm{ch}}_{G,k} := U(\widehat{\mathfrak{g}}_k) \underset{U(\mathfrak{g}[\![z]\!])}{\otimes} \mathbb{C}[J_\infty G]$$
$$\simeq \bigoplus_{\lambda \in P_+ \cap Q} \mathbb{V}^{k^L}_{\lambda^\dagger} \otimes \mathbb{V}^{k^R}_\lambda, \quad \frac{1}{k^L + h^\vee} + \frac{1}{k^R + h^\vee} = 0, \ (k \notin \mathbb{Q})$$

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- Gluing level $(=0)$ can be shifted [Moriwaki '21]

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Remark
(i) We need a VOA for all $k \in \mathbb{C}$ with integral form to make it rigorous.
(ii) For $\mathcal{W}_\mathfrak{g}(p)$, we have such a VOA by using $\mathscr{D}^{\mathrm{ch},\mathcal{W}}_{G,k}$ [CN '22].

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$$0 \longrightarrow V^k(\mathfrak{g}) \xrightarrow{\mu} \mathbb{W}_0^k \xrightarrow{\bigoplus \mathcal{Q}_i} \bigoplus_{i \in I} \mathbb{W}_{\alpha_i}^k \quad (\mathbb{W}_\lambda^k := \beta\gamma^{\Delta_+} \otimes \pi_{Q,\lambda}^{k+h^\vee})$$

$$\mathcal{Q}_i = \int Y(\underbrace{\Psi(e_i) \otimes e^{\frac{1}{k+h^\vee}\alpha_i}}_{\mathbf{Q}_i}, z)dz, \quad \Psi \colon \begin{array}{ccc} N_+ & \curvearrowright & N_+ \\ \mathfrak{n}_+ & \simeq & \mathrm{Vect}_{N_+}^R \end{array}$$

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Definition/Proposition 1.1 (CNS)

For $k + h^\vee = \frac{1}{p}$ $(p \in r^\vee \mathbb{Z})$, $\mathbf{Q}_i \in \mathbb{W}_{\sqrt{p}Q} := \bigoplus_{\lambda\in Q} \mathbb{W}_\lambda^k (\simeq \beta\gamma^{\Delta_+} \otimes V_{\sqrt{p}Q})$,

$$B \curvearrowright \mathbb{W}_{\sqrt{p}Q}, \quad (p > 0).$$

Feigin–Tipunin type extension for $\mathcal{W}$-algebras are defined as

$$\mathrm{FT}_p(\mathfrak{g}) := H^0(G \times_B \mathbb{W}_{\sqrt{p}Q}), \quad \mathrm{FT}_p(\mathfrak{g}, f) := H^0(G \times_B H_f^0(\mathbb{W}_{\sqrt{p}Q}))$$

Basic conjecture

Conjecture 1.2 (CNS)

(1) $G \curvearrowright \mathrm{FT}_p(\mathfrak{g}, f) \curvearrowleft H^0(G \times_B H^0_f(\mathbb{W}_{\sqrt{p}Q}))^G \supset \mathcal{W}^k(\mathfrak{g}, f)$

$$\Rightarrow \mathrm{FT}_p(\mathfrak{g}, f) \simeq \bigoplus_{\lambda \in P_+ \cap Q} L_{\lambda^\dagger} \otimes H_f(\mathbb{V}^k_\lambda) \quad \text{as } (G, \mathcal{W}^k(\mathfrak{g}, f))\text{-bimod}$$

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(2) $H_f(H(G \times_B \text{-})) \simeq H(G \times_B H_f(\text{-}))$ i.e.

$$\mathrm{FT}_p(\mathfrak{g}, f) \simeq H_f(\mathrm{FT}_p(\mathfrak{g})) \quad \text{as vertex algebras}$$

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(3) $\mathrm{FT}_p(\mathfrak{g}, f)$ is quasi-lisse (esp. $\mathcal{W}_{\mathfrak{g}}(p)$ is C_2-cofinite)

$$X_{\mathrm{FT}_p(\mathfrak{g})} \subset \mathcal{N}, \quad X_{\mathrm{FT}_p(\mathfrak{g},f)} \subset \mathcal{N} \cap S_f.$$

Rank one case

Fix $k = -2 + \frac{1}{p}$ $(p \geq 1)$.

Algebra	$\mathrm{FT}_p(\mathfrak{sl}_2, f_{\mathrm{prin}})$	$\mathrm{FT}_p(\mathfrak{sl}_2, 0)$
Construction	$H^0(\mathrm{SL}_2 \times_B V_{\sqrt{p}A_1})$ $F = \int Y(e^{\sqrt{p}\alpha}, z)dz$	$H^0(\mathrm{SL}_2 \times_B \beta\gamma \otimes V_{\sqrt{p}A_1})$ $F = \int Y(\beta \otimes e^{\sqrt{p}\alpha}, z)dz$
Subalgebra	$\mathcal{W}^k(\mathfrak{sl}_2) = L(c_{1,p}, 0)$	$V^k(\mathfrak{sl}_2)$
Branching	$\bigoplus L_{n\alpha} \otimes L(c_{1,p}, h_{1,2n+1})$	$\bigoplus L_{n\alpha} \otimes \mathbb{V}^k_{n\alpha}$
F.F.R $(p \geq 2)$	$\mathrm{Ker} \int Y(e^{-\frac{1}{\sqrt{p}}\alpha}, z)dz$ $\subset V_{\sqrt{p}A_1}$	$\mathrm{Ker} \int Y(e^{-\frac{1}{p}(u+v) - \frac{1}{\sqrt{p}}\alpha}, z)dz$ $\subset \beta\gamma \otimes V_{\sqrt{p}A_1}$
Name	Triplet algebra $\mathcal{W}(p)$	$\mathcal{W}(2, (2p)^{\times 3 \times 3})$, $(\mathcal{V}^{(p)})^{\mathbb{Z}_2}$
Ass. Var	$\{\mathrm{pt}\}$	$\mathcal{N}$
Quantum Grp.	"$u_q(\mathfrak{sl}_2)$"	"$u_q(\mathfrak{sl}_{2\|1})$"

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Quantum Grp.	"$u_q(\mathfrak{sl}_2)$"	"$u_q(\mathfrak{sl}_{2\|1})$"

Simple $\mathrm{FT}_p(\mathfrak{sl}_2)$-modules

Theorem 1 (CNS)

(1) $\mathrm{FT}_p(\mathfrak{sl}_2)$ *is simple and (for $p \geq 2$)*

$$H^0(SL_2 \times_B \beta\gamma \otimes V_{\sqrt{p}A_1}) \simeq \mathrm{Ker}_{\beta\gamma\otimes V_{\sqrt{p}A_1}} \int Y(e^{-\frac{1}{p}(u+v)} \otimes e^{-\frac{1}{\sqrt{p}}\alpha}, z)dz.$$

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Theorem 1 (CNS)

(1) $\mathrm{FT}_p(\mathfrak{sl}_2)$ *is simple and (for $p \geq 2$)*

$$H^0(SL_2 \times_B \beta\gamma \otimes V_{\sqrt{p}A_1}) \simeq \mathrm{Ker}_{\beta\gamma\otimes V_{\sqrt{p}A_1}} \int Y(e^{-\frac{1}{p}(u+v)} \otimes e^{-\frac{1}{\sqrt{p}}\alpha}, z)dz.$$

(2) *Global sec's of sheaves of $V^k(\mathfrak{sl}_2)$-mod give simple $\mathrm{FT}_p(\mathfrak{sl}_2)$-mod:*

$$\mathcal{X}^+_{r,s} = H^0(SL_2 \times_B \beta\gamma \otimes V_{r,s}) \simeq \bigoplus \mathbb{C}^{2n+s} \otimes L_k(\lambda_{r,(2n+s)})$$

$$\mathcal{X}^-_{r,s} = H^0(SL_2 \times_B \tau(\Pi[\tfrac{r}{p}] \otimes V_{r,s})) \simeq \bigoplus \mathbb{C}^{2n+s} \otimes L_k(\lambda_{-r,-(2n+s)})$$

$$\mathcal{W}^{[a]}_{r,s} = H^0(SL_2 \times_B \Pi[a] \otimes V_{r,s}) \simeq \bigoplus \mathbb{C}^{2n+s} \otimes \mathcal{R}^{[a]}_{r,2n+s}.$$

Remark

(i) The simplicity of $\mathcal{X}^\pm_{r,s}$ and $\mathcal{W}^{[a]}_{r,s}$ ($[a] \in \mathbb{Z}_p$) is conjectured in [ST '13].

(ii) It is important that $\mathrm{FT}_p(\mathfrak{sl}_2)$ has a continuous family of simple modules.

$V^k(\mathfrak{sl}_2) \dashrightarrow L(c_{1,p}, 0)$

The geometric construction of simple modules gives character formulas:

$$\mathrm{ch}_{L_k(\lambda_{r,s})}(z, q) = \frac{z^{\lambda_{r,s}} q^{\Delta_{r,s}} (1 - z^{-(s+1)\alpha} q^{r(s+1)})}{(z^{\alpha} q, z^{-\alpha}, q; q)_{\infty}}$$

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$\Rightarrow$ Resolution by affine Verma modules

$$0 \to \mathbb{M}_k(-s-2-\tfrac{r}{p}) \to \mathbb{M}_k(s-\tfrac{r}{p}) \to L_k(s-\tfrac{r}{p}) \to 0$$
$$0 \to \mathbb{M}_k(s+\tfrac{r+2}{p}) \to \mathbb{M}_k(-s-2+\tfrac{r+2}{p}) \to L_k(-s-2+\tfrac{r+2}{p}) \to 0$$

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$\Rightarrow$ Equiv. to the Feigin–Fuchs resolution via H_{DS}

$$0 \to M_{r+1,-(s+1)} \to M_{r+1,s+1} \to L_{r+1,s+1} \to 0.$$

$\mathrm{FT}_p(\mathfrak{sl}_2) \dashrightarrow \mathcal{W}(p)$

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(1) [ACGY] $H_{DS}(\mathrm{FT}_p(\mathfrak{sl}_2)) \simeq \mathcal{W}(p)$.

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$$H_{DS}(H(\mathrm{SL}_2 \times_B M)) \simeq H(\mathrm{SL}_2 \times_B H_{DS}(M)), \textit{ i.e. } H_{DS}(\mathcal{X}^{\pm}_{r,s}) \simeq \mathcal{W}_{r,s}$$

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(3) *Compatibility of Felder complexes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{DS}(\mathcal{X}^{+}_{r,s}) & \longrightarrow & H^0_{DS}(\beta\gamma \otimes V_{r,s}) & \longrightarrow & H_{DS}(\mathcal{X}^{-}_{p-r,3-s}(-\varpi)) & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathcal{W}_{r,s} & \longrightarrow & V_{r,s} & \longrightarrow & \mathcal{W}_{p-r,3-s}(-\varpi) & \longrightarrow & 0 \end{array}$$

Screenings and quantum groups

$$V \simeq \bigcap \mathrm{Ker}_A S_i, \quad S_i = \int Y(e^{\beta_i}, z)dz,$$

$$A : \text{free field algebra } (\pi^Q,\ V_L, \cdots)$$

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- $\mathrm{Rep}^{\mathrm{wt}}(A) \simeq \mathrm{Rep}^{\mathrm{wt}}(\mathcal{H})$ ($\mathcal{H}$: quasi-Hopf algebra)
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- [Lentner '21] $\{S_i\}$ generates the Nichols algebra $\mathfrak{N}^{\mathrm{scr}} := \mathfrak{N}(\oplus \beta_i)$
- If we are good enough, the extension theory $V \hookrightarrow A$ implies

$$\mathrm{Rep}^{\mathrm{wt}}(V) \simeq \mathcal{Z}_{\mathcal{C}_A^0}(\mathcal{C}_A) \simeq {}^{\mathfrak{N}^{\mathrm{cat}}}_{\mathfrak{N}^{\mathrm{cat}}}\mathcal{YD}(\mathrm{Rep}^{\mathrm{wt}}(A))$$
$$\simeq {}^{\mathfrak{N}^{\mathrm{scr}}}_{\mathfrak{N}^{\mathrm{scr}}}\mathcal{YD}(\mathrm{Rep}^{\mathrm{wt}}(\mathcal{H})) \simeq \mathrm{Rep}^{\mathrm{wt}}(\mathscr{U})$$

Free field algebras

Ex ([GN,CLR]) $\mathcal{W}(p)$ case

$$\mathcal{H}$$

$$\begin{array}{llllll} \mathcal{W}(p) & \simeq \mathrm{Ker} S \subset V_{\sqrt{p}A_1} & \Rightarrow u_q(\mathfrak{sl}_2) & \mathbb{C}[K^{\pm 1}]/(K^{2p}-1) \\ \cup & \qquad\qquad\quad \cup & & \\ \mathcal{M}(p) & \simeq \mathrm{Ker} S \subset \pi^{\alpha} & \Rightarrow u_q^H(\mathfrak{sl}_2) & \mathbb{C}[K^{\pm 1}, H] \end{array}$$

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Ex ([CNS]) $\mathrm{FT}_p(\mathfrak{sl}_2)$ case

$$\mathrm{FT}_p(\mathfrak{sl}_2) \simeq \bigcap_{i=1,2} \mathrm{Ker} S_i|_{\Pi[0]\otimes V_{\sqrt{p}A_1}}, \quad \Pi[0] := \bigoplus_{n\in\mathbb{Z}} \pi^{u,v}_{n(u+v)} \subset V_{\mathbb{Z}\oplus\sqrt{-1}\mathbb{Z}}$$

$$S_1 = \begin{cases} \displaystyle\int Y(e^{-\frac{1}{p}(u+v)-\frac{1}{\sqrt{p}}\alpha}, z)dz, & (p \geq 2), \\ \displaystyle\int Y(e^{-(v+\alpha)}, z)dz & (p = 1), \end{cases} \qquad S_2 = \int Y(e^{u}, z)dz.$$

Kazama–Suzuki dual $V \leftrightarrow \mathrm{Com}(\pi^{\mathrm{diag}}, V \otimes V_{\mathbb{Z}})$

$$
\begin{array}{c|ccccccc}
 & & & & \text{KS dual} & & & \\
\mathcal{W}(p) & \Pi[0] \otimes V_{\sqrt{p}A_1} & \supset & \mathrm{FT}(\mathfrak{sl}_2) & \longleftrightarrow & s\mathcal{W}_p(\mathfrak{sl}_{2|1}) & \subset & V_{\mathbb{Z}} \otimes V_{\sqrt{p}A_1} \otimes \pi^{\alpha^\dagger} \\
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L(c_{1,p}, 0) & \pi^{u,v,\alpha} & \supset & V^k(\mathfrak{sl}_2) & \longleftrightarrow & \mathcal{W}^\ell(\mathfrak{sl}_{2|1}) & \subset & \pi^{u,,\alpha,\alpha^\dagger}
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\cup & & \cup & & \cup & \\
L(c_{1,p},0) & \pi^{u,v,\alpha} \supset & V^k(\mathfrak{sl}_2) & \longleftrightarrow & \mathcal{W}^\ell(\mathfrak{sl}_{2|1}) \subset & \pi^{u,,\alpha,\alpha^\dagger}
\end{array}$$

Theorem 3 (CNS)

(1)

$$\begin{array}{ccc}
\mathrm{Rep}^{\mathrm{wt}}(\mathrm{FT}_p(\mathfrak{sl}_2))^{[\varepsilon\lambda]} & \xrightarrow{\simeq H_{\lambda_\theta}} & \mathrm{Rep}^{\mathrm{wt}}(s\mathcal{W}_p(\mathfrak{sl}_{2|1}))^{[\frac{1}{\varepsilon}\lambda_\theta]} \\
\downarrow \simeq S_\bullet & & \downarrow \simeq S_\bullet \\
\mathrm{Rep}^{\mathrm{wt}}(\mathrm{FT}_p(\mathfrak{sl}_2))^{[\varepsilon\lambda']} & \xrightarrow{\simeq H_{\lambda'_{\theta'}}} & \mathrm{Rep}^{\mathrm{wt}}(s\mathcal{W}_p(\mathfrak{sl}_{2|1}))^{[\frac{1}{\varepsilon}\lambda'_{\theta'}]}
\end{array}$$

(2) $$\mathcal{I}_{\mathrm{FT}(\mathfrak{sl}_2)}\begin{pmatrix} M_3 \\ M_1 \;\; M_2 \end{pmatrix} \xrightarrow{\simeq} \mathcal{I}_{s\mathcal{W}_p(\mathfrak{sl}_{2|1})}\begin{pmatrix} \mathrm{H}_{\lambda+\mu}(M_3) \\ \mathrm{H}_\lambda(M_1) \;\; \mathrm{H}_\mu(M_2) \end{pmatrix}.$$

Quantum supergroup for $s\mathcal{M}_p(\mathfrak{sl}_{2|1}) \subset V_{\mathbb{Z}} \otimes \pi^{\alpha,\alpha^\dagger}$

- Hopf algebra for free field algebra

$$\mathcal{H} = \mathbb{C}[H_i, K_i^{\pm 1}, K_0 \mid i = 1, 2]/(K_0^2 - 1)$$

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- Braiding matrix of the screeings ($q = \mathrm{e}^{\pi\sqrt{\text{-1}}/p}$)

$$\underset{p \geq 2}{\mathcal{B}^{\mathbf{a}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} q^2 & q^{-1} \\ q^{-1} & 1 \end{pmatrix}} \quad \underset{p = 1}{\mathcal{B}^{\mathbf{s}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & q^{-1} \\ q^{-1} & 1 \end{pmatrix}}$$

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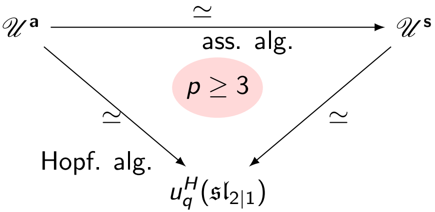

(i) The module cagtegories are all tensor equivalence.

(ii) $p = 1$ is well-defined only for $\mathscr{U}^{\mathbf{s}}$

$$u_{-1}^H(\mathfrak{sl}_{2|1}) := \mathscr{U}^{\mathbf{s}}(p = 1)$$

Logarithmic Kazhdan–Lusztig correspondence

Theorem 4 (CNS)

For $p = 1$ the logarithmic Kazhdan–Lusztig correspondence holds:

$$\mathrm{Rep}^{\mathrm{wt}}(V) \simeq {}^{\mathfrak{N}^{\mathrm{scr}}}_{\mathfrak{N}^{\mathrm{scr}}}\mathcal{YD}(\mathrm{Rep}^{\mathrm{wt}}(A)).$$

$$(V, A) = (\mathrm{FT}_1(\mathfrak{sl}_2), \Pi[0] \otimes V_{A_1}), \quad (s\mathcal{W}_1(\mathfrak{sl}_{2|1}), V_{\mathbb{Z}} \otimes V_{A_1} \otimes \pi^{\alpha^\dagger}),$$
$$(s\mathcal{M}_1(\mathfrak{sl}_2), \Pi[0] \otimes \pi^{\alpha}), \quad (s\mathcal{M}_1(\mathfrak{sl}_{2|1}), V_{\mathbb{Z}} \otimes \pi^{\alpha,\alpha^\dagger}).$$

In particular,

$$\mathrm{Rep}^{\mathrm{wt}}(s\mathcal{M}_1(\mathfrak{sl}_{2|1})) \simeq \mathrm{Rep}^{\mathrm{wt}}(u^H_{-1}(\mathfrak{sl}_{2|1})).$$

Remark For $\mathcal{W}(p)$ the (quasi-) Hopf algebra $u_q(\mathfrak{sl}_2)$ is finite dim'l. For $s\mathcal{W}_p(\mathfrak{sl}_{2|1})$ and $\mathrm{FT}_p(\mathfrak{sl}_2)$, it is NOT the case: unrolled in the odd root direction reflecting we have simple modules $\mathcal{W}^{[a]}_{r,s}$. ($[a] \in \mathbb{C}/\mathbb{Z}\backslash\{2pt\}$)

Thank you for your attention!