Orbifold Theory for Vertex Algebras

Chongying Dong, Li Ren, Chao Yang

Dubrovnik

June 25-July 1, 2023

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1. Introduction

Orbifold theory

- V is a vertex algebra
- G is a finite automorphism group of V
- Orbifold theory: Study the V^G-modules

Twisted modules

- Main feature: Appearance of g-twisted V-module: A g-twisted V-module is not a V-module but restricts to a V^G-module
- Problem: Do not know how to construct twisted modules in mathematics. This poses a great challenge in studying orbifold theory

History

- (Frenkel-Lepowsky-Measman 1988) The first obiforld construction is moonshine vertex operator algebra
- (Dijkgraff-Vafa-Verlinde-Verlinde 1989) General orbifold theory (physics point of view)
- (Dijkgraaf-Pasquier-Roche 1990) Connnection between holomorphic orbifold theory and twisted Drinfeld doubles
- (Dong-Mason, Dong-Li-Mason 1996-1997) Orbifold theory for a vertex operator algebra, Schur-Weyl duality for G and V^G on V
- (Dong-Mason 1997, Hanaki-Miyamoto-Tambara 1999, Dong-Jiao-Xu 2013) Quantum Galois theory

- (Dong-Yamskulna 2002) Duality results on modules
- (Miyamoto-Tanabe 2004) Algebeas $A_{G,n}(V)$ and duality
- (Carnahan-Miyamoto 2017) If V is rational and C_2-cofinite and G is a solvable automorphism group of V then V^G is rational and C_2-cofinite
- (Dong-R-Xu 2017) Classification of irreducible V^G-modules
- (Kirillov 2002, Dong-Ng-R 2021) Orbifold theory and minimal modular extensions (Dijkgraaf-Pasquier-Roche Conjecture)
- Study of general orbifold construction for vertex algebra is very limited

Our work

Assumptions

V is a simple vertex algebra of countable dimension such that

- G is a finite automorphism group of V and $\sigma \in G$ is a central element
- $\mathcal{S}$ is a finite set of inequivalent irreducible σ-twisted V-modules such that $\mathcal{S}$ is invariant under the action of G
- $\mathcal{M} = \bigoplus_{M \in \mathcal{S}} M$

The assumption is always valid in this talk

Main results

1 The actions of finite dimensional semisimple associative algebra $\mathcal{A}_\alpha(G,\mathcal{S})$ and V^G on $\mathcal{M}$ commute
2 Every irreducible $\mathcal{A}_\alpha(G,\mathcal{S})$-module occurs in $\mathcal{M}$
3 The multiplicity space of each irreducible $\mathcal{A}_\alpha(G,\mathcal{S})$-module is an irreducible V^G-module
4 The multiplicity spaces of different irreducible $\mathcal{A}_\alpha(G,\mathcal{S})$-modules are inequivalent V^G-modules
5 A Galois correspondence is established

Remark

Results 1-4 in the case that V is a vertex operator algebra were obtained by Dong-Yamskulna in 2002

2. Basics

- $V = (V, Y, \mathbf{1})$ a vertex algebra
- g is an automorphism of V of order T, and $o(g) = T$

$$V = \bigoplus_{r \in \mathbb{Z}/T\mathbb{Z}} V^r$$

where $V^r = \{v \in V \mid gv = e^{-2\pi i r/T} v\}$, $r = 0, \ldots, T-1$

- A g-twisted V-module $M = (M, Y_M)$:

$$Y_M : V \to (\operatorname{End} M)\{z^{1/T}, z^{-1/T}\}$$
$$v \mapsto Y_M(v,z) = \sum_{n \in \frac{1}{T}\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \operatorname{End} M)$$

+axioms

- If $g = 1$, M is called a V-module

Remark

- In the case that V is a vertex operator algebra, such g-twisted V-module is called a weak g-twisted module in the literature
- If G is an automorphism group of V then for $g \in G$, a g-twisted V-module is V^G-module

3.Associative algebras

V: a vertex algebra, t: an indeterminate

- $\mathcal{L}(V) = V \otimes \mathbb{C}[t^{\frac{1}{T}}, t^{-\frac{1}{T}}]$ tensor product of vertex algebras
- g automorphism of V of order T: $g(u \otimes t^m) = e^{-2\pi i m}(gu \otimes t^m)$ defines an automorphism of $\mathcal{L}(V)$
- $\mathcal{L}(V, g) = \oplus_{r=0}^{T-1} V^r \otimes t^{r/T}\mathbb{C}[t, t^{-1}]$: fixed point vertex subalgebra
- D is the endomorphism of V defined by $D(v) = v_{-2}\mathbf{1}$ for $v \in V$, $\mathcal{D} = 1 \otimes \frac{d}{dt} + D \otimes 1$ is the endomorphism of $\mathcal{L}(V, g)$
- $V[g] = \mathcal{L}(V, g)/\mathcal{D}\mathcal{L}(V, g)$ is a Lie algebra (Borcherds)
- $\mathcal{U}(V[g])$ is the universal enveloping algebra of the Lie algebra $V[g]$. (This associative algebra will be used when we need Jacobson density theorem)

Lemma

Let M be a g-twisted V-module. Let X be a finite dimensional subspace of M. Let $u^1, \ldots, u^k \in V$ and $n_1, \ldots, n_k \in \frac{1}{T}\mathbb{Z}$. Then there exist $a^1, \ldots, a^t \in V$ and $m_1, \ldots, m_t \in \frac{1}{T}\mathbb{Z}$ such that

$$u^1_{n_1} \cdots u^k_{n_k}(w) = a^1_{m_1} w + \cdots + a^t_{m_t} w$$

for any $w \in X$

Remark

This result essentially follows from the associativity of twisted vertex operators on M

Jacobson's density theorem

Let A be an associative algebra over $\mathbb{C}$. Let M be an simple A-module of countable dimension. Then for any finite dimensional subspace X of M and any $f \in \mathrm{Hom}_{\mathbb{C}}(X, M)$, there exists an element $a \in A$ such that $f(x) = ax$ for any $x \in X$

Remark

In the case that V is a vertex operator algebra, the main tool is the associative algebras $A_{g,n}(V)$ introduced and studied by Don-Li-Mason in 1998. The algebras $A_{g,n}(V)$ only work for vertex operator algebras and not for vertex algebras which are not graded in general. The Jacobson's density theorem will be used in the proof for vertex algebra

4. Duality I

- V : simple vertex algebra of countable dimension
- H : finite automorphism group of V.
- $V[[z_1^{\pm 1}, \cdots, z_n^{\pm 1}]]$: an H-module such that H acts on V
- Let $n \geq 0$. Define a linear mapping

$$\varphi_n : V^{\otimes(n+1)} \to V[[z_1^{\pm 1}, \cdots, z_n^{\pm 1}]]$$

by

$$\varphi_n(v^n \otimes \cdots \otimes v^1 \otimes v^0) = Y(v^n, z_n) \cdots Y(v^1, z_1)v^0.$$

Note that $\varphi_0 = Id_V$.

Lemma

The mapping φ_n is an injective H-homomorphism

Corollary

Every irreducible H-modules appears in V

Remark

- The proof of Corollary follows from a well-known result that if W is a faithful H-module then any irreducible H-module appears in $W^{\otimes n}$ for some $n \geq 0$
- If V is a vertex operator algebra, this result was proved by Dong-Li-Mason in 1996

Definition

Let A be an associative algebra and let V be a vertex algebra. If M is both an A-module and a V-module such that the actions of A and the actions of V on M commute with each other, M is called an $A \otimes V$-module

Some basics notations:

- A : a finite dimensional semisimple associative algebra
- M : an $A \otimes V$-module
- Λ : set of all irreducible characters of A
- W_λ : simple A-module associated to $\lambda \in \Lambda$
- $M_\lambda = \mathrm{Hom}_A(W_\lambda, M)$ is a V-module such that $(v_n f)(w) = v_n f(w)$ for $v \in V, n \in \mathbb{Z}, f \in M_\lambda, w \in W_\lambda$

As $A\otimes V$-module, M has the following decomposition:

$$M=\bigoplus_{\lambda\in\Lambda}W_\lambda\otimes M_\lambda.$$

Key Lemma

Assume $\dim A<\infty$. Let M be an $A\otimes V$-module such that for any finite dimensional A-submodule X of M and $f\in \mathrm{Hom}_A(X,M)$, there exist $v^1,\dots,v^n\in V$ and $i_1,\dots,i_n\in\mathbb{Z}$ such that

$$f=v^1_{i_1}+\cdots+v^n_{i_n}$$

Then

1. M_λ is an irreducible V-module if $M_\lambda\neq 0$,
2. $M_\lambda\cong M_\mu$ if and only if $\lambda=\mu$

Remark

- In the case V is a vertex operator algebra, A is the group algebra $\mathbb{C}^{\alpha_M}[G_M]$, the proofs of ① and ② need associative algebras $A_{\sigma,n}(V)$ or $A_{G,n}(V)$ which was defined and studied by Miyamoto-Tanabe
- The key Lemma is a replacement of using associative algebras $A_{\sigma,n}(V)$ or $A_{G,n}(V)$

Some basics notations:

- M is an irreducible σ-twisted V-module
- Define a new σ-twisted V-module $M \circ h$: $M \circ h \cong M$ as vector spaces, $Y_{M\circ h}(v,z) = Y_M(hv,z)$ for $h \in G$, $v \in V$
- $G_M = \{h \in G | M \circ h \cong M\}$ acts on M projectively: for $h \in G_M$ there exists $\phi(h) \in GL(M)$ such that

$$\phi(h)Y_M(v,z)\phi(h)^{-1} = Y_M(hv,z)$$

for all $v \in V$ and

$$\phi(h)\phi(k) = \alpha_M(h,k)\phi(hk)$$

for $h,k \in G_M$ and some $\alpha_M \in H^2(G_M, \mathbb{C}^*)$

- $\mathbb{C}^{\alpha_M}[G_M] = \oplus_{h\in G_M}\mathbb{C}\hat{h}$ is the twisted group algebra: $\hat{h}\hat{k} = \alpha_M(h,k)\widehat{hk}$
- M is a $\mathbb{C}^{\alpha_M}[G_M]$-module such that $\hat{h}$ acts as $\phi(h)$
- Λ_M is the set of all irreducible characters of $\mathbb{C}^{\alpha_M}[G_M]$
- W_λ is the corresponding irreducible module for $\lambda \in \Lambda_M$
- $M_\lambda = \mathrm{Hom}_{\mathbb{C}^{\alpha_M}[G_M]}(W_\lambda, M)$ is V^{G_M}-module
- M is $\mathbb{C}^{\alpha_M}[G_M] \otimes V^{G_M}$-module

Theorem

Assume that $G_M = G$.

1. We have decomposition

$$M = \oplus_{\lambda \in \Lambda_M} W_\lambda \otimes M_\lambda$$

   and each simple $\mathbb{C}^{\alpha_M}[G]$-module appears in M
2. M_λ is an irreducible V^G-module. In particular, M_λ is nonzero for any $\lambda \in \Lambda_M$
3. M_λ and M_γ are equivalent V^G-modules iff $\lambda = \gamma$

- The main idea in the proof is using the Key Lemma
- In the case that V is a vertex operator algebra, this result was due to Dong-Yamskulna

Galois correspondence

Theorem

The map $H \to V^H$ gives a bijection between the set of subgroups of G and the set of subalgebras of V containing V^G

- If V is a vertex operator algebra, the theorem were obtained by Dong-Mason(injective, 1997), Hanaki-Miyamoto-Tambara (surjective, 1999)
- A full Galois correspondence was given by Dong-Jiao-Xu by using the quantum dimension (including $o(G) = \mathrm{qdim}_{V^G} V$ and Galois extensions, 2013)
- A full Galois correspondence for vertex operator superalgebra was obtained by Dong-R-Yang (2022)

5. Duality II

Some notations

- $\mathcal{S}$ is a finite set of inequivalent irreducible σ-twisted V-modules which is G-invariant
- $\mathcal{S}$ is G-invariant: for any $M \in \mathcal{S}$ and $h \in G$, there exists $N \in \mathcal{S}$ such that $N \cong M \circ h$
- $\mathcal{A}_\alpha(G, \mathcal{S}) = \mathbb{C}[G] \otimes \mathbb{C}\mathcal{S} = \bigoplus_{g \in G, M \in \mathcal{S}} \mathbb{C}(g \otimes e(M))$ is an associative algebra:

$$g \otimes e(M) \cdot h \otimes e(N) = \alpha_N(g,h) gh \otimes e(M \circ h)e(N)$$

- $\mathcal{M} = \bigoplus_{M \in \mathcal{S}} M$
- $\mathcal{M}$ is $\mathcal{A}_\alpha(G, \mathcal{S})$-module such that

$$(g \otimes e(M))w = \delta_{M,N} \phi_N(g) w$$

for $M, N \in \mathcal{S}$, $w \in N$, $g \in G$, where $\phi_N(g) : N \to N \circ g^{-1}$

- $\mathcal{S} = \cup_{j\in J}\mathcal{O}_j$, and $\mathcal{O}_j = M^j \circ G$
- $G = \cup_{j=1}^{k} G_M g_j$ is a right coset decomposition
- $S(M) = \mathrm{Span}\{g \otimes e(M) \mid g \in G_M\}$
- $D(M) = \mathrm{Span}\{g \otimes e(M) \mid g \in G\}$
- $D(\mathcal{O}_j) = \mathrm{Span}\{g \otimes e(M^j \circ g_i) \mid i = 1, \ldots, k, g \in G\}$
- $\mathcal{A}_\alpha(G, \mathcal{S}) = \bigoplus_{j\in J} D(\mathcal{O}_j)$

Theorem (Dong-Yamskulna 2002)

1. $S(M) \cong \mathbb{C}^{\alpha_M}[G_M]$ is a semisimple associative algebra
2. The functors $W \mapsto \mathrm{Ind}_{S(M)}^{D(M)} W$ gives an equivalence between the category of $\mathbb{C}^{\alpha_M}[G_M]$-modules and the category of $D(\mathcal{O}_M)$-modules.
3. $\mathcal{A}_\alpha(G, \mathcal{S})$ is a semisimple associative algebra $\{\mathrm{Ind}_{S(M^j)}^{D(M^j)} W \mid W$ is irreducible $\mathbb{C}^{\alpha_{M^j}}[G_{M^j}]-$module, $j \in J\}$ gives a complete list of irreducible $\mathcal{A}_\alpha(G, \mathcal{S})$-modules

Recall:

- $\mathcal{S} = \cup_{j\in J}\mathcal{O}_j$ and M^j is a representative of $\mathcal{O}_j$
- $\mathbb{C}^{\alpha_{M^j}}[G_{M^j}]$ is the twisted group algebra
- Λ_j is the set of irreducible characters of $\mathbb{C}^{\alpha_{M^j}}[G_{M^j}]$
- For any $\lambda \in \Lambda_j$, $W_{j,\lambda}$ the corresponding irreducible module
- $M^j = \bigoplus_{\lambda\in\Lambda_j} W_{j,\lambda} \otimes M_\lambda^j$ as $\mathbb{C}^{\alpha_{M^j}}[G_{M^j}] \otimes V^{G_{M^j}}$-module
- $W_\lambda^j = \mathrm{Ind}_{S(M^j)}^{D(M^j)} W_{j,\lambda}$

Theorem

1. As $\mathcal{A}_\alpha(G,\mathcal{S}) \otimes V^G$-module, we have the decomposition
$$\mathcal{M} = \bigoplus_{j\in J,\lambda\in\Lambda_j} W_\lambda^j \otimes M_\lambda^j$$
and M_λ^j is an irreducible V^G-module. In particular M_λ^j is nonzero for any $j \in J$ and $\lambda \in \Lambda_j$
2. $M_{\lambda_1}^{j_1}$ and $M_{\lambda_2}^{j_2}$ are isomorphic V^G-modules if and only if $j_1 = j_2$ and $\lambda_1 = \lambda_2$
3. In particular, any irreducible σ-twisted module is completely reducible V^G-module

- The main idea is to use Key Lemma instead of algebras $A_{\sigma,n}(V)$

Corollary (Adamović-Lam-Pedicć-Yu 2022)

Let M be an irreducible σ-twisted V-module. Then M is an irreducible V^G-module if and only if $G_M = \{1\}$

Remark

- Miyamoto-Tanabe(2004) defined associative algebras $A_{G,n}(V)$ for a vertex operator algebra V and $n \in \frac{1}{o(G)}\mathbb{Z}$ which are isomorphic to direct sum of $A_{g,n}(V)$ for $g \in G$. These algebras are good enough to study any finite set (of twisted modules) which is G-invariant, and are important to classify irreducible V^G-modules if V^G is rational and C_2-cofinite (Dong-R-Xu 2017). But these algebras do not work for vertex algebra
- Tanabe (very recently) defines a notion of weak (V,T)-module for any vertex algebra V and positive integer T. It turns out that any g-twisted V-module is a (V,T) module with $T = o(G)$. Using our proof and his new modules, he can take $\mathcal{S}$ in our discussion be any twisted modules to obtain similar results

THANKS