SVEUČILIŠTE U ZAGREBU
PMF
PRIRODOSLOVNO - MATEMATIČKI FAKULTET

Whittaker modules for $\widehat{\mathfrak{gl}}$ and $W_{1+\infty}$–modules which are not tensor products

QuantiXLie
PROVEDBA VRHUNSKIH ISTRAŽIVANJA U SKLOPU ZNANSTVENOG CENTRA IZVRSNOSTI ZA KVANTNE I KOMPLEKSNE SUSTAVE TE REPREZENTACIJE LIEJEVIH ALGEBRI

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This talk is mostly based on the following papers:

- [AP21] D. Adamović, V. Pedić Tomić, Whittaker modules for $\widehat{\mathfrak{gl}}$ and $W_{1+\infty}$-modules which are not tensor products, Letters in Mathematical Physics 113 (2023), no. 2, 39 pp
- [ALPY19] D. Adamović, C.-H. Lam, V. Pedić, N. Yu, On irreducibility of modules of Whittaker type for cyclic orbifold vertex algebras, Journal of Algebra 539 (2019), 1-23

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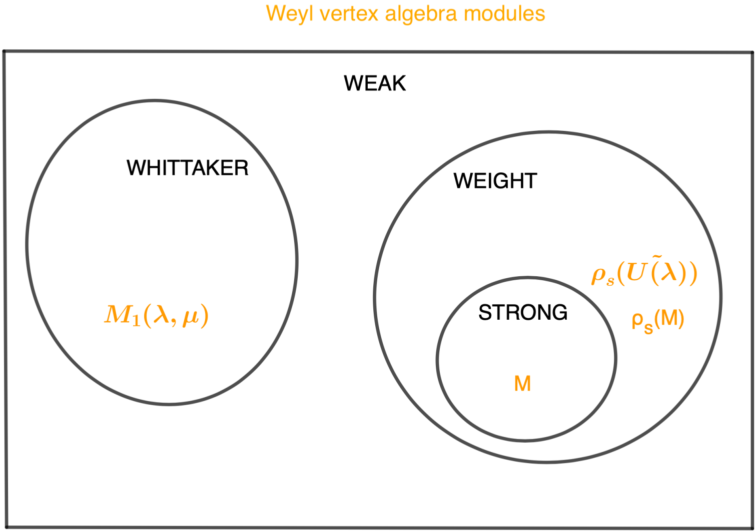
Weyl vertex algebra: notes on representation theory



Contents

Universal Whittaker module

- $\mathfrak{g}$ a Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.
- W a $\mathfrak{g}$–module
- Vector $v \in W$ is a Whittaker vector if there is a Lie functional $\lambda : \mathfrak{n}_+ \to \mathbb{C}$ such that $xv = \lambda(x)v$ for all $x \in \mathfrak{n}_+$.
- Let $\lambda : \mathfrak{n}_+ \to \mathbb{C}$ be a Lie functional. Let $\mathbb{C}v_\lambda$ be a 1–dimensional $\mathfrak{n}_+$–module such that $xv_\lambda = \lambda(x)v_\lambda$ for all $x \in \mathfrak{n}_+$.

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}_+)} \mathbb{C}v_\lambda$$

is a $U(\mathfrak{g})$–module, which is called the universal (or standard) Whittaker module.

Generalized Whittaker module

- $\mathfrak{g}$ a Lie algebra, and $\mathfrak{n}$ *any nilpotent subalgebra of* $\mathfrak{g}$.
- W a $\mathfrak{g}$–module
- Vector $v \in W$ is a Whittaker vector for the pair $(\mathfrak{g}, \mathfrak{n})$ if there is a Lie functional $\lambda : \mathfrak{n} \to \mathbb{C}$ such that $xv = \lambda(x)v$ for all $x \in \mathfrak{n}$.
- Let $\lambda : \mathfrak{n} \to \mathbb{C}$ be a Lie functional. Let $\mathbb{C}v_\lambda$ be a 1–dimensional $\mathfrak{n}$–module such that $xv_\lambda = \lambda(x)v_\lambda$ for all $x \in \mathfrak{n}$

  $$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}v_\lambda$$

  is a $U(\mathfrak{g})$–module, which is called *the universal Whittaker module for the pair* $(\mathfrak{g}, \mathfrak{n})$.
- Such Whittaker modules and corresponding Whittaker vectors are sometimes called the generalized Whittaker modules/Whittaker vectors.

Example: $\mathfrak{gl}(2\ell, \mathbb{C})$

- usual basis $e_{i,j}$, $i, j = 1, \dots, n$
- triangular decomposition: $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Here

$$\begin{aligned} \mathfrak{n}_+ &= \text{span}_{\mathbb{C}}\{e_{i,j} \mid i < j\}, \\ \mathfrak{n}_- &= \text{span}_{\mathbb{C}}\{e_{i,j} \mid i > j\}, \\ \mathfrak{h} &= \text{span}_{\mathbb{C}}\{e_{i,i} \mid i = 1, \dots, 2\ell\}. \end{aligned}$$

- $\mathfrak{n}_+$ not commutative, e.g. $[e_{1,2}, e_{2,3}] = e_{1,3}$
- we can take a commutative Lie subalgebra:

$$\mathfrak{n} = \text{span}_{\mathbb{C}}\{e_{i,j+\ell} \mid i, j = 1, \dots, \ell\} \subset \mathfrak{n}_+$$

- $\lambda \in \mathfrak{n}^*$, generalized universal Whittaker module

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C} v_\lambda.$$

- two natural questions:
  - Is M_λ irreducible?
  - If M_λ is reducible, describe simple quotients of M_λ.

Example: $\mathfrak{gl}(2\ell, \mathbb{C})$

- usual basis $e_{i,j}$, $i, j = 1, \dots, n$
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- $\lambda \in \mathfrak{n}^*$, generalized universal Whittaker module

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}v_\lambda.$$

- two natural questions:
  - Is M_λ irreducible? $\longrightarrow$ no
  - If M_λ is reducible, describe simple quotients of M_λ $\longrightarrow$ later

Contents

Weyl algebra

- $\mathcal{L}$ infinite-dimensional Lie algebra with generators

$$K, a(n), a^*(n), \quad n \in \mathbb{Z},$$

such that K is in the center and the only notrivial relations are

$$[a(n), a^*(m)] = \delta_{n+m,0} K, \qquad n, m \in \mathbb{Z}.$$

- Weyl algebra $\widehat{\mathcal{A}}$ is given as:

$$\widehat{\mathcal{A}} = \frac{U(\mathcal{L})}{\langle K - 1 \rangle},$$

where $\langle K - 1 \rangle$ is the two sided ideal generated by $K - 1$.

Weyl vertex algebra

- M simple Weyl algebra module generated by a cyclic vector $\mathbf{1}$ such that
$$a(n)\mathbf{1} = a^*(n+1)\mathbf{1} = 0 \quad (n \geq 0),$$
- unique vertex algebra $(M, Y, \mathbf{1})$, where the vertex operator Y is given by:
$$Y(a(-1)\mathbf{1}, z) = a(z), \quad Y(a^*(0)\mathbf{1}, z) = a^*(z),$$
$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n) z^{-n}.$$
- $(M, Y, \mathbf{1}, \omega)$ has the structure of a $\frac{1}{2}\mathbb{Z}_{\geq 0}$–graded vertex operator algebra for
$$\omega = \frac{1}{2}(a(-1)a^*(-1) - a(-2)a^*(0))\mathbf{1} \quad \text{(cf. [KR96])}$$

Whittaker modules for the Weyl (vertex) algebra

- Whittaker module for $\widehat{\mathcal{A}}$:

$$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \widehat{\mathcal{A}}/I,$$

where $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$ and I is the left ideal:

$$\left\langle a(0)-\lambda_0, .., a(n)-\lambda_n, a^*(1)-\mu_1, .., a^*(m)-\mu_m, a(n+1), a^*(n+1), .. \right\rangle.$$

- $\mathfrak{n} \subset \mathcal{L}$ generated by $a(n), a^*(n+1)$, $n \in \mathbb{Z}_{\geq 0}$ $\longrightarrow$ comm., nilp.

Proposition

[ALPY19] We have:

1. *$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a universal Whittaker module for the Whittaker pair $(\mathcal{L}, \mathfrak{n})$.*
2. *$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible $\widehat{\mathcal{A}}$–module.*
3. *$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is an irreducible weak module for the Weyl vertex operator algebra M.*

Irreducibility of Whittaker modules for M

- For $s \in \mathbb{Z}$, Weyl algebra automorphism ρ_s:

$$\rho_s(a(n)) = a(n+s), \quad \rho_s(a^*(n)) = a^*(n-s), \quad (n \in \mathbb{Z}).$$

- For $\widehat{\mathcal{A}}$–module N, $\rho_s(N) = N$ as a vector space and

$$x.v = \rho_s(x)v, \quad x \in \mathcal{A}, \; v \in N.$$

- For $\zeta_p = e^{2\pi i/p}$, M–automorphism g_p uniquely determined by the $\widehat{\mathcal{A}}$–automorphism:

$$a(n) \mapsto \zeta_p a(n), \quad a^*(n) \mapsto \zeta_p^{-1} a^*(n) \quad (n \in \mathbb{Z}).$$

Theorem

[ALPY19] Assume that $\Lambda = (\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq 0$ and $s \in \mathbb{Z}$. Then $\rho_s(M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}))$ is an irreducible weak module for the orbifold subalgebra $M^{\mathbb{Z}_p} = M^{\langle g_p \rangle}$, for each $p \geq 1$.

Additional realizations of the orbifold M^0

- Orbifold M^0 is isomorphic to the following two algebras[KR96] :
  - vertex algebra $\mathcal{W}_{1+\infty,c}$ at central charge $c = -1$.
    This is the simple quotient of the universal vertex algebra $\mathcal{W}^c_{1+\infty}$ generated by the fields

$$J^k(z) = \sum_{n\in\mathbb{Z}} J^k(n) z^{-n-k-1}, k \in \mathbb{Z}_{\geq 0}$$

    whose components satisfy commutation relations for the Lie algebra $\widehat{\mathcal{D}}$ with central charge c
  - Lie algebra $\widehat{\mathfrak{g}l}$.
    This is the infinite-dimensional general linear Lie algebra.
    - Basis: $C, E_{i,j}, i, j \in \mathbb{Z}$.
    - Triangular decomposition: $\widehat{\mathfrak{gl}} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where

$$\begin{aligned} \mathfrak{n}_+ &= span_{\mathbb{C}}\{E_{i,j} \mid i < j\}, \\ \mathfrak{n}_- &= span_{\mathbb{C}}\{E_{i,j} \mid i > j\}, \\ \mathfrak{h}_+ &= \mathbb{C}C \oplus span_{\mathbb{C}}\{E_{i,i} \mid i \in \mathbb{Z}\}. \end{aligned}$$

Additional realizations of the orbifold M^0

Proposition

Assume that $\boldsymbol{\lambda} \neq 0$ *and* $\boldsymbol{\mu} \neq 0$. *Then* $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ *is a Whittaker* $\widehat{\mathfrak{gl}}$*–module for the pair* $\mathfrak{p} \subset \widehat{\mathfrak{gl}}$ *at central charge* $K = -1$*, generated by the Whittaker vector* $\mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$.

Theorem

Assume that $\boldsymbol{\lambda} \neq 0$ *and* $\boldsymbol{\mu} \neq 0$. *Then* $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ *is a Whittaker* $\mathcal{W}_{1+\infty,c=-1}$*–module generated by the cyclic vector* $\mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$. *In particular,* $M_1(\boldsymbol{\lambda},\boldsymbol{\mu})$ *is a cyclic module for the orbifold vertex algebra* M^0.

The structure of $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ as a $\widehat{\mathfrak{gl}}$–module

Theorem

$M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a reducible cyclic $\widehat{\mathfrak{gl}}$–module.

Corollary

We have:

1. *$\rho_s(M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}))$ is a reducible, cyclic $\widehat{\mathfrak{gl}}$–module*
2. *Let $\mathcal{L}$ be any irreducible quotient of $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$. Then $\widetilde{\rho}_s(\mathcal{L})$ is an irreducible quotient of $\rho_s(M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}))$.*

Let $\mathcal{L}$ be any irreducible quotient of $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$. Two important problems:

(A). Find an explicit realization of $\mathcal{L}$ if possible

(B). Determine the complete set of Whittaker vectors in $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ which generate the maximal submodule of $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$

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(A). Find an explicit realization of $\mathcal{L}$ if possible $\longrightarrow$ solved

(B). Determine the complete set of Whittaker vectors in $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ which generate the maximal submodule of $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ $\longrightarrow$ solved

Whittaker vectors in $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$

Proposition

Assume that v is a Whittaker vector in $\mathcal{M}$. Then $v \in \mathbb{C}[\mathbf{I}]\mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}}$.

Theorem

For each $d \in \mathbb{C}$:

$$L(d, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \frac{M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})}{\langle (I - d)\mathbb{C}[I]\mathbf{w}_{\boldsymbol{\lambda},\boldsymbol{\mu}} \rangle}$$

is irreducible.

Corollary

For any $s \in \mathbb{Z}$, $\widetilde{\rho}_s(L(d, \boldsymbol{\lambda}, \boldsymbol{\mu}))$ is an irreducible quotient of $\rho_s(M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}))$.

Whittaker vectors in $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$

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is irreducible.

Corollary

For any $s \in \mathbb{Z}$, $\widetilde{\rho}_s(L(d, \boldsymbol{\lambda}, \boldsymbol{\mu}))$ is an irreducible quotient of $\rho_s(M_1(\boldsymbol{\lambda}, \boldsymbol{\mu}))$.

$\longrightarrow$ solved (B)

Contents

Bosonic realization of $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$

- Tanabe[Tan17]: Whittaker modules for the rank one Heisenberg VOA remain irreducible when restricted to the singlet vertex algebra [Ada03]
- $\mathcal{W}_{1+\infty, c=-1} \cong \mathcal{M}(2) \otimes M_1(1) \subset M_2(1)$
- We have two cases:
  - $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a $\Pi(0)$–module $\longrightarrow$ $L(d, \boldsymbol{\lambda}, \boldsymbol{\mu})$ are $M_2(1)$–modules
  - General case: $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is different from Whittaker modules in [Tan17]

Case $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is a $\Pi(0)$–module

Proposition

Assume that $\lambda \neq 0$, $\mu_n \neq 0$, $\boldsymbol{\lambda} = (\lambda, 0, 0, \dots)$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_n)$. *Then* $M_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$ *has the structure of an irreducible* $\Pi(0)$*–module.*

Proposition

Identify $\mathcal{W}_{1+\infty,-1}$ *as a subalgebra of the Heisenberg vertex algebra* $M_{\gamma,\overline{\delta}}(1)$. *Assume that*

$$\boldsymbol{\lambda} = (\lambda, 0, 0, \dots), \ \boldsymbol{\mu} = (\mu_1, \cdots, \mu_n).$$

Then the Whittaker $\mathcal{W}_{1+\infty,-1}$*–module* $L(d, \boldsymbol{\lambda}, \boldsymbol{\mu})$ *has the structure of an irreducible Whittaker module for the Hiesenberg vertex algebra.*

The general case and non-tensor product modules

- Whittaker modules for M^0 can be obtained by using [Tan17]
- $L(d, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is not an irreducible Whittaker $M_2(1)$–module in general $\longrightarrow$ [Tan17] does not work

Proposition

Assume that $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n)$, $\mu = (\mu_{\mathbf{1}}, \ldots, \mu_{\mathbf{m}})$, $n > 0$ *and* $\lambda_n \cdot \mu_m \neq 0$. *Then* $L(d, \boldsymbol{\lambda}, \boldsymbol{\mu})$ *does not have the tensor product form, i.e, it is not realized as a Whittaker* $M_2(1)$*–module.*

$\longrightarrow$ solved (A)

Contents

Generalized Whittaker modules for $\mathfrak{g} = \mathfrak{gl}(2\ell, \mathbb{C})$

- Triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$
- $\mathfrak{n} = \mathrm{span}_{\mathbb{C}}\{e_{i,j+\ell} \mid i,j = 1, \dots, \ell\} \subset \mathfrak{n}_+$
- Weyl algebra $\mathcal{A}_{2\ell}$, gen. $a_i, a_i^*, i = 1, \dots, 2\ell$, rel. $[a_i, a_j^*] = \delta_{i,j}$
- For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\ell)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_\ell) \in \mathbb{C}^\ell$, Whittaker module:

$$W(\boldsymbol{\alpha}, \boldsymbol{\beta}) = {}^{\mathcal{A}_{2\ell}}/{}_{I_\ell(\boldsymbol{\alpha}, \boldsymbol{\beta})},$$

where $I_\ell(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the left ideal

$$I_\ell(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \langle a_1 - \alpha_1, \dots, a_\ell - \alpha_\ell, a_{\ell+1}^* - \beta_1, \dots, a_{2\ell}^* - \beta_\ell \rangle.$$

Theorem

For each $d \in \mathbb{C}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}^\ell$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \neq 0$,

$$L(d, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{W(\boldsymbol{\alpha}, \boldsymbol{\beta})}{\langle (I - d)\mathbb{C}[I]\mathbf{w}_{\boldsymbol{\alpha}, \boldsymbol{\beta}} \rangle}$$

is an irreducible $\mathfrak{g}$–module.

Thank you!

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