# Some results on unitary highest weight modules for minimal $W$-algebras

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joint work with D. Adamović, V. Kac, P. Möseneder Frajria

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# Plan of the talk

1. Introduction
2. Minimal $W$-algebras
3. Unitarity for minimal $W$-algebras
4. Main results
5. Some details on proofs and techniques

# $W$-algebras

Kac and Wakimoto used quantum Hamiltonian reduction to build up a vertex algebra $W^k(\mathfrak{g}, x, f)$, associated to a datum $(\mathfrak{g}, x, f)$ and $k \in \mathbb{C}$. Here

- $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a basic Lie superalgebra, i.e. $\mathfrak{g}$ is simple, its even part $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra and $\mathfrak{g}$ carries an even invariant non-degenerate supersymmetric bilinear form $(\cdot|\cdot)$,
- $x$ is an *ad*-diagonalizable element of $\mathfrak{g}_{\bar{0}}$ with eigenvalues in $\frac{1}{2}\mathbb{Z}$, $f \in \mathfrak{g}_{\bar{0}}$ is such that $[x, f] = -f$ and the eigenvalues of $ad\, x$ on the centralizer $\mathfrak{g}^f$ of $f$ in $\mathfrak{g}$ are non-positive
- $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

# $W$-algebras

The most important examples are provided by $x$ and $f$ to be part of an $sl_2$ triple $\{e, x, f\}$, where $[x, e] = e, [x, f] = -f, [e, f] = x$. In this case $(\mathfrak{g}, x, f)$ is called a *Dynkin datum*.

We recently proved that if $\phi$ is a conjugate linear involution of $\mathfrak{g}$ such that

$$\phi(x) = x, \quad \phi(f) = f \text{ and } \overline{(\phi(a)|\phi(b))} = (a|b),\ a, b \in \mathfrak{g}, \tag{1.1}$$

then $\phi$ induces a conjugate linear involution of the vertex algebra $W^k(\mathfrak{g}, x, f)$.

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then $\phi$ induces a conjugate linear involution of the vertex algebra $W^k(\mathfrak{g}, x, f)$.
Moreover, if $\phi$ is a conjugate linear involution of $W^k(\mathfrak{g}, x, f)$, this vertex algebra carries a non-zero $\phi$-invariant Hermitian form $H(\cdot, \cdot)$ for all $k \neq -h^\vee$ if and only if $(\mathfrak{g}, x, f)$ is a Dynkin datum; such an $H$ is unique, up to a real constant factor, and we normalize it by the condition $H(\mathbf{1}, \mathbf{1}) = 1$.

# Unitarity

**Definition**

A module $M$ over a conformal vertex algebra $V$ is called *unitary* if there is a conjugate linear involution $\phi$ of $V$ such that there is a positive definite $\phi$–invariant Hermitian form on $M$. A simple vertex algebra $V$ is called unitary if the adjoint module is.

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Denote by $W^k_{\min}(\mathfrak{g})$ the vertex algebra $W^k(\mathfrak{g}, x, f)$ when $f$ is a minimal nilpotent element. In collaboration with Kac and Möseneder Frajria we have recently classified the pairs $(\mathfrak{g}, k)$ such that the simple quotient $W_k^{\min}(\mathfrak{g})$ of $W^k_{\min}(\mathfrak{g})$ is unitary. For each $\mathfrak{g}$ the corresponding set of values of $k$ is called the *unitary range*.

## Goals of the talk

1. Detect necessary and sufficient conditions, in the unitary range, in order that a irreducible highest weight $W^k_{\min}(\mathfrak{g})$-module descends to $W_k^{\min}(\mathfrak{g})$.
2. Classify irreducible positive energy representations of $W_k^{\min}(\mathfrak{g})$ in the unitary range.

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1. Detect necessary and sufficient conditions, in the unitary range, in order that a irreducible highest weight $W^k_{\min}(\mathfrak{g})$-module descends to $W_k^{\min}(\mathfrak{g})$.
2. Classify irreducible positive energy representations of $W_k^{\min}(\mathfrak{g})$ in the unitary range.
3. Important note: the solution of problem 2 is related to the following problem, which we are able to solve

Classify the irreducible highest weight $V_k(\mathfrak{g})$-modules in the unitary range

# Minimal $W$-algebras

Let $\mathfrak{g}^\natural$ be the centralizer of the $sl_2$ subalgebra $\mathfrak{s} = span\{e, x, f\}$ in $\mathfrak{g}_{\bar{0}}$; it is a reductive subalgebra; we let $\mathfrak{g}^\natural = \bigoplus_{i \in S} \mathfrak{g}_i^\natural$ denote its irreducible decomposition.

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*Definition: minimal W-algebra*

We ask that for the $ad\, x$-gradation $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ one has

$$\mathfrak{g}_j = 0 \text{ if } |j| > 1, \text{ and } \mathfrak{g}_{-1} = \mathbb{C}f. \tag{2.1}$$

In this case $(\mathfrak{g}, x, f)$ is automatically a Dynkin datum. The corresponding $W$-algebra is called *minimal*, and denoted by $W^k_{\min}(\mathfrak{g})$.

# Universal minimal $W$-algebras: structure

Recall that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x$.

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Theorem (KRW)

*(a) The vertex algebra $W^k_{\min}(\mathfrak{g})$ is strongly and freely generated by elements $J^{\{a\}}$, where $a$ runs over a basis of $\mathfrak{g}^\natural$, $G^{\{v\}}$, where $v$ runs over a basis of $\mathfrak{g}_{-1/2}$, and the Virasoro vector $L$.*

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*(b) The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight $1$ and $3/2$, respectively, with respect to $L$.*

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*(b) The elements* $J^{\{a\}}$*,* $G^{\{v\}}$ *are primary of conformal weight* 1 *and* 3/2*, respectively, with respect to* $L$*.*

*(c) The following* $\lambda$*-brackets hold:*

$$[J^{\{a\}}{}_\lambda J^{\{b\}}] = J^{\{[a,b]\}} + \lambda \underbrace{\left((k+h^\vee/2)(a|b) - \tfrac{1}{4}\kappa_0(a,b)\right)}_{\beta_k(a,b)}, \; a,b \in \mathfrak{g}^\natural,$$

$$[J^{\{a\}}{}_\lambda G^{\{u\}}] = G^{\{[a,u]\}}, \; a \in \mathfrak{g}^\natural, \; u \in \mathfrak{g}_{-1/2}.$$

*(d) There are explicit formulas yielding* $[G^{\{u\}}{}_\lambda G^{\{v\}}]$ *for* $u,v \in \mathfrak{g}_{-1/2}$*.*

# Presentation of $W^k_{\min}(\mathfrak{g})$

**Proposition (AKMPP, J-alg)**

Let $u, v \in \mathfrak{g}_{-1/2}$. Then

$$G^{\{u\}}{}_{(2)}G^{\{v\}} = 4(e_\theta|[u,v])p(k)\mathbf{1}.$$

Moreover, the linear polynomial $k_i(k)$, $i \in S$, defined by $k_i(k) = k + \frac{1}{2}(h^\vee - h_i^\vee)$, divides $p(k)$ and

$$G^{\{u\}}{}_{(1)}G^{\{v\}} = 4\sum_{i\in S}\frac{p(k)}{k_i(k)}J^{\{([[e_\theta,u],v])_i^\natural\}}$$

where $(a)_i^\natural$ denotes the orthogonal projection of $a \in \mathfrak{g}_0$ onto $\mathfrak{g}_i^\natural$ and $p(k)$ is an explicit monic quadratic polynomial.

# Collapsing levels

**Definition**

We say that a level $k$ is *collapsing* if

$$W_k^{\min}(\mathfrak{g}) = V_{\beta_k}(\mathfrak{g}^\natural).$$

Here $V_{\beta_k}(\mathfrak{g}^\natural)$ denotes the simple affine vertex algebra relative to the cocycle

$$\beta_k(a,b) = \left((k + h^\vee/2)(a|b) - \tfrac{1}{4}\kappa_0(a,b)\right), \quad a,b \in \mathfrak{g}^\natural$$

**Theorem (AKMPP)**

*Let $\mathfrak{g}^\natural = \bigoplus_{i \in S} \mathfrak{g}_i^\natural$. Then $k$ is a collapsing level if and only if $p(k) = 0$, and*

$$W_k^{\min}(\mathfrak{g}) = \bigotimes_{i \in S : k_i \neq 0} V_{k_i}(\mathfrak{g}_i^\natural). \tag{3.1}$$

# Unitarity: basic definitions

Let $W = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}_+} W(n)$, be a conformal vertex algebra and let $\phi$ is a conjugate linear involution of $W$ such that $\phi(L) = L$. Set

$$g(z) = e^{-\pi\sqrt{-1}(\frac{1}{2}p(a)+\Delta_a)}\phi(a), \quad a \in W(\Delta_a).$$

## Definition

A Hermitian form $(\cdot\,,\cdot)$ on $W$ is said to be $\phi$*–invariant* if, for all $a \in W$,

$$(v, Y(a,z)u) = (Y(A(z)a, z^{-1})v, u), \quad u, v \in W.$$

where

$$A(z) = e^{zL_1} z^{-2L_0} g.$$

# Restrictions on $\mathfrak{g}$ for unitarity

## Proposition (KMP, CCM2022)

*If $W_k^{\min}(\mathfrak{g})$ is unitary and $k$ is not a collapsing level, then either $\mathfrak{g} = sl_2$ or $\mathfrak{g}$ is not a Lie algebra and $\mathfrak{g}^\natural$ is a Lie algebra. In particular, the parity of $\mathfrak{g}$ is compatible with the ad $x$-gradation, i.e.*

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-1}}_{even} \oplus \underbrace{\mathfrak{g}_{-1/2}}_{odd} \oplus \underbrace{\mathfrak{g}_0}_{even} \oplus \underbrace{\mathfrak{g}_{1/2}}_{odd} \oplus \underbrace{\mathfrak{g}_1}_{even} \qquad (\clubsuit)$$

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## Corollary

*The complete list of the $\mathfrak{g}$ for which $(\clubsuit)$ holds is*

$$sl(2|m) \text{ for } m \geq 3, \quad psl(2|2), \quad spo(2|m) \text{ for } m \geq 0,$$
$$osp(4|m) \text{ for } m > 2 \text{ even}, \quad D(2,1;a) \text{ for } a \in \mathbb{C}, \quad F(4), \quad G(3).$$

# Unitarity

In these cases

$$\mathfrak{g}_{\bar{0}} = \mathfrak{g}^{\natural} \oplus \mathfrak{s}, \quad \mathfrak{s} \cong sl_2.$$

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with $\mathfrak{g}^\natural$ a reductive Lie algebra.

- We proved that the conjugate linear involutions of $W_k^{\min}(\mathfrak{g})$ that fix the Virasoro vector $L$ are in one-to-one correspondence with the conjugate linear involutions $\phi$ of $\mathfrak{g}$ that fix pointwise the triple $\{e, x, f\}$.
- It is easy to see that, in order to have $W_k^{\min}(\mathfrak{g})$ unitary, $\phi_{|\mathfrak{g}^\natural}$ must be the conjugation corresponding to a compact real form. Such a conjugate linear involution is called an *almost compact involution*.
- An almost compact involution $\phi_{ac}$ exists in all the cases listed above

# Representations

$W$-algebras have a highest weight module theory. Let $L^W(\nu, \ell_0)$, the simple highest weight $W^k_{\min}(\mathfrak{g})$–module with highest weight $(\nu, \ell_0)$, where

- $\nu$ is a weight of $\mathfrak{g}^\natural$;
- $\ell_0$ is the eigenvalue of $L_0$.

## Fact

We prove that $L^W(\nu, \ell_0)$ admits a $\phi$–invariant nondegenerate Hermitian form (unique up to normalization).

# Conditions for unitarity: rough statement

### Basic remark

Unitarity of $L^W(\nu, \ell_0)$ implies that

- the levels $M_i(k)$ of the affine Lie algebras $\widehat{\mathfrak{g}}_i^\natural$ in $W^k_{\min}(\mathfrak{g})$ (they are explicit linear polynomials in $k$, related to $p(k)$), where $\mathfrak{g}_i^\natural$ are the simple components of $\mathfrak{g}^\natural$, are non-negative integers;
- $\nu$ is a dominant integral weight of level $M_i(k)$;
- a certain inequality $(*)$ holds. Moreover, $(*)$ must be an equality when $\nu$ is an "extremal" weight.

# Classification: preparation

## Notation

Recall that

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1.$$

- Let $\theta_i^\vee$ be the coroots of the maximal roots $\theta_i$ of the simple components $\mathfrak{g}_i^\natural$ of $\mathfrak{g}^\natural = \bigoplus_i \mathfrak{g}_i^\natural$.
- Let $2\rho^\natural$ be the sum of positive roots of $\mathfrak{g}^\natural$,
- Let $\xi$ be a highest weight of the $\mathfrak{g}^\natural$-module $\mathfrak{g}_{-1/2}$ (this module is irreducible, except for $\mathfrak{g} = psl(2|2)$ when it is $\mathbb{C}^2 \oplus \mathbb{C}^2$).
- Let $\nu$ be a dominant integral weight for $\mathfrak{g}^\natural$ and $\ell_0 \in \mathbb{R}$.

# Classification

### Theorem (KMP)

*Let $L^W(\nu, \ell_0)$ be a simple highest weight $W^k_{\min}(\mathfrak{g})$-module over $\mathfrak{g} = psl(2|2)$, $spo(2|m)$ with $m \geq 3$, $D(2,1;a)$, $F(4)$ or $G(3)$.*

1. *This module can be unitary only if the following conditions hold:*
   1. *$M_i(k)$ are non-negative integers,*
   2. *$\nu(\theta_i^\vee) \leq M_i(k)$ for all $i$,*
   3. $$\ell_0 \geq \frac{(\nu|\nu + 2\rho^\natural)}{2(k+h^\vee)} + \frac{(\xi|\nu)}{k+h^\vee}((\xi|\nu) - k - 1) =: A(k,\nu), \qquad (*)$$

      *and equality holds in (*) if $\nu(\theta_i^\vee) > M_i(k) + \chi_i$ for $i = 1$ or $2$.*
2. *This module is unitary if the following conditions hold:*
   1. *$M_i(k) + \chi_i \in \mathbb{Z}_+$ for all $i$,*
   2. *$\nu(\theta_i^\vee) \leq M_i(k) + \chi_i$ for all $i$ (i.e. $\nu$ is not extremal),*
   3. *inequality (*) holds.*

# Summary

For the module $L^W(0,0) = W_k^{\min}(\mathfrak{g})$, both (1) and (2) hold, so one can determine the unitary range, displayed in the second row of the following table

| | | | | | |
|---|---|---|---|---|---|
| $psl(2\|2)$ | $spo(2\|3)$ | $spo(2\|m), m > 4$ | $D(2,1;\frac{m}{n}),\ m,n \in \mathbb{N}$ | $F(4)$ | $G(3)$ |
| $-(\mathbb{N}+1)$ | $-\frac{1}{4}(\mathbb{N}+2)$ | $-\frac{1}{2}(\mathbb{N}+1)$ | $-\frac{mn}{m+n}\mathbb{N},\ (m,n)=1$ | $-\frac{2}{3}(\mathbb{N}+1)$ | $-\frac{3}{4}(\mathbb{N}+1)$ |
| $sl(2)$ | $sl(2)$ | $so(m)$ | $sl(2) \oplus sl(2)$ | $so(7)$ | $G_2$ |
| $k+1$ | $4k+2$ | $2k+1$ | $\frac{m+n}{n}k+1, \frac{m+n}{m}k+1$ | $\frac{3}{2}k+1$ | $\frac{4}{3}k+1$ |
| $0$ | $\frac{1}{2}$ | $2-\frac{1}{2}m$ | $0$ | $-2$ | $-\frac{3}{2}$ |

Table: Unitarity range, $\mathfrak{g}^\natural, -M_i(k)$, and $h^\vee$

# Conjecture

**Conjecture** *The modules $L^W(\nu, \ell_0)$ are unitary if $\nu$ is extremal and $l_0 = R.H.S.$ of $(*)$. In other words, the necessary conditions of the above Theorem are sufficient.*

We are able to prove this conjecture only for $\mathfrak{g} = psl(2|2)$ and $spo(2|3)$, obtaining thereby a complete classification of unitary simple highest weight $W^k_{\min}(\mathfrak{g})$-modules in these two cases.

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We are able to prove this conjecture only for $\mathfrak{g} = psl(2|2)$ and $spo(2|3)$, obtaining thereby a complete classification of unitary simple highest weight $W^k_{\min}(\mathfrak{g})$-modules in these two cases.
The proof is a nice application of the theory of collapsing levels, and uses the fact that superconformal algebras are linearizable, i.e. up to adding some extra fields the modes of the fields form a Lie superalgebra, hence we can tensor their modules.

# Example: $N = 3$, beginning of proof

We have $M_1(k) = -4k - 2 \in \mathbb{N}$. The extremal modules are

$$L^W\left(\frac{M_1-1}{2}\alpha, \frac{M_1-1}{4}\right), \quad L^W\left(\frac{M_1}{2}\alpha, \frac{M_1}{4}\right)$$

Proceed by induction on $M_1$. If $M_1 = 1$ then $k = -3/4$, which is collapsing

$$W^{\min}_{-3/4}(spo(2|3)) = V_1(sl_2).$$

Now $V_1(sl_2)$ has two irreducible modules, both unitary with highest weights $0, \alpha/2$. To conclude use induction, the relationship between $W^{\min}_{-3/4}(spo(2|3))$ and the $N = 3$ superconformal algebra $\mathcal{W}^k_{N=3}$, and the fact that tensor product of unitary modules for $\mathcal{W}^k_{N=3}$ is unitary.

# Main results (Adamović, Kac, Möseneder, P.)

### Theorem

*Let $k$ be in the unitary range. Then all irreducible highest weight $W^k_{\min}(\mathfrak{g})$–modules $L^W(\nu, \ell_0)$ with $\ell_0 \in \mathbb{C}$ when $\nu \in P^+_k$ is not extremal, and $\ell_0 = A(k, \nu)$ otherwise, descend to $W_k^{\min}(\mathfrak{g})$.*

### Corollary

*Any unitary $W^k_{\min}(\mathfrak{g})$–module $L^W(\nu, \ell_0)$ descends to $W_k^{\min}(\mathfrak{g})$. Hence Conjecture 4 from [KMP23] holds.*

### Theorem

The modules appearing above form the complete list of inequivalent irreducible positive energy representations of $W_k^{\min}(\mathfrak{g})$ in the unitary range.

# Some details on AKMP results

Let $P^+ \subset (\mathfrak{h}^\natural)^*$ be the set of dominant integral weights for $\mathfrak{g}^\natural$ and let

$$P^+_k = \left\{\nu \in P^+ \mid \nu(\theta_i^\vee) \leq M_i(k) \text{ for all } i \geq 1\right\}. \tag{8.1}$$

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Recall that $\nu \in P^+_k$ an *extremal weight* if $\nu + \xi$ doesn't lie in $P^+_k$. Denote by $L(\lambda)$ the irreducible highest weight $V^k(\mathfrak{g})$-module of highest weight $\lambda$. For $h \in \mathbb{C}$ and $\nu \in (\mathfrak{h}^\natural)^*$, set

$$\widehat{\nu}_h = k\Lambda_0 + h\theta + \nu. \tag{8.2}$$

Note that every highest weight module for $V^k(\mathfrak{g})$ has highest weight $\widehat{\nu}_h$ for some $\nu \in (\mathfrak{h}^\natural)^*$ and $h \in \mathbb{C}$.

# Some details on AKMP results

### Theorem (T1)

*Let $k$ be in the unitary range. Then, up to isomorphism, the irreducible highest weight $V_k(\mathfrak{g})$-modules are as follows:*

1. $L(\widehat{\nu}_h)$ *with* $\nu \in P_k^+$ *non-extremal and* $h \in \mathbb{C}$ *arbitrary;*
2. $L(\widehat{\nu}_h)$ *with* $\nu$ *extremal and* $h$ *from the set*

$$E_{k,\nu} = \{(\xi|\nu), k + 1 - (\xi|\nu)\}.$$

# Idea of proof

Gorelik-Kac computed the characters of highest weight $\widehat{\mathfrak{g}}$-modules with highest weight $k\Lambda_0$ using only their integrability with respect to $\widehat{\mathfrak{g}}^\natural$, which implies that such modules are irreducible. Gorelik-Serganova deduced from this that, if $V_k(\mathfrak{g})$ is integrable as a $\widehat{\mathfrak{g}}^\natural$-module, then the $V^k(\mathfrak{g})$–modules, which are integrable over $\widehat{\mathfrak{g}}^\natural$, descend to $V_k(\mathfrak{g})$.

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- One first shows that $V_k(\mathfrak{g})$ is integrable for $\widehat{\mathfrak{g}}^\natural$, by checking that $(x_{\theta_i})_{(-1)}^N \mathbf{1} = 0$ for $N \gg 0$. This is done by the *method of odd reflections*.

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- Then we prove that the modules listed in (1), (2) are $\widehat{\mathfrak{g}}^\natural$-integrable. This part is technical.
- We are left with proving that any irreducible highest weight module for $V_k(\mathfrak{g})$ is of the form (1) or (2). Let $L(\widehat{\nu}_h)$ be an irreducible highest weight $V_k(\mathfrak{g})$-module with h.w. vector $v$. We prove that, necessarily, $\nu \in P_k^+$. Indeed, the action of $\mathfrak{g}^\natural$ on $v$ should be locally finite, so that $\nu \in P^+$.

# Idea of proof

If $h \notin E_{k,\nu}$ then $(\widehat{\nu}_h|\alpha_1) \neq 0$ and $(\widehat{\nu}_h - \alpha_1|\alpha_0 + \alpha_1) \neq 0$. It follows that $(x_{\theta-\alpha_1})_{(-1)}(x_{-\alpha_1})_{(0)}v$ is a highest weight vector with respect to the set of simple roots $r_{\alpha_0+\alpha_1}r_{\alpha_1}(\widehat{\Pi})$ of highest weight $\Lambda' = \widehat{\nu}_h - \alpha_0 - 2\alpha_1$. If $L(\widehat{\nu}_h)$ is integrable with respect to $\widehat{\mathfrak{g}}^\natural$, then a direct computation shows that

$$m_i = ((\delta - \theta_i)^\vee|\Lambda') = M_i(k) + \chi_i - (\nu|\theta_i^\vee) \in \mathbb{Z}_+,$$

hence

$$(\nu|\theta_i^\vee) \leq M_i(k) + \chi_i \leq M_i(k).$$

It follows that $\nu \in P_k^+$ and it is not extremal, i.e. $L(\widehat{\nu}_h)$ is of type (1).
If $h = (\xi|\nu)$ (resp. $h = k + 1 - (\xi|\nu)$), then $(\widehat{\nu}_h|\alpha_1) = 0$ (resp. $(\widehat{\nu}_h|\alpha_0 + \alpha_1) = 0$ ) and in turn we get that $\nu \in P_k^+$. In particular, $L(\widehat{\nu}_h)$ is of type (1) if $\nu$ is not extremal and of type (2) if $\nu$ is extremal.

# Explicit description of the maximal ideal of $W^k_{\min}(\mathfrak{g})$

Let $I^k$ be the maximal ideal of $W^k_{\min}(\mathfrak{g})$. Set

$$v_i = (J^{\{x_{\theta_i}\}}_{(-1)})^{M_i(k)+1}\mathbf{1}.$$

If an irreducible highest weight $W^k_{\min}(\mathfrak{g})$–module $L^W(\nu, \ell_0)$ is unitary, then, restricted to the affine subalgebra $V^{\beta_k}(\mathfrak{g}^\natural)$ it is unitary, hence a direct sum of irreducible integrable highest weight $\widehat{\mathfrak{g}}^\natural_i$–modules of levels $M_i(k)$. But it is well-known that all these modules descend to the simple affine vertex algebra $V_{\beta_k}(\mathfrak{g}^\natural)$, and are annihilated by the elements $v_i$. In particular,

$$v_i \in I^k.$$

**Theorem**

*The maximal ideal $I^k$ is generated by the singular vectors*

$$\widetilde{v}_i = \begin{cases} (J^{\{x_{\theta_1}\}}_{(-1)})^{M_1(k)-1} G^{\{x_{-\alpha_1}\}}_{(-3/2)}\mathbf{1}, & \text{if } \mathfrak{g} = spo(2|3), \\ v_i, \ i \in S & \text{otherwise.} \end{cases}$$

# Idea of proof (assume $|S| = 1$)

1. We explicitly provide a singular vector in the universal affine vertex algebra $V^k(\mathfrak{g})$ which generates the maximal ideal $J^k$ of $V^k(\mathfrak{g})$.
2. From (1) and exactness of the quantum Hamiltonian reduction functor $H_0$, we deduce that $I^k = H_0(J^k)$ is a highest weight module.
3. The highest weight of $I^k$ is $((M_1(k)+1)\theta_1, M_1(k)+1)$.
4. But also $v_1$ has weight $I^k$ is $((M_1(k)+1)\theta_1, M_1(k)+1)$, hence it is singular and generates $I^k$.

# Classification

**Theorem (T2)**

*Let $k$ be in the unitary range. Then all irreducible highest weight $W^k_{\min}(\mathfrak{g})$–modules $L^W(\nu, \ell_0)$ with $\ell_0 \in \mathbb{C}$ when $\nu \in P^+_k$ is not extremal, and $\ell_0 = A(k, \nu)$ otherwise, descend to $W_k^{\min}(\mathfrak{g})$.*

**Proof.**

By Arakawa's theorem, $H_0(L(\widehat{\nu}_h)) = 0$ if $\widehat{\nu}_h(\alpha_0^\vee) = k - 2h \in \mathbb{Z}_{\geq 0}$, and $H_0(L(\widehat{\nu}_h) = L^W(\nu, \ell_0(h))$ otherwise. Here $\ell_0(h) = \frac{(\widehat{\nu}_h | \widehat{\nu}_h + 2\widehat{\rho})}{2(k+h^\vee)} - h$. If $k - 2h \in \mathbb{Z}_{\geq 0}$, then for $h' = k + 1 - h$ we have $\ell_0 := \ell_0(h) = \ell_0(h')$. Since $k - 2h' \notin \mathbb{Z}_+$, $H_0(L(\widehat{\nu}_{h'})) = L^W(\nu, \ell_0)$. So for each $\ell_0$ there is $\tilde{h} \in \mathbb{C}$ such that $L^W(\nu, \ell_0) = H_0(L(\widehat{\nu}_{\tilde{h}}))$. By T1, if $\nu$ is not extremal, then $L(\widehat{\nu}_{\tilde{h}})$ is a $V_k(\mathfrak{g})$–module, hence $L^W(\nu, \ell_0(\tilde{h}))$ is a $W_k^{\min}(\mathfrak{g})$-module. But $h \in E_{k,\nu}$ if and only if $\ell_0(h) = A(k, \nu)$. T1 implies that, if $\nu$ is extremal, then $L(\widehat{\nu}_{\tilde{h}})$ is a $V_k(\mathfrak{g})$–module, hence $L^W(\nu, A(k, \nu))$ is a $W_k^{\min}(\mathfrak{g})$-module. □

# Preparation

Recall that a module $M$ over a vertex operator superalgebra is called *positive energy* if admits an $\mathbb{R}$-grading :

$$M = \bigoplus_{j \geq 0} M_j$$

with $a_n^M M_j \subset M_{j-n}$. The subspace $M_0$ is called the *top component* of $M$. Recall that there is one-to-one correspondence between irreducible positive energy $V$–modules and irreducible modules for the Zhu algebra $Zhu(V)$, which associates to a $V$-module $M$ the $Zhu(V)$-module $M_0$. Namely, to $Y^M(a,z) = \sum_j a_j^M z^{-j-\deg a}$ one associates $a^M_{0\,|M_0}$. We have

$$Zhu(W^k_{\min}(\mathfrak{g})) \simeq \mathbb{C}[L] \otimes U(\mathfrak{g}^\natural).$$

Indeed $Zhu(W^k_{\min}(\mathfrak{g}))$ is generated by the image of the strong generators of $W^k_{\min}(\mathfrak{g})$. But $G$–generators of conformal weight $\frac{3}{2}$ are zero in $Zhu(W^k_{\min}(\mathfrak{g}))$. Thus, $Zhu(W^k_{\min}(\mathfrak{g}))$ is generated only by generators of conformal weights 1 and 2.

# Theorem

## Theorem (T3)

*The modules appearing in Theorem T2 form the complete list of inequivalent irreducible highest weight representations of $W_k^{\min}(\mathfrak{g})$. Moreover they are exactly all the irreducible positive energy representations of $W_k^{\min}(\mathfrak{g})$.*

We present a sketch of proof of the first statement for $\mathfrak{g} = spo(2|3)$. Let $k = -\frac{m}{4}$, where $m \in \mathbb{Z}_{\geq 3}$, so that $M_1(k) = m - 2$. Set $\mathcal{W}^k = W^k_{\min}(spo(2|3)), \mathcal{W}_k = W_k^{\min}(spo(2|3))$, and note that $\mathcal{W}^k$ is generated by $L, G^+, G^-, G^0, J^+, J^-, J^0$ with conformal weights $2, 3/2, 1$.

Let $L_k[j, q]$ be the irreducible highest weight $\mathcal{W}_k$-module of level $k$:

$$L_n v_{j,q} = q\delta_{n,0} v_{j,q},\ \ G^+_{(n+1/2)} v_{j,q} = G^-_{(n+1/2)} v_{j,q} = G^0_{(n+1/2)} v_{j,q} = 0$$
$$J^0_{(n)} v_{j,q} = j\delta_{n,0} v_{j,q},\ \ J^+_{(n)} v_{j,q} = J^-_{(n+1)} v_{j,q} = 0 \quad (n \geq 0)$$

# proof

Under the Zhu correspondence, the module $L^W(\nu, h)$ goes to the irreducible highest weight $\mathfrak{g}^\natural$-module with highest weight $\nu$ on which $L$ acts by the scalar $h$, which we denote by $V(\nu, h)$. We have

$$Zhu(W_k^{\min}(\mathfrak{g})) \simeq \left(\mathbb{C}[L] \otimes U(\mathfrak{g}^\natural)\right) / J(\mathfrak{g}),$$

for a certain 2-sided ideal $J(\mathfrak{g})$. So any non-zero element in $J(\mathfrak{g})$ imposes a condition on the highest weight $(\nu, h)$ of $V(\nu, h)$. Using the explicit expression of $\tilde{v}_1$ one proves

## Lemma

*Denote by* $[a]$ *the class of* $a$ *in the Zhu algebra. Set*

$$\Omega = -\tfrac{m-2}{4}([L] + \frac{[J^0]}{4}) + \tfrac{1}{8}[J^+] * [J^-].$$

*Then*

$$\Omega * [J^-]^{m-3} \in J(\mathfrak{g}). \tag{8.3}$$

# proof

**Proposition ( first claim of T3 for $\mathfrak{g} = spo(2|3)$)**

Let $L_k[j,q]$ be an irreducible highest weight $\mathcal{W}_k$–module. Then it is isomorphic to exactly one of the following modules:

- $L_k[j,q]$ with $0 \le j \le m-4$ and $q \in \mathbb{C}$;
- $L_k[m-3, \frac{m-3}{4}]$;
- $L_k[m-2, \frac{m-2}{4}]$.

Note that $L_k[j,q]_{top} = \mathbb{C}_q \otimes V(j\omega_1) = V(j\omega_1, q)$. If $j = m-3, m-2$, then $(J^-)_{(0)}^{m-3}$ acts non-trivially on $L_k[j,q]_{top}$. Hence there exists $w \in L_k[j,q]_{top}$ such that $w' = (J^-)_{(0)}^{m-3} w$ is a lowest weight vector for $sl(2)$, i.e. $(J^0)_{(0)} w' = -jw'$, $(J^-)_{(0)} w' = 0$.

# End of the proof

By the Lemma :

$$
\begin{aligned}
0 &= (\Omega * [J^-]^{m-3})w \\
&= \left(-\tfrac{m-2}{4}(L_0 + \tfrac{1}{4}(J^0)_{(0)}) + \tfrac{1}{8}(J^+)_{(0)}(J^-)_{(0)}\right)(J^-)_{(0)}^{m-3} w \\
&= \left(-\tfrac{m-2}{4}(L_0 + \tfrac{1}{4}(J^0)_{(0)}) + \tfrac{1}{8}(J^+)_{(0)}(J^-)_{(0)}\right) w' \\
&= -\tfrac{m-2}{4}(q - \tfrac{j}{4})w'.
\end{aligned}
$$

This implies that for $m-3 \leq j \leq m-2$, we need to have $q = \frac{j}{4}$.

# Irreducible highest weight / Positive Energy

Let $M$ be an irreducible positive energy $W_k^{\min}(\mathfrak{g})$–module.

- $M_{top}$ is an irreducible $Zhu(W_k^{\min}(\mathfrak{g}))$-module.
- By using the embedding $\mathcal{V}_k(\mathfrak{g}^\natural) = \bigotimes_{i\in S} V_{M_i(k)}(\mathfrak{g}_i^\natural) \to W_k^{\min}(\mathfrak{g})$, we see that $M_{top}$ is also a $Zhu(\mathcal{V}_k(\mathfrak{g}^\natural))$–module.
- Since $[L]$ is central element in $Zhu(W_k^{\min}(\mathfrak{g}))$ it doesn't affect irreducibility.
- Since $Zhu(W^k_{\min}(\mathfrak{g})) \simeq \mathbb{C}[L] \otimes U(\mathfrak{g}^\natural)$, $M_{top}$ is an irreducible $U(\mathfrak{g}^\natural)$–module, and therefore an irreducible $Zhu(\mathcal{V}_k(\mathfrak{g}^\natural))$–module.
- Since the $V_{M_i(k)}(\mathfrak{g}_i^\natural)$ are rational, $Zhu(\mathcal{V}_k(\mathfrak{g}^\natural))$ is a semi-simple f.d. associative algebra, hence $M_{top}$ is finite-dimensional.
- Hence $M_{top}$ contains a highest weight $\mathcal{V}_k(\mathfrak{g}^\natural)$–vector $w$, which is then a highest weight vector for the action of $W_k^{\min}(\mathfrak{g})$.
- Therefore $W_k^{\min}(\mathfrak{g})w$ is a highest weight submodule of $M$.
- Irreducibility of $M$ implies that $M = W_k^{\min}(\mathfrak{g})w$ is a highest weight $W_k^{\min}(\mathfrak{g})$–module.